Courant Algebroids and generating operators

1 COURANT ALGEBROIDS

[LIU-WEINSTEIN-XU]

$$\begin{array}{ll} E & \text{vector bundle} \\ \downarrow & \\ M & \dim M = m \ , \ \dim E_x = 2n \end{array}$$

- $\rho : E \to TM$ anchor
- $(\cdot,\cdot) : E_x \times E_x \to \mathbb{R}$

  non-degenerate symmetric biliner form (signature (n,n))
- $[\![\cdot,\cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

  bracket on sections of E

Remark: $\rho^* : T^*M \longrightarrow E^* \cong E$

For $e_i \in \Gamma(E)$, $f \in C^\infty(M)$

- $[\![e_1, f e_2]\!] = f [\![e_1, e_2]\!] + (\rho(e_1) f) e_2$

LEIBNIZ RULE

- $\rho(e_1)(e_2, e_2) = 2 ([\![e_1, e_2]\!], e_2)$

INVARIANCE OF $(\cdot,\cdot)$

- $[\![e, e]\!] = \frac{1}{2} \rho^* d (e, e)$

SYMMETRIC PART OF $[\![\cdot, \cdot]\!]$

- $[\![e, [\![e_1, e_2]\!]]\!] =$

$$= [\![[\![e, e_1]\!], e_2]\!] + [\![e_1, [\![e, e_2]\!]]\!]$$

Jacobi

$\Rightarrow$ $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]_{Lie}$

algebroid morphism

EXAMPLES

- $E = \mathfrak{g} \to M = pt$
  - $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
    Lie bracket
  - $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$
    invariant scalar product
  - $\rho : \mathfrak{g} \longrightarrow 0$

- $E = TM \oplus T^*M \to M$, $\quad H \in \Omega^3(M)$, $\quad dH = 0$
  - $\rho : TM \oplus T^*M \longrightarrow TM$
    natural projection
  - $(u + \alpha, v + \beta) = \alpha(v) + \beta(u)$
    $u, v \in \mathfrak{X}(M)$, $\alpha, \beta \in \Omega^1(M)$

Courant bracket

$$[\![u+\alpha , v+\beta]\!] = [u,v]_{Lie} +$$

$$+ L_u \beta - L_v \alpha +$$

$$+ d(\alpha(v)) + H(u,v,\cdot)$$

Remark :

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0$$

exact

ρ is surjective, ρ^* is injective

Thm [Severa] Classification by

$$[H] \in H^3(M, \mathbb{R})$$

- G connected Lie group

$$\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$$

invariant scalar product

$$E = (\mathfrak{g}\oplus\mathfrak{g})\times G$$
$$\downarrow$$
$$M = G$$

- $\rho : (x,y)\longmapsto x^L - y^R$

- $((x_1,y_1),(x_2,y_2)) =$

$$\langle x_1,x_2\rangle - \langle y_1,y_2\rangle$$

- $[\![(x_1,y_1),(x_2,y_2)]\!] = ([x_1,x_2],[y_1,y_2])$

Prop: $E \cong (TG\oplus T^*G,\ H = \text{Cartan 3-form})$

Dirac structures

Def $F \subset E$ is Dirac if

- $\dim F_x = n$
- $(\cdot,\cdot)|_F = 0$
- $[\![\cdot,\cdot]\!] : \Gamma(F) \times \Gamma(F) \longrightarrow \Gamma(F)$

$\Rightarrow$ $\begin{array}{c} F \\ \downarrow \\ M \end{array}$, $\rho : F \longrightarrow TM$, $[\![\cdot,\cdot]\!]$ is a Lie algebroid

Examples:

- $\mathfrak{f} \subset \mathfrak{g}$ Lagrangian Lie subalgebra

- $\omega \in \Omega^2(M)$, $d\omega = 0$

$$F_x^{\omega} = \{ u + \omega(u, \cdot) \ ; \ u \in T_x M \}$$

- $\pi \in \Gamma(\Lambda^2 TM)$, $[\pi, \pi]_{Sch} = 0$

$$F_x^{\pi} = \{ \pi(\alpha, \cdot) + \alpha \ ; \ \alpha \in T_x^* M \}$$

- $F_{\mathfrak{g}} = \{ (x, x) \in \mathfrak{g} \oplus \mathfrak{g} \ ; \ x \in \mathfrak{g} \}$

Cartan-Dirac structure

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$

$$F_{\mathfrak{g}} = \{ (x + z, y - z) \ ; \ x \in \mathfrak{n}_+, y \in \mathfrak{n}_-, z \in \mathfrak{h} \}$$

Gauss-Dirac structure

(2) GENERATING OPERATORS

[XU - AA]

[GRÜTZMANN - MICHEL - XU]

$$\begin{array}{ccccc} (E, (\cdot,\cdot)) & \rightsquigarrow & Cl(E) & \circlearrowright & \mathcal{S} \\ \downarrow & & \downarrow & & \downarrow \\ M & & M & & M \end{array}$$

- bundle of Clifford algebras
- Clifford module

- ∇^E connection on E preserving $(\cdot,\cdot)$

$$v(e_1, e_2) = (\nabla^E_v e_1, e_2) + (e_1, \nabla^E e_2)$$

- $\Rightarrow$ connection $\nabla^{Cl(E)}$

$$\nabla^{Cl(E)}_v (a \cdot b) = (\nabla^{Cl(E)}_v a) \cdot b + a \cdot (\nabla^{Cl(E)}_v b)$$

- compatible connection on $\mathcal{S}$:

$$\nabla^{\mathcal{S}}_v (a \cdot s) = (\nabla^{Cl(E)}_v a) \cdot s + a \cdot (\nabla^{\mathcal{S}} s)$$

Differential operators

$$\mathrm{Diff}^{(n)}(\mathcal{S}) = \{ \delta \in \mathrm{Diff}(\mathcal{S}) ;$$
$$[\ldots[\delta, e_1], e_2], \ldots, e_{n+1}] = 0$$
$$\forall\ e_1, \ldots, e_{n+1} \in \Gamma(E) \}$$

- increasing filtration

$$\mathrm{Diff}^{(0)}(\mathcal{S}) = C^\infty(M)$$
$$\mathrm{Diff}^{(1)}(\mathcal{S}) = \Gamma(E) \qquad \ldots$$

- associated graded

$$\mathcal{O}(T^*[2]M \oplus E[1])$$

$$\sigma_n : \mathrm{Diff}^{(n)}(\mathcal{S}) / \mathrm{Diff}^{(n-1)}(\mathcal{S}) \longrightarrow \mathcal{O}^n(T^*[2]M \oplus E[1])$$

symbol map

Thm [GMX]

$D \in \mathrm{Diff}^{(3)}(\mathcal{S})$ such that

- $\delta_3(D) \neq 0$
- $D^2 \in \mathrm{Diff}^{(2)}(\mathcal{S})$

$$\Rightarrow \quad \rho(e)f = [[D, e], f]$$

$$[\![e_1, e_2]\!] = [[D, e_1], e_2] \qquad (*)$$

is a Courant algebroid

Remark :

- $(*)$ is the derived bracket [Kosmann-Schwarzbach]

- $\tilde{D} = D + e$ defines the same structure

EXAMPLE

$$\rho(e) f = [[D,e], f]$$

$$\Rightarrow \quad \rho^* d f = [D, f]$$

- $$\llbracket e, e \rrbracket = [[D,e], e]$$

$$= \frac{1}{2} [D, [e,e]]$$

$$= \frac{1}{2} [D, (e,e)]$$

$$= \frac{1}{2} \rho^* d (e,e)$$

$$\Theta = \sigma_3(D) \in \mathcal{O}_3\left(T^*[2]M \oplus E[1]\right)$$

Thm [Roytenberg]

$\exists!\ \Theta \in \mathcal{O}_3\left(T^*[2]M \oplus E[1]\right)$

such that

- $\rho(e)f = \{\{\Theta, e\}, f\}$
- $[\![e_1, e_2]\!] = \{\{\Theta, e_1\}, e_2\}$

Question : existence & uniqueness of D

Assume $\mathcal{S} \times \mathcal{S} \to \Omega^{top}(M)$

Hermitian scalar product

$\Rightarrow$ $D^* \in \mathrm{Diff}(\mathcal{S})$

formal adjoint

Thm [Xu-AA, Severa, DMX]

- $\exists!$ D generating operator

  such that $D^* = -D$

- $D^2 \in \mathrm{Diff}^{(0)}(\mathcal{S}) = C^\infty(M)$

  is an invariant of Courant algebroid

EXAMPLES

- $\mathfrak{g} \quad \mathfrak{h} \subset \mathfrak{g}$ Dirac

$$\begin{array}{c} \mathfrak{g} \\ \downarrow \\ pt \end{array} \qquad S = \Lambda(\mathfrak{g}/\mathfrak{h})$$

$$\mathcal{S} = \frac{1}{6} \sum_{abc} C_{abc}\, e_a e_b e_c$$

$\uparrow$ structure constants

$$D^2 \sim (C, C)$$

- $E = TM \oplus T^*M$

$$\Gamma(S) = \Omega(M)$$

$$D = d + H\wedge \,, \quad D^2 = 0$$

$$\rho(e) f = [[D, u + \alpha], f]$$

$$= [L_u + H(u, \cdot, \cdot), f] = L_u f = u(f)$$

- $E = (\mathfrak{g} \oplus \mathfrak{g}) \times G$

$$\Gamma(\mathcal{S}) = C(G, Cl(\mathfrak{g}))$$

$$D = \sum_a \left(e_a^L \otimes e_a^L - e_a^R \otimes e_a^R\right) + \delta^L - \delta^R$$

$D^2 = 0$ ($\Leftarrow$ equivalence to $(TG \oplus T^*G, H)$)

- general case:

$$D = \sum_a e^a \nabla^{\mathcal{S}}_{\rho(e_a)} - T^\nabla + \frac{1}{2} e^\nabla$$

$\{e_a\}, \{e^a\}$ pair of dual bases of $\Gamma(E)$

- $T^\nabla(e_1, e_2, e_3) = \frac{1}{2} \mathrm{Cycl}_{123} \left(\frac{1}{3} \left([\![e_1, e_2]\!] - [\![e_2, e_1]\!] \right) - \right.$
$\left. - \left(\nabla^E_{\rho(e_1)} e_2 - \nabla^E_{\rho(e_2)} e_1\right), e_3 \right)$

$T^\nabla \in \Gamma(\Lambda^3 E) \longrightarrow \Gamma(Cl(E))$ Courant torsion

- $e^\nabla = \sum_a \nabla^E_{\rho(e_a)} e^a$

Applications to Dirac structures

Def $\psi \in \Gamma(S)$ is a pure spinor

if $\ker(\psi_x) = \{ e \in E_x \, ; \, e . \psi_x = 0 \}$

is of $\dim = n \quad \forall x \in M$

$\Rightarrow \quad (\cdot , \cdot)|_{F_x} = 0$

Thm
- $\psi \in \Gamma(S)$ is a pure spinor
- $\exists e \in \Gamma(E)$ such that $\boxed{D\psi = e\psi}$

$\Rightarrow \ker(\psi) \subset E$ is a

Dirac structure

Examples

- $E = \mathfrak{g} \supset \mathfrak{h}\,, \quad S = \Lambda(\mathfrak{g}/\mathfrak{h})$

$$|$$

$\underline{\psi = 1}$

$$D\psi = e\psi\,, \quad e = \frac{1}{2}\sum_a \left(\mathrm{Tr}_{\mathfrak{h}}\, \mathrm{ad}_{e_a}\right) e^a$$

$$\cap$$

$$\mathfrak{g}/\mathfrak{h}$$

modular element

- $E = TM \oplus T^*M \supset \{u + \omega(u,\cdot)\}\,, \quad \underline{H = 0}$

$$\psi = e^{-\omega} \in \Omega(M)$$

$$D\psi = d\psi = 0$$

- $E = TM \oplus T^*M\,, \quad H = 0$

$$\cup$$

$$\{\pi(\alpha,\cdot) + \alpha\}\,, \quad \sigma \in \Omega^{top}(M)$$

$$\psi = e^{\iota(\pi)}\sigma\,, \quad d\psi = \iota(u)\,\psi$$

modular vector field

[Bursztyn - Meinrenken - AA]

- Cartan - Dirac

$$E = (\mathfrak{g} \oplus \mathfrak{g}) \times G$$

$$\mathcal{S} = Cl(\mathfrak{g}) \times G$$

$$\boxed{\psi(g) = 1}$$

- Gauss - Dirac

$N_+ H N_- \longrightarrow G^{\mathbb{C}}$ open & dense

$$\psi = g_0^{\rho} \exp\left[\frac{1}{2} \langle \theta_-^L, \mathrm{Ad}_{g_0} \theta_+^R \rangle\right]$$

extends to $G^{\mathbb{C}}$

$$(d + H)\psi = \langle \rho, dg_0 g_0^{-1} \rangle \psi$$

Thank you !