TM is symplectically trivial so $c_1 = 0$.

- Other 4D solvmanifolds $T^2 \hookrightarrow X \to T^2$
- Will explain how to find examples in 6D in a bit.

Conjecture (folklore)

Let (X^4, ω) be closed symplectic 4-manifold with $c_1 = 0$. Then X is one of the known examples above. ($K3$ or $T^2 \hookrightarrow X \to T^2$)

In particular if $c_1(X) = 0 = \pi_1(X)$ then (X, ω) is actually Kähler.

What we know so far:

- $b_+(X) \leq 3$ (via Seiberg-Witten)
- $\sigma(X) = 0$ or -16
- If $\sigma(X) = -16$ then $b_1(X) = 0$ and $b_+(X) = 3$ $\rightsquigarrow$ should be $K3$
- If $\sigma(X) = 0$ then $b_1 = 1 + b_+$
  $\rightsquigarrow$ $b_+ = 3$ should be T^4
  $\rightsquigarrow$ $b_+ = 2$ should be Kodaira-Th.
  $\rightsquigarrow$ $b_+ = 1$ should be solvmanifold.

Nothing like this is true in higher dims.

Theorem (F.–Panov 2011).

Let G be fin. presented group, $b \in \mathbb{N}$. There exists a closed 6-dimensional symplectic Calabi–Yau (X, ω) with

$$\begin{cases} \pi_1(X) = G \\ b_2(X) > b \\ b_3(X) > b \end{cases}$$

If $G = \{Id\}$ we can also find examples with $b_2 > 0$ and $b_3 = 0$

In particular there are infinitely many simply-connected symplectic Calabi–Yau 6-manifolds that are not Kähler.

Moreover there can be no "classification" of general closed symplectic CY 6-manifolds. (because finitely presented groups are algorithmically unrecognisable).

III A source of 6D symplectic CYs

Basic idea: Let $E \to M^4$ be an $SO(3)$ vector bundle over a 4-manifold.

$Z :=$ unit sphere bundle in E

$$S^2 \hookrightarrow Z \xrightarrow{\pi} M.$$

Each fibre $F_x := \pi^{-1}(x)$ has area form ω_x

Want to fit all of these fibrewise forms ω_x together into a global closed 2-form. i.e.

find $\omega \in \Omega^2(Z)$ st

$$\begin{cases} d\omega = 0 \\ \omega|_{F_x} = \omega_x \end{cases}$$

Then see if ω is actually symplectic (ie $\omega^3 \neq 0$).

Recall one way to get area form of S^2

- $S^2 \subseteq \mathbb{R}^3$
- $TS^2 \leq S^2 \times \mathbb{R}^3$
- $\nabla^{LC} =$ projection of trivial connection
- $\frac{1}{2\pi i} F_{\nabla^{LC}} = \omega_{S^2}$

We will do the same "relative to M^4"

- Let A be an $SO(3)$ connection in E
  Let $V \to Z$ be vertical tgt bd.

  ie $V|_{F_x} = TF_x$ is tangent bundle of fibre

- $TS^2 \leq S^2 \times \mathbb{R}^3$
  $V \leq \pi^* E$ via $\pi : Z \to M$.

  (over F_x, $\pi^* E \cong F_x \times E_x$ is trivial)

- $\pi^* A$ is $SO(3)$ – connection in $\pi^* E$

  Over F_x, $\pi^* A$ is trivial connection

- Can project $\pi^* A$ to V to get metric connection ∇ on V

Let $\omega_A := \frac{1}{2\pi i} F_\nabla$

By construction $\begin{cases} d\omega_A = 0 \\ \omega_A|_{F_x} = \omega_x \end{cases}$

Question: when is ω_A symplectic?