Symplectic Calabi-Yau Manifolds

Outline of what I hope to cover:

I CYs in Kähler geometry

II CYs in symplectic geometry

III Examples in 6D from 4D gauge theory / Riemannian geom.

IV Gromov-Witten & Minimal surfaces in 4D Riemannian manifolds.

V Gromov-Witten & Minimal surfaces in H^4 filling links in $S^3 = \partial_\infty H^4$.

I. CYs in Kähler geometry

Definition. A Calabi-Yau manifold is a compact Kähler manifold X for which $K := \Lambda^{top} T^* X$ is holomorphically trivial.

Why care?

Riemannian geometry: if X is CY then there exists a Kähler Ricci-flat metric on X (deep theorem of Yau, proving conjecture by Calabi).

This gave the first examples of compact Ricci flat metrics that aren't simply flat.

Algebraic geometry: try to classify smooth varieties. Only hol. line bundle we ALWAYS have is $K = \Lambda^{top} T^* X$.

- K^{-1} ample, "Fano variety" VERY SPECIAL
  In each dim. there are only finitely many diffeo types. Complete list in $\dim_{\mathbb{C}} = 1, 2, 3$ (1, 10, 105 diffeo types)

- K ample, "General type", somehow generic, too large a family to classify.

So $K \cong \mathcal{O}$, "Calabi-Yau" is a sort of phase transition. Might hope to have a classification.

$\dim_{\mathbb{C}} = 1$ they are elliptic curves $\cong T^2$

$\dim_{\mathbb{C}} = 2$ they are $\mathbb{C}^2/\Lambda \cong T^4$ or K3 surface (all diffeomorphic)

$\dim_{\mathbb{C}} = 3$ still unknown if there are infinitely many diffeomorphism types. Know of $\geq 30\,000$ non-diffeomorphic examples.

Physics Supersymmetry, String theory, M theory, ...

"Mirror symmetry" given a CY (X, ω, J) it should have a "mirror" $(\check{X}, \check{\omega}, \check{J})$ and the complex geometry of (X, J) should be related to the symplectic geometry of $(\check{X}, \check{\omega})$ (and vice versa)

II CYs in symplectic geometry

Natural question: can one make sense of "Calabi–Yau" in either purely complex or purely symplectic setting?

Definition. A symplectic manifold (X, ω) is called "Calabi–Yau" if $c_1(X, \omega) = 0$. Some authors also require $\pi_1(X) = 0$.

- To define $c_1(X, \omega)$ pick J compatible with ω. Then $c_1(X, \omega) := c_1(TX, J)$. Doesn't depend on choice of J.

- If X is Kähler, then $c_1(X) = 0 = \pi_1(X)$ implies K holomorphically trivial.

Examples

- All the Kähler CYs.

- Kodaira–Thurston manifold M

$$H_{\mathbb{R}} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$M = \frac{H_{\mathbb{R}}}{H_{\mathbb{Z}}} \times S^1$$

Nil manifold with left invariant ω