Lemma ω_A is symplectic iff the following inequality holds:

$\forall\, u,v \in T_xM$ independent, $F_A(u,v) \neq 0 \in so(E_x)$

ie F_A is non-zero on EVERY tangent 2-plane

("fat connection" introduced by Weinstein)

Definition If F_A is non-zero on every tangent 2-plane then we call A "definite".

Turns out that definite connections come in two "types" positive and negative:

$F_x \subseteq (Z, \omega_A)$ is symplectic submanifold

So has symplectic normal bundle $N \to F$

Can ask what is $\int_F c_1(N)$?

Lemma There are two possibilities:

1. $\int_F c_1(N) = 2.$

In this case $c_1(Z, \omega_A) = 4\omega_A$ and we say A is "positive definite"

2. $\int_F c_1(N) = -2$

In this case $c_1(Z, \omega_A) = 0$ Calabi-Yau!
and we say A is "negative definite."

Examples

$(M^4, g, \text{orientation})$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$
$$* = 1 \oplus -1$$

$\Lambda^+ \to M$ is $SO(3)$-bd

LC connection A is metric connection in Λ^+

When is A definite?

$Rm : \Lambda^2 \to \Lambda^2$

$$Rm = \left(\begin{array}{c|c} W^+ + \frac{S}{12} & Rc_0 \\ \hline Rc_0^* & W^- + \frac{S}{12} \end{array}\right) = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}$$

Let $\mathcal{B} = \left(W^+ + \frac{S}{12}\right)^2 - Rc_0^* Rc_0$

$$= R_{++}^2 - R_{+-}^* R_{+-}$$

Lemma If $D > 0$ then A is definite.

When this happens, $W^+ + \frac{s}{12} : \Lambda^+ \to \Lambda^+$ is invertible and the sign of A agrees with that of $\det(W^+ + \frac{s}{12})$.

In particular if g is anti-self-dual, Einstein and $s \neq 0$ then A is definite because $D = \frac{s^2}{144}$.

The sign of A in this case agrees with that of s.

Two key examples

1. $(S^4, g = g_{round})$

$$Rc_0 = 0$$
$$s = 12$$
$$W_+ = 0 = W_-$$

So LC on Λ^+ is positive definite

Get S^2 bundle $S^2 \hookrightarrow Z \to S^4$ with symplectic form ω st $c_1(Z) = 4\omega$

2. $(H^4, g = g_{hyperbolic})$

$$Rc_0 = 0$$
$$s = -12$$
$$W_+ = 0 = W_-$$

Get S^2 bundle $S^2 \hookrightarrow Z \to H^4$
with symplectic form ω st $c_1(Z,\omega)=0$

For S^4 the structure is standard:
$(Z,\omega) \cong (\mathbb{CP}^3, \omega_{FS})$

(To see $\mathbb{CP}^3$ as an S^2 bundle over S^4
we use quaternions: $\mathbb{C}^4 \cong \mathbb{H}^2$
Each complex line generates a quaternionic line. This gives a map $\mathbb{CP}^3 \to \mathbb{HP}^1$
Need to convince yourself that $\mathbb{HP}^1 \cong S^4$
just as $\mathbb{CP}^1 \cong S^2$. Also need to convince yourself the fibres are S^2.)

For H^4 we get a new interesting (non compact)
symplectic Calabi–Yau.

For compact examples we can take
$M = H^4/\Gamma$ to be a compact orientable
hyperbolic 4-manifold.

Then $X = Z/\Gamma$ is a compact symplectic
Calabi–Yau.

The examples I mentioned before
(showing symplectic CY 6-manifolds