Higher Holonomy

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Classical connections

Let $E \to M$ be a vector bundle.

- A **connection** on E is a degree 1 derivation
$$d^\nabla : \Omega(M, E) \to \Omega(M, E)$$
with symbol d, i.e., satisfying the Leibniz rule
$$d^\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d^\nabla \beta.$$
- **Curvature** $R^\nabla := (d^\nabla)^2 \in \Omega^2(M, \mathrm{End}(E))$.
- **Holonomy**: $\mathrm{Hol}(\nabla) := \{\tau_\gamma : \gamma \text{ is a smooth loop at } x\}$.

Theorem (Ambrose–Singer, 1953)

$$\mathfrak{hol}(\nabla) = \mathrm{span}\{\tau_\gamma^{-1} R^\nabla(X, Y)\tau_\gamma : \gamma \textit{ is a smooth path with } \gamma(0) = x \textit{ and } X, Y \in T_{\gamma(1)}M\}.$$

Lie algebroid connections

Let $A \to M$ be a Lie algebroid and $E \to M$ a vector bundle.

- An A-**connection** on E is a degree 1 derivation
$$d^\nabla : \Omega(A, E) \to \Omega(A, E)$$
with symbol d_A, i.e., satisfying the Leibniz rule
$$d^\nabla(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d^\nabla \beta.$$
- **Curvature** $R^\nabla := (d^\nabla)^2 \in \Omega^2(A, \mathrm{End}(E))$.
- **Holonomy**: $\mathrm{Hol}(\nabla) := \{\tau_a : a \text{ is an } A\text{-loop at } x\}$.

Theorem (Fernandes, 2002)

$$\begin{aligned} \mathfrak{hol}(\nabla) = \mathrm{span}\{\tau_e^{-1} R^\nabla(a,b)\tau_e : &\ e \textit{ is an } A\textit{-path with } \gamma_e(0) = x \\ &\textit{and } a, b \in A_{\gamma_e(1)}\} \\ + \mathrm{span}\{\tau_e^{-1}\nabla_c \tau_e : &\ e \textit{ is an } A\textit{-path with } \gamma_e(0) = x \\ &\textit{and } c \in \ker \rho_{\gamma_e(1)}\} \end{aligned}$$

Lie algebroid connections

We can define

$$\mathfrak{curv}(\nabla) := \mathrm{span}\{\tau_e^{-1} R^\nabla(a,b)\tau_e : e \text{ is an } A\text{-path with } \gamma_e(0) = x \\ \text{and } a, b \in A_{\gamma_e(1)}\}.$$

- It is an ideal of $\mathfrak{hol}(\nabla)$.
- There is a normal subgroup $\mathrm{Curv}(\nabla) \subseteq \mathrm{Hol}^0(\nabla)$ integrating it.

Conclusion: there is an inclusion of Lie groups $\mathrm{Curv}(\nabla) \hookrightarrow \mathrm{Hol}(\nabla)$.

- For a Lie theorist, this is just an inclusion of Lie groups.
- For a higher Lie theorist, this is the easiest example of a strict 2-group.

Question

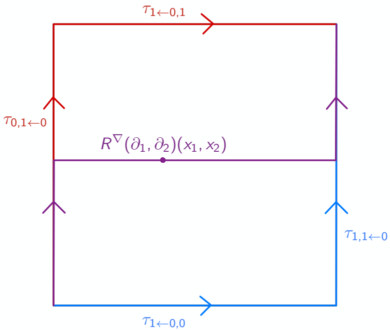
Does the inclusion $\mathrm{Curv}(\nabla) \hookrightarrow \mathrm{Hol}(\nabla)$ have a higher life? Is it geometric?

The proof of Ambrose–Singer

Lemma

Let ∇ be a connection on $E \to I^2$. Then

$$\tau_{1\leftarrow 0,1}\tau_{0,1\leftarrow 0} - \tau_{1,1\leftarrow 0}\tau_{1\leftarrow 0,0} =$$
$$= \int_0^1\int_0^1 \tau_{1,1\leftarrow x_2}\tau_{1\leftarrow x_1,x_2} R^{\nabla}(\partial_1,\partial_2)(x_1,x_2)\tau_{x_1\leftarrow 0,x_2}\tau_{0,x_2\leftarrow 0}\,dx_1\,dx_2.$$

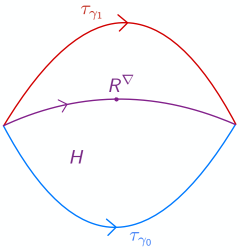


The proof of Ambrose–Singer

Corollary

Let ∇ be a connection on $E \to M$ and H a homotopy between paths γ_0 and γ_1. Then

$$\tau_{\gamma_1} - \tau_{\gamma_0} = \int_H R^\nabla.$$



Corollary

$\mathrm{Hol}^0(\nabla) = \{\mathbf{1} + \int_H R^\nabla : H \text{ is a null-homotopy based at } x\}$.

Riemann–Hilbert correspondence

Corollary

If ∇ is flat and $\gamma_0 \sim \gamma_1$, then $\tau_{\gamma_0} = \tau_{\gamma_1}$. In particular, ∇ defines a Lie groupoid morphism $\tau : \Pi_1(M) \to \mathrm{GL}(E)$.

Recall:

- $\Pi_1(M) \rightrightarrows M$ with $\Pi_1(M) = \{[\gamma] : \gamma \text{ path}\}$,
- $\mathrm{GL}(E) \rightrightarrows M$ with $\mathrm{GL}(E) = \{(y, A, x) : x, y \in M, A : E_x \xrightarrow{\cong} E_y\}$.

Theorem (Riemann–Hilbert correspondence I)

There is an equivalence of categories

$$\mathrm{Flat}_M(E) \to [\Pi_1(M), \mathrm{GL}(E)]_*^\infty.$$

Riemann–Hilbert correspondence

Question

What if ∇ is not flat?

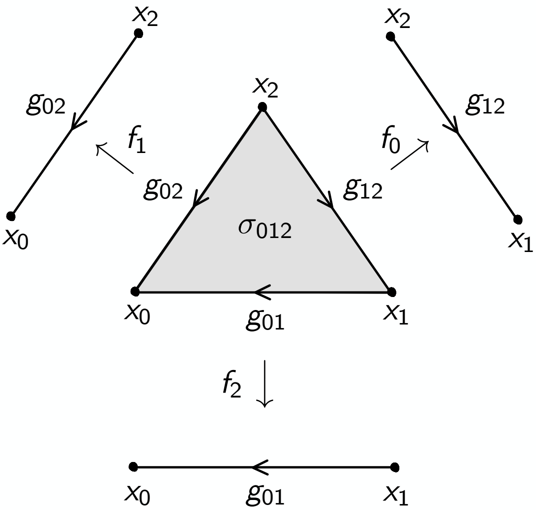
- We still have a functor $\tau : \mathcal{P}_1(M) \to \mathrm{GL}(E)$. *Anything else?*
- Idea: if H_0 and H_1 are two homotopies between γ_0 and γ_1 and L is a homotopy from H_0 to H_1, then a non-abelian Stokes's theorem (??) would give, by the Bianchi identity,

$$\int_{H_1} R^\nabla - \int_{H_0} R^\nabla = \int_L d^\nabla R^\nabla = 0.$$

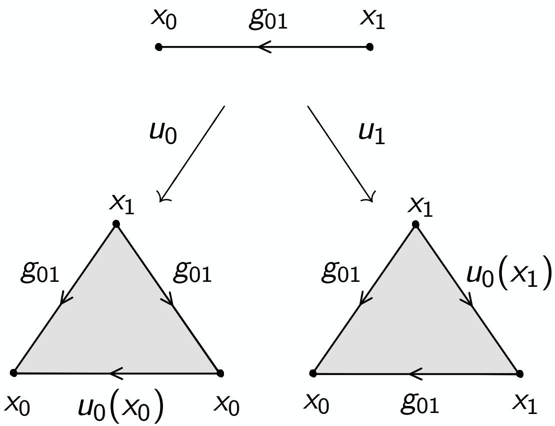
Question

Do we get a functor $\Pi_2(M) \to$?? ?

Higher (Lie) groupoids



Higher (Lie) groupoids



Higher (Lie) groupoids

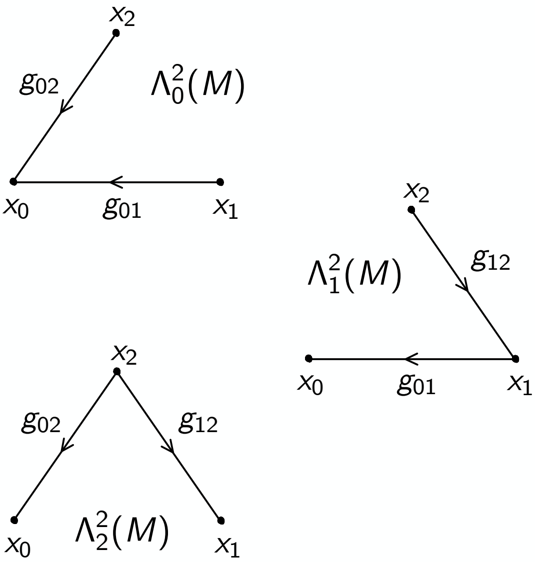
Definition

A **simplicial manifold** M is given by

- manifolds $\{M_n\}_n$, where M_n is called the space of n-**simplices**,
- surjective submersions $f_i : M_n \to M_{n-1}$, for $i = 0, \ldots, n$, called the **face maps**,
- smooth maps $u_i : M_{n-1} \to M_n$, for $i = 0, \ldots, n-1$, called the **units** or **degeneracy maps**,

satisfying the **simplicial relations**.

Higher (Lie) groupoids



Higher (Lie) groupoids

Definition

An (n, k)-**horn** is an n-simplex with the interior and the kth face removed. The space of (n, k)-horns of M is denoted by $\Lambda^n_k(M)$.

There are natural **horn maps** $\lambda^n_k : M_n \to \Lambda^n_k(M)$.

Definition

A simplicial manifold G is

- a **Lie ∞-groupoid** if all the horn maps are surjective submersions,
- a **Lie n-groupoid** if it is a Lie ∞-groupoid and the horn maps λ^m_k are diffeomorphisms for $m > n$.

Higher (Lie) groupoids

Example

If $G \rightrightarrows M$ is a Lie groupoid, its **nerve** $\mathrm{N}(G)$ is the Lie 1-groupoid defined by

$$\mathrm{N}(G)_k := G^{(k)} := G \times_M \overset{(k)}{\cdots} \times_M G.$$

Example

If $A \to M$ is a Lie algebroid, its **homotopy ∞-groupoid** $\Pi_\infty(A)$ is the ∞-groupoid defined by

$$\Pi_\infty(A)_k := \mathrm{Hom}_{\mathrm{LA}}(T\Delta^k \to A).$$

Its **homotopy** 2-**groupoid** $\Pi_2(A)$ we define by $\Pi_2(A)_i = \Pi_\infty(A)_i$ for $i = 0, 1$, and

$$\Pi_2(A)_2 := \mathrm{Hom}_{\mathrm{LA}}(T\Delta^2 \to A)/\mathrm{LA}\text{ homotopy}.$$

General linear 2-groupoid

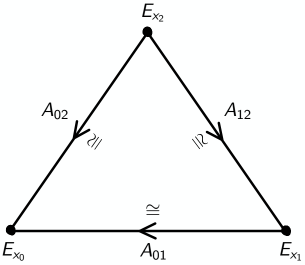
Definition

Let $E \to M$ be a vector bundle. Its **general linear** 2-**groupoid** is the Lie 2-groupoid $\mathrm{GL}_2(E)$ defined by

$$\mathrm{GL}_2(E)_0 := M,$$

$$\mathrm{GL}_2(E)_1 := \{(x_0, A_{01}, x_1) : x_0, x_1 \in M, A_{01} : E_{x_1} \xrightarrow{\cong} E_{x_0}\}$$

$$\mathrm{GL}_2(E)_2 := \{(x_0, x_1, x_2, A_{01}, A_{02}, A_{12}) : x_i \in M, A_{ij} : E_{x_j} \xrightarrow{\cong} E_{x_i}\}.$$



Very important: we do not impose

$$A_{01}A_{12} - A_{02} = 0.$$

General linear 2-groupoid

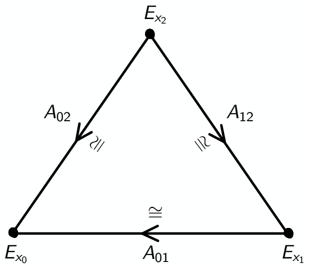
Definition

The **curvature bundle** $\mathrm{Curv}_2(E) \to \mathrm{GL}_2(E)$ is the simplicial vector bundle given by $\mathrm{Curv}_2(E)_0 = \mathrm{Curv}_2(E)_1 := 0$ and

$$\left(\mathrm{Curv}_2(E)_2\right)_{(x_i, A_{ij})} := \{H_{012} : E_{x_2} \to E_{x_0}\}.$$

The **curvature section** $K : \mathrm{GL}_2(E) \to \mathrm{Curv}_2(E)$ is defined by

$$K(x_i, A_{ij}) := A_{01}A_{12} - A_{02}.$$



Notice: $K^{-1}(0) \cong \mathrm{N}(\mathrm{GL}(E))$.

Parallel transport

Proposition

An A-connection ∇ on E induces a 2-functor $\tau : \Pi_2(A) \to \mathrm{GL}_2(E)$ as follows:

- $\tau_0 := \mathrm{id}$,
- $\tau_1(a) := \tau_a$,
- $\tau_2(\sigma) := (x_0(\sigma), x_1(\sigma), x_2(\sigma), \tau_{g_{01}(\sigma)}, \tau_{g_{02}(\sigma)}, \tau_{g_{12}(\sigma)})$.

Moreover, τ is such that

$$K(\tau(\sigma)) = \int_\sigma R^\nabla$$

$$\begin{array}{ccc} \tau^*\mathrm{Curv}_2(E) & & \mathrm{Curv}_2(E) \\ {\scriptstyle \tau^*K}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle K} \\ \Pi_2(A) & \xrightarrow[\tau]{} & \mathrm{GL}_2(E) \end{array}$$

$$\tau^* K = \int R^\nabla$$

Riemann–Hilbert correspondence revisited

Theorem (Riemann–Hilbert correspondence II)

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Conn}_A(E) & \longrightarrow & [\Pi_2(A), \mathrm{GL}_2(E)]_*^\infty \\ \uparrow & & \uparrow \\ \mathrm{Flat}_A(E) & \longrightarrow & [\Pi_2(A), \mathrm{GL}_2(E)]_{*,\mathrm{flat}}^\infty \end{array}$$

where the vertical arrows are inclusions of full subcategories and the horizontal arrows are equivalences of categories.

A functor $\tau : \Pi_2(A) \to \mathrm{GL}_2(E)$ is **flat** if $\tau^* K = 0$.

Moreover, there is an equivalence of categories

$$[\Pi_2(A), \mathrm{GL}_2(E)]_{*,\mathrm{flat}}^\infty \cong [\Pi_1(A), \mathrm{GL}(E)]_*^\infty.$$

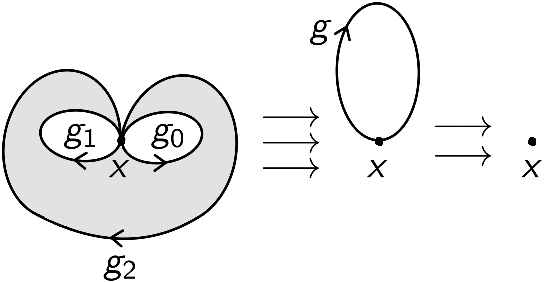
Holonomy

Definition

Let ∇ be an A-connection on a vector bundle $E \to M$. We define its **holonomy 2-group** at $x \in M$ as

$$\mathrm{Hol}_2(\nabla) := \tau(\Pi_2(A))_x.$$

If G is a Lie ∞-groupoid, its isotropy over $x \in G_0$ is the Lie ∞-group G_x given by $(G_x)_n := \{\sigma \in G_n : x_i(\sigma) = x, \text{for all } i = 0, \ldots, n\}$.



To compute its Lie 2-algebra, we first compute its tangent complex.

Tangent complex of higher Lie groupoids

How to differentiate higher Lie groupoids?

- In general, very hard problem.
- First step: tangent complex.

Definition

If G is a Lie ∞-groupoid, its **tangent complex** is the complex of vector bundles $(A \to G_0, \partial)$ defined by

$$A_n := \bigcap_{k=1}^{n} \ker f_{k*}|_{G_0},$$

$$\partial := f_{0*}.$$

Example

If $G \rightrightarrows M$ is a Lie groupoid, then its tangent complex is $\rho : A \to TM$.

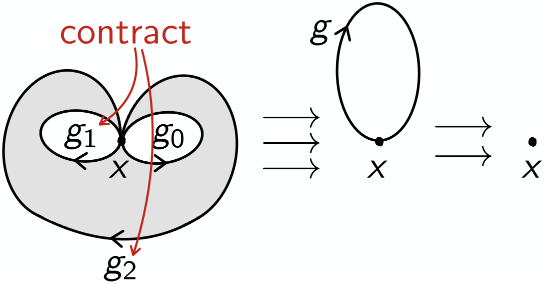
Tangent complex of the isotropy

If G is a Lie ∞-groupoid and $x \in G_0$, we can compute the tangent complex of G_x as follows:

- consider
$$H_n := \{\sigma \in (G_x)_n : f_k(\sigma) = x, \text{for all } k = 1, \dots, n\} \subseteq (G_x)_n,$$
- then the tangent complex of G_x is obtained by differentiating
$$\dots \xrightarrow{f_0} H_n \xrightarrow{f_0} H_{n-1} \xrightarrow{f_0} \dots \xrightarrow{f_0} H_1$$
at x.

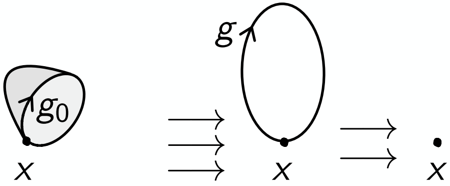
Tangent complex of the holonomy

In the case of $\mathrm{Hol}_2(\nabla)$:



Tangent complex of the holonomy

In the case of $\mathsf{Hol}_2(\nabla)$:

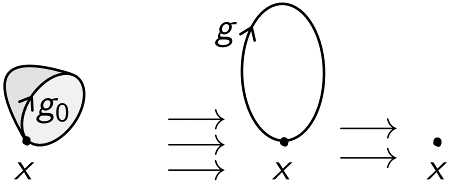


The tangent complex $\mathfrak{hol}_2(\nabla)$ we obtain by differentiating at the identity

$$\{\tau_a : a \text{ is a null } A\text{-homotopic } A\text{-path}\} \hookrightarrow \mathsf{Hol}(\nabla)$$

Tangent complex of the holonomy

In the case of $\mathrm{Hol}_2(\nabla)$:



The tangent complex $\mathfrak{hol}_2(\nabla)$ we obtain by differentiating at the identity

$$\mathrm{Curv}(\nabla) \hookrightarrow \mathrm{Hol}(\nabla)$$

Monodromy

As a 2-group, we have that

$$\pi_1(\mathrm{Hol}_2(\nabla)) = \mathrm{Hol}(\nabla)/\mathrm{Curv}(\nabla).$$

We could call this the **monodromy group** of ∇.

- If $A = TM$ we have that
$$\pi_1(\mathrm{Hol}_2(\nabla)) = \mathrm{Hol}(\nabla)/\mathrm{Hol}^0(\nabla) = \mathrm{Mon}(\nabla),$$
which is discrete.
- If A has isotropy, then
$$\pi_1(\mathrm{Hol}_2(\nabla)) = \mathrm{Hol}(\nabla)/\mathrm{Curv}(\nabla)$$
is non-discrete. These are the isotropy terms from Fernandes's version of Ambrose–Singer.

Generalized geometry?

Where is generalized geometry?

$$\text{Courant algebroids} = \text{symplectic Lie 2-algebroids.}$$

Notice:

- Parallel transport $\tau : \Pi_2(A) \to \mathsf{GL}_2(E)$ is fully determined by what it does on $\Pi_2(A)_1$.
- If A is a Courant algebroid, we need to genuinely see $\Pi_2(A)$, so A-connections on vector bundles are not enough, we need to go higher!

We need to consider

higher A**-connections** on **graded vector bundles** $E = \bigoplus_{i \geq 0} E^i$.

Going higher

Theorem

Let $\mathscr{A} \to M$ be a higher Lie algebroid and $E \to M$ a graded vector bundle. Then there is a commutative diagram

$$\begin{array}{ccc} \mathsf{HConn}_{\mathscr{A}}(E) & \longrightarrow & [\Pi_\infty(\mathscr{A}), \mathsf{GL}_\infty(E)]_*^\infty \\ \uparrow & & \uparrow \\ \mathsf{HFlat}_{\mathscr{A}}(E) & \longrightarrow & [\Pi_\infty(\mathscr{A}), \mathsf{GL}_\infty(E)]_{*,\mathsf{flat}}^\infty \end{array}$$

where the vertical arrows are inclusions of full subcategories. Moreover, if $E = \bigoplus_{i=0}^{n-1} E^i$, then the diagram becomes

$$\begin{array}{ccc} \mathsf{HConn}_{\mathscr{A}}(E) & \longrightarrow & [\Pi_{n+1}(\mathscr{A}), \mathsf{GL}_{n+1}(E)]_*^\infty \\ \uparrow & & \uparrow \\ \mathsf{HFlat}_{\mathscr{A}}(E) & \longrightarrow & [\Pi_{n+1}(\mathscr{A}), \mathsf{GL}_{n+1}(E)]_{*,\mathsf{flat}}^\infty. \end{array}$$

Thanks

Smoothness of the holonomy 2-group

Recall: $\mathrm{Hol}_2(\nabla)$ is the 2-group

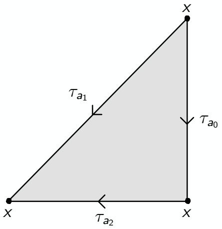
$$\mathrm{Hol}_2(\nabla)_2 \Rrightarrow \mathrm{Hol}(\nabla) \rightrightarrows x,$$

where

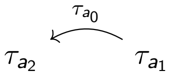
$$\mathrm{Hol}_2(\nabla)_2 = \{(\tau_{a_0}, \tau_{a_1}, \tau_{a_2}) : a_i \text{ are } A\text{-loops at } x \text{ such that there is a 2-}A\text{-simplex whose boundary is } (a_0, a_1, a_2)\}$$

Proposition

There is a groupoid structure $\mathrm{Hol}_2(\nabla)_2 \rightrightarrows \mathrm{Hol}(\nabla)$.



We picture this as an arrow



Smoothness of the holonomy 2-group

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where

$$\mathrm{Hol}_2(\nabla)_2 = \{(\tau_{a_0}, \tau_{a_1}, \tau_{a_2}) : a_i \text{ are } A\text{-loops at } x \text{ such that there is a 2-}A\text{-simplex whose boundary is } (a_0, a_1, a_2)\}$$

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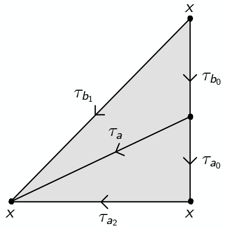
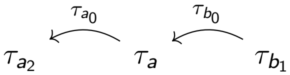


Multiplication



Smoothness of the holonomy 2-group

Is $\mathrm{Hol}_2(\nabla)_2 \rightrightarrows \mathrm{Hol}(\nabla)$ a Lie groupoid?

Proposition

As set-theoretic groupoids, $\mathrm{Hol}_2(\nabla)_2 \cong \mathrm{Curv}(\nabla) \ltimes \mathrm{Hol}(\nabla) \rtimes \mathrm{Hol}(\nabla)$.

Definition

For commuting left G-action and right H-action on M, the **biaction groupoid** is the groupoid $G \ltimes M \rtimes H \rightrightarrows M$ given by

$$s(g, x, h) := x,$$
$$t(g, x, h) := gxh,$$
$$(g', gxh, h') \cdot (g, x, h) := (g'g, x, hh').$$

It is enough to prove that both $\mathrm{Curv}(\nabla)$ and $\mathrm{Hol}(\nabla)$ are Lie groups.

Smoothness of the holonomy 2-group

Theorem (Yamabe, 1950)

Any path-connected subgroup of a Lie group is an analytic subgroup.

Proposition

$\mathrm{Curv}(\nabla)$ *and* $\mathrm{Hol}^0(\nabla)$ *are path-connected subgroups of* $\mathrm{GL}(E_x)$.

Proof.

$$\mathrm{Curv}(\nabla) = \{\tau_a : a \text{ is a null } A\text{-homotopic } A\text{-path}\},$$
$$\mathrm{Hol}^0(\nabla) = \{\tau_a : a \text{ is a null anchored-homotopic } A\text{-path}\}.$$