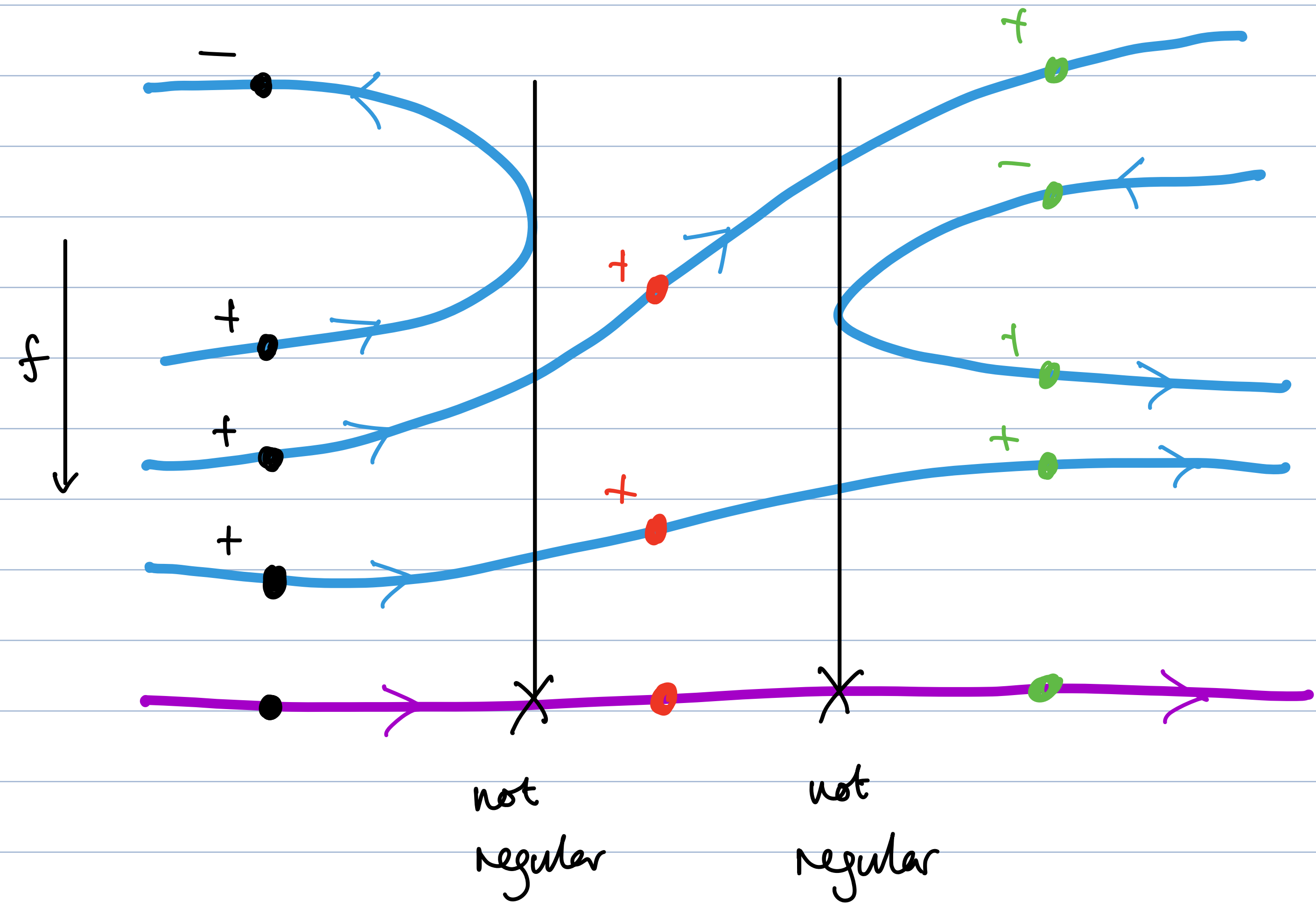
Can try to count points in $\beta^{-1}(L)$ ie genus g minimal fillings of L.

Compare definition of degree of map $f: M \to N$ between oriented manifolds of same dimension:

$$\text{degree}(f) = \begin{array}{l}\text{signed count of points} \\ \text{in } f^{-1}(x) \text{ for} \\ x \text{ a regular value of } f\end{array}$$



To make this work one needs f to be PROPER ie if $C \subseteq N$ is compact then $f^{-1}(C) \subseteq M$ is also compact.

Two reasons this matters:

1. If $x \in N$ is regular value, $f^{-1}(x)$ is compact 0-dim manifold and so FINITE.

2. If x_0 and x_1 are regular values of f in same component of N then we can find path x_t joining them which is transverse to f

So $f^{-1}(x_t) \subseteq M$ is compact 1-dim manifold with boundary $f^{-1}(x_1) - f^{-1}(x_0)$

ie collection of intervals pairing up the pts in $f^{-1}(x_0)$ and $f^{-1}(x_1)$.

So # pts in $f^{-1}(x_1)$ = # pts in $f^{-1}(x_0)$

Theorem (F.)

Fix $g=0$, $k=1$ (minimal discs filling knots)

There is a codim 2 subset $\mathcal{B} \subseteq \mathcal{L}_1$ s.t. $\beta: \mathcal{M}_{0,1} \to \mathcal{L}_1$ is proper over $\mathcal{L}_1 \setminus \mathcal{B}$.

Consequence The number of minimal discs in H^4 filling a knot in S^3 is a topological invariant of K.

($\mathcal{B}$ is codim 2 so if $K_0, K_1 \in \mathcal{L}_1 \setminus \mathcal{B}$ are isotopic in $\mathcal{L}_1$, they are isotopic in $\mathcal{L}_1 \setminus \mathcal{B}$.)

Conjecture Same thing works for every g, k ie $\exists$ codim 2 subset $\mathcal{B}_{g,k} \subseteq \mathcal{L}_k$ st β is proper over $\mathcal{L}_k \setminus \mathcal{B}_{g,k}$.

What happens in $\mathcal{B}$?

$\Sigma_n \in \mathcal{M}_{g,k}$ seq of minimal surfaces with ideal boundaries $L_n \in \mathcal{L}_k$

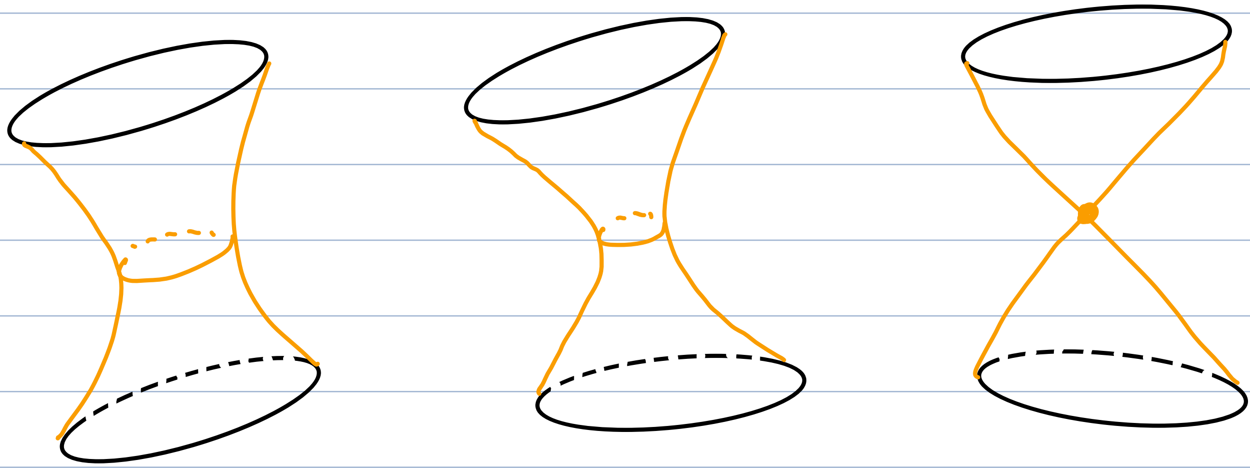
Suppose $L_n \longrightarrow L$. Want to show subsequence of Σ_n converges to filling of L.

Theorem (F.)

A subsequence of Σ_n does indeed converge to a minimal filling Σ of L

BUT

Σ might be nodal, ie some closed geodesic in Σ_n shrinks to a point giving a node in Σ.



$L_n \longrightarrow L$ $\quad$ $\Sigma_n \longrightarrow \Sigma$, node

Assume conjecture. Then we have "new" invariants $n_g(L)$ of an oriented link.

Put $\quad P(L) = \sum_g n_g(L)\, z^{2g+k-1}$

"minimal surface polynomial"

Conjecture $\quad P(L)$ is the Alexander polynomial of L.

Why I find this exciting:

- Alexander polynomial is relatively easy to compute