Self crossing stable generalized complex structures

Aldo Witte, Hamburg

February 20, 2026

Motivation

Goal

Construct examples of GC structures on manifolds which are neither symplectic nor complex.

1. $n\#\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}$ if n is odd (Cavalcanti-Gualteri '09).
2. $n\#\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}\#(S^1 \times S^3)$ if n is even (Torres '12);
3. $n\#(S^2 \times S^2)$ if n is odd (Torres '12);
4. $n\#(S^2 \times S^2)\#(S^1 \times S^3)$if n is even (Torres '12);
5. Most elliptic surfaces (Goto-Hayano '16, Torres-Yazinski '14).

These are all special GC structures called **stable**.

Motivation

Goal

Obtain a connected sum procedure for stable GC structures.

To do this, we will need to:

- Enlarge the definition of stable to **Self-crossing stable** (X-stable).
- Work with **elliptic symplectic structures**.

Joint work with Gil Cavalcanti and Ralph Klaasse.

Stable structures

Let $(\mathbb{J}, H)$ be a generalized complex structure. Consider the spinor line bundle $K < \wedge^\bullet T^*M$ defined by the condition

$$K \cdot L_{\mathbb{J}} = 0.$$

Let $s \in \Gamma(K^*)$ denote the section $\Gamma(K) \ni \rho \mapsto \rho_0$.

Definition

We say that $\mathbb{J}$ is **stable** if s vanishes transversely.

Example

$\mathbb{C}^2$ with $\rho = z_1 + dz_1 \wedge dz_2$.

Type 0 everywhere except type 2 at $D := s^{-1}(0)$.

X-stable structures

Definition

We say that $\mathbb{J}$ is **X-stable** if s is locally of the form $z_1 z_2$.

Example

$\mathbb{C}^2$ with $\rho = z_1 z_2 + dz_1 \wedge dz_2$.

Type 0 except everywhere type 2 at D.

Divisors

The ideal $s(\Gamma(K^*)) \subset C^\infty(M, \mathbb{C})$ is an example of a **divisor**.

Definition

- A **complex log** divisor is $I_D \subset C^\infty(M, \mathbb{C})$ which is locally generated by a single transverse vanishing function. $D = I_D^{-1}(\{0\})$.
- A **complex X-log** divisor is I_D such that locally $I_D = I_{D_1} \cdot I_{D_2}$ a product of complex log, with D_1 transverse to D_2.

For an X-stable structure $s(\Gamma(K^*)$ is a complex X-log divisor.

Real divisors

Definition

- An **elliptic divisor** is a real divisor $I_q \subset C^\infty(M, \mathbb{R})$ which is locally generated by a definite Morse-Bott function with codimension two critical set D.
- An **X-elliptic divisor** is I_q with locally $I_q = I_{q_1} \otimes I_{q_1}$ a product of elliptic, with D_1 transverse to D_2.

Given I_D complex X-log, then $I_{|D|} := I_D \otimes \overline{I}_D$ is X-elliptic. Moreover, I_D induces a co-orientation $\mathfrak{o}$ on $D[1]$.

Proposition

The association $I_D \to (I_{|D|}, \mathfrak{o})$ is a bijection.

Algebroids

Definition

Define the Lie algebroids:

- The **complex X-log tangent bundle** $A_D = \{X \in \mathfrak{X}(M, \mathbb{C}) | X(I_D) \subset I_D)\}$.
- The **X-elliptic tangent bundle** $A_{|D|} = \{X \in \mathfrak{X}(M) | X(I_{|D|}) \subset I_{|D|}\}$.

Example

$M = \mathbb{C}^2$, $I_D = \langle z_1 z_2 \rangle$, $I_{|D|} = \left\langle r_1^2 r_2^2 \right\rangle$.
$A_D = \langle z_1 \partial_{z_1}, z_2 \partial_{z_2}, \partial_{\overline{z_1}}, \partial_{\overline{z_2}} \rangle$, $A_{|D|} = \langle r_1 \partial_{r_1}, \partial_{\theta_1}, r_2 \partial_{r_2}, \partial_{\theta_2} \rangle$.
$r_1 \partial_{r_1} = x_1 \partial_{x_1} + y_1 \partial_{y_1}$, $\partial_{\theta_1} = x_1 \partial_{y_1} - y_1 \partial_{x_1}$.

Let Q be holomorphic Poisson on (M, J), then $\rho = e^{Q}(\Omega)$ is GC.

Example

Let $Q = z_1 z_2 \partial_{z_1} \wedge \partial_{z_2}$. Then $\rho = z_1 z_2 + dz_1 \wedge dz_2$ and

$$Im(Q) = r_1 \partial_{r_1} \wedge \partial_{\theta_2} + \partial_{\theta_1} \wedge \partial_{r_2}.$$
$$\omega = Im(Q)^{-1} = d\theta_2 \wedge d\log r_1 + d\log r_2 \wedge d\theta_1 \in \Omega^2(\mathcal{A}_{|D|}).$$

Definition

We call $\omega \in \Omega^2(\mathcal{A}_{|D|})$ an **X-elliptic symplectic form** if $d\omega = 0$ and $\omega^\flat$ is an isomorphism.

Given an X-stable structure, we get an X-elliptic symplectic. When can we get back?

Residues

We single out singular coefficients:

Definition

Given $\omega \in \Omega^2(\mathcal{A}_{|D|})$ we define

- $\mathrm{Res}_q(\omega) \in \Omega^0(D[1])$, to be $\omega_p(r\partial_r, \partial_\theta)$ for all $p \in D[1]$.
- $\mathrm{Res}_{\theta_1\theta_2}(\omega) \in \Omega^0(D[2])$, to be $\omega_p(\partial_{\theta_1}, \partial_{\theta_2})$ for all $p \in D[2]$.
- $\mathrm{Res}_{r_1 r_2}(\omega), \mathrm{Res}_{r_1\theta_2}(\omega), \mathrm{Res}_{r_2\theta_1}(\omega) \in \Omega^0(D[1])$.

$$Im(d\log z_1 \wedge d\log z_2) = d\log r_1 \wedge d\theta_2 + d\theta_1 \wedge d\log r_2,$$
$$Im(id\log z_1 \wedge d\log z_2) = d\log r_1 \wedge d\log r_2 - d\theta_1 \wedge d\theta_2.$$

$$\Omega^2_{\mathfrak{J}}(\mathcal{A}_{|D|}) = \ker(\mathrm{Res}_q) \cap \ker(\mathrm{Res}_{\theta_i r_j} - \mathrm{Res}_{r_i\theta_j}) \cap \ker(\mathrm{Res}_{r_i r_j} - \mathrm{Res}_{r_i\theta_j}).$$

Equivalence with elliptic symplectic

Theorem

The association

$$(\mathbb{J}, H) \mapsto (I_{|D|}, \mathfrak{o}, \pi_{\mathbb{J}}^{-1})$$

is such that $\pi_{\mathbb{J}}^{-1} \in \Omega^2_{\mathfrak{J}}(\mathcal{A}_{|D|})$*. And is a 1-1 correspondence between gauge-equivalence classes of X-stable structures and such elliptic symplectic structures.*

$\Omega^2_{\mathfrak{J}}(\mathcal{A}_{|D|}) = Im^*(\Omega^2(A_D)).$

Local model

Theorem

Let M^4 be X-stable and $p \in D[2]$, then there are complex coordinates (z_1, z_2) such that

$$\rho = e^B(\lambda z_1 z_2 + dz_1 \wedge dz_2),$$

for some $\lambda \in \mathbb{C}^*$ and $B \in \Omega^2(M; \mathbb{R})$.

Definition

We say that $\omega \in \Omega^2(\mathcal{A}_{|D|})$ is **locally complex** if every point has a neighbourhood and complex log divisor I_D such that $\omega = Im^*(\sigma)$ for some $\sigma \in \Omega^2(A_D)$.

Connected sum of divisors

Lemma

Let (M^4, I_{D_M}), (N^4, I_{D_N}) *be oriented manifolds with elliptic divisors, and* $p \in D_M[2], q \in D_N[2]$. *Then* $M\#_{p,q}N$ *admits a natural elliptic divisor.*

Proof.

Consider

$$F : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^4 \setminus \{0\}, (r, \phi_1, \phi_2, \phi_3) \mapsto (r^{-1}, \phi_1, \phi_2, -\phi_3).$$

Locally I_{D_M} and I_{D_N} are of the form $I =< r_1^2 r_2^2 >$. You can check that $r^8 F^*(r_1^2 r_2^2) = r_1^2 r_2^2$, and hence $F^* I = I$. □

Connected sum of elliptic symplectic

Lemma

Let $\omega = Im^*(id \log z_1 \wedge d \log z_2)$. *Then* $F^*\omega = -\omega$.

Definition

We say that $\omega in \Omega^2(\mathcal{A}_{|D|})$ locally complex has **imaginary parameter** at $p \in D[2]$ if $\mathrm{Res}_{r_1\theta_2}\, \omega(p) = \mathrm{Res}_{r_2\theta_1}\, \omega(p) = 0$.

Theorem

Let $(M, I_{D_M}, \omega_1), (N, I_{D_N}, \omega_2)$ *be locally complex, with imaginary parameter at* $p \in D_M[2], q \in D_N[2]$. *Then* $M\#_{p,q}N$ *admits a natural locally complex elliptic symplectic form.*

X-stable structures

Example (Stable examples)

- $\mathbb{C}P^2$ together with $\pi = z_1 z_2 \partial_{z_1} \wedge \partial_{z_2}$.
- $S^2 \times S^2$ together with same π.

Example (Non GC examples)

- $\overline{\mathbb{C}P^2}$.
- S^4.

Many examples

Theorem

The manifolds in the following two families admit self-crossing elliptic symplectic structures

1. $X_{n,\ell} := \#n(S^2 \times S^2)\#\ell(S^1 \times S^3)$, *with* $n, \ell \in \mathbb{N}$*;*
2. $\hat{X}_{n,m,\ell} := \#n\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}\#\ell(S^1 \times S^3)$, *with* $n, m, \ell \in \mathbb{N}$*,*

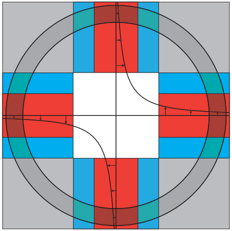
whenever the Euler characteristic is non-negative. They correspond to X-stable generalized complex structures as long as $1 - b_1 + b_2^+$ *is even.*

Smoothing

Theorem

Let $(M^4, I_{|D|}, \omega)$ *be co-oriented X-elliptic symplectic. Then:*

- *If* ω *is locally complex, then it can be deformed into a smooth elliptic symplectic form* $\widetilde{\omega}$.
- *The resulting structure is induced by a GC structures iff the original one was.*



T-duality

Definition

Let $M, \widetilde{M}$ be principal T^k-bundles with base B, and $H, \widetilde{H}$ T^k-invariant closed three forms. They are **T-dual** if there exists a T^{2k}-invariant form $F \in \Omega^2(M \times_B \widetilde{M})$ such that

$$dF = p^* H - \widetilde{p}^* H$$

and

$$F : \mathfrak{t}_M \times \mathfrak{t}_{\widetilde{M}} \to \mathbb{R},$$

with $\mathfrak{t}$ the tangent to the fibre of p, is non-degenerate.

Consequences

Theorem (Bouwknegt–Evslin–Mathai)

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual via F, then

$$\tau : (\Omega^\bullet_{T^k}(M), d_H) \to (\Omega^\bullet_{T^k}(\widetilde{M}), d_{\widetilde{H}}), \quad \rho \mapsto \int_{T^k} e^F \wedge \rho,$$

with integral over fibres of p, is an isomorphism of chain complexes.

Theorem (Cavalcanti-Gualteri)

There is an isomorphism of Courant algebroids

$$(TM \oplus T^*M)/T^k \to (T\widetilde{M} \oplus T^*\widetilde{M})/T^k,$$

intertwining the Clifford actions of $\Omega^\bullet_{T^k}(M)$ and $\Omega^\bullet_{T^k}(\widetilde{M})$.

Consequences 2

Theorem (Cavalcanti-Gualteri)

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual, then τ sends Dirac structures to Dirac structures, generalized complex structures to generalized complex structures.

Toric actions

Definition

A **toric action** is an effective T^n-action on M^{2n} with connected isotropy groups.

Proposition

$\mu : M \to B := M/T^n$ is a manifold with corners, and if $I_{\partial B}$ are the functions vanishing at ∂B, then $I_{|D|} := \mu^* I_{\partial B}$ is an elliptic divisor.

Lemma

The infinitesimal generators $X_1, \ldots, X_n$ all lift to nowhere vanishing sections of $\mathcal{A}_{|D|}$.

Toric T-duality

Definition

Let $M^{2n}, \widetilde{M}^{2n}$ be toric actions with base B, and $H \in \Omega^3_{T^n}(M, \mathcal{A}_{|D|})$ closed, and similar for $\widetilde{H}$. They are **T-dual** if there exists a T^{2k}-invariant form $F \in \Omega^2(M \times \widetilde{M}, \mathcal{A}_{|D|} \times \mathcal{A}_{|\widetilde{D}|})$ such that

$$\iota^*_{(M\backslash D)\times_B(\widetilde{M}\backslash\widetilde{D})}(dF - p^*H + \widetilde{p}^*H) = 0$$

and

$$F : \mathfrak{t}_M \times \mathfrak{t}_{\widetilde{M}} \to \mathbb{R},$$

with $\mathfrak{t}$ the tangent to the fibre of p, is non-degenerate.

Consequences

Theorem

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual via F, then

$$(\Omega^{\bullet}_{T^k}(M, \mathcal{A}_{|D|}), d_H) \simeq (\Omega^{\bullet}_{T^k}(\widetilde{M}, \mathcal{A}_{|\widetilde{D}|}), d_{\widetilde{H}}).$$

Theorem

There is an isomorphism of Courant algebroids

$$(\mathcal{A}_{|D|} \oplus \mathcal{A}^*_{|D|})/T^k \simeq (\mathcal{A}_{|\widetilde{D}|} \oplus \mathcal{A}^*_{|\widetilde{D}|})/T^k,$$

Generalized complex structures on $\mathcal{A}_{|D|}$ are send to generalized complex structures on $\mathcal{A}_{|\widetilde{D}|}$. But these might not descend to M.

Example

Example

$\mathbb{C}^2 : (\theta_1, \theta_2) \cdot (z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$

Example

- $\omega_1 = d \log r_1 \wedge d\theta_2 + d\theta_1 \wedge d \log r_2$ is T-dual to the standard complex structure on $\mathbb{C}^2$.
- $\omega_2 = d \log r_1 \wedge d \log r_2 - d\theta_1 \wedge d\theta_2$ is self T-dual.