Above and beyond generalized complex geometry

Roberto Rubio

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Geometry and TACoS

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Looking back: geometry and TACoS 2021

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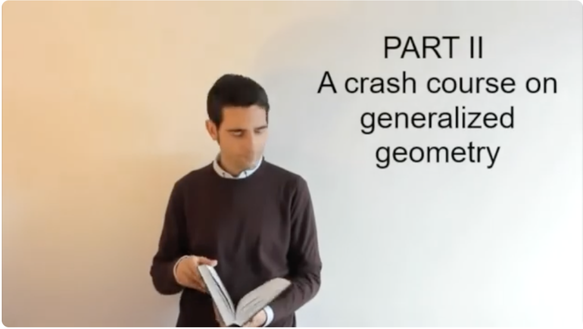

The Calabi system and non-Kähler Calabi-Yau manifolds through generalized geometry - Roberto Rubio

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Join us on Gitter to be part of the discussion: https://gitter.im/GTACOS-Jan2021/

Looking back: geometry and TACoS 2021

and even further back to 2010.

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JCAS Lecture Series

Generalized Geometry

Professor Nigel Hitchin
Department of Mathematics
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I read Hitchin's notes, Gualtieri's thesis... what we saw in Cavalcanti's talk

Part I

Introduction

The generalized complex trinity

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$\mathcal{J}$ skew for $\langle \, , \rangle$

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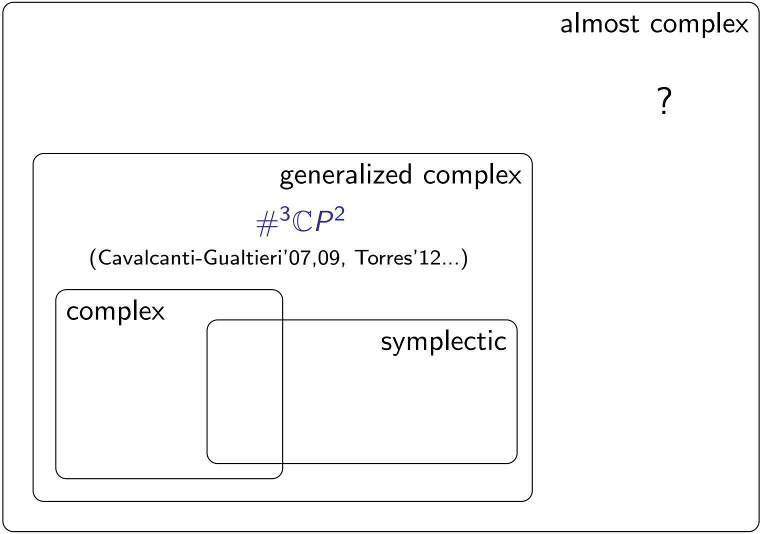
My two favourite
facts of generalized
complex structures
(actually examples):

Within generalized complex structures

The interior of the curve is e^B-equivalent to symplectic structures.

Examples coming from hyperKähler or holomorphic symplectic structures.

Beyond complex and symplectic



My spiritual beliefs on generalized complex structures

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Most basic object:
differential forms.
Best for type-change,
$z + dz \wedge dw$.

Hitchin's advice: best way to understand, develop...

2.4 B_k-generalized geometry

We will finish this section with one last example which is an original construction. On an n-manifold M consider the bundle

$$E = TM \oplus 1 \oplus T^*M$$

equipped with the natural pairing

$$\langle X + f + \xi, Y + g + \eta \rangle = i_X \eta + i_Y \xi + fg$$

of signature $(n+1, n)$ where $X, Y \in \Gamma(TM)$, $f, g \in \mathcal{C}^\infty(M)$ and $\xi, \eta \in \Gamma(T^*M)$. We define a Dorfman bracket as follows

$$\{X + f + \xi, Y + g + \eta\} = [X, Y] + X(g) - Y(f) + \mathcal{L}_X \eta - i_Y d\xi + gdf$$

⋆ ⋆ ⋆

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[Submitted on 5 Jan 2011 (v1), last revised 25 Apr 2011 (this version, v2)]

Leibniz algebroids, twistings and exceptional generalized geometry

David Baraglia

B

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Let us see it concretely

Quadratic form on $TM \oplus T^*M$ given by $Q(X+\alpha) = \alpha(X)$

$\mathrm{Cl}_{\mathbb{C}}(TM \oplus T^*M)$-module structure on $\wedge^\bullet T^*_{\mathbb{C}}M$

$$(X+\alpha)\cdot\rho = \iota_X\rho + \alpha\wedge\rho$$

($\wedge^\bullet T^*_{\mathbb{C}}M \simeq$ the spinor representation)

Pure spinors are pointwise $\sim e^{B+i\omega}\theta_1\wedge\ldots\wedge\theta_r$

($\leftrightarrow$ $\mathrm{Ann}(\rho)$ max. isotropic) $B,\omega\in\wedge^2$, $\theta_j\in\wedge^1_{\mathbb{C}}$

Chevalley pairing on spinors $(\rho,\psi) = (\rho^T\wedge\psi)_{top}$

($\wedge^{top}T^*_{\mathbb{C}}M$-valued)

$$+$$

Weakening of $d\rho = 0 \to d\rho = v\cdot\rho$ for $v = X+\alpha$

$$\updownarrow$$

$\Gamma(\mathrm{Ann}\,\rho)$ involutive for **Dorfman bracket**

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$$

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The world of B_n-generalized geometry

- A-fields e^A, for A a closed 1-form, join B-fields e^B.
- Generalized diffeomorphisms are enlarged to $\mathrm{Diff}(M) \ltimes \Omega^{2+1}_{cl}(M)$.

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- Toy model for string or heterotic Courant algebroids [Garcia-Fernandez,R.,Tipler], [Baraglia,Hekmati].
- Metric side of the story developed by [Cortes,David].

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- Toy model for string or heterotic Courant algebroids [Garcia-Fernandez,R.,Tipler], [Baraglia,Hekmati].
- Metric side of the story developed by [Cortes,David].

But let us look at examples!

Examples!

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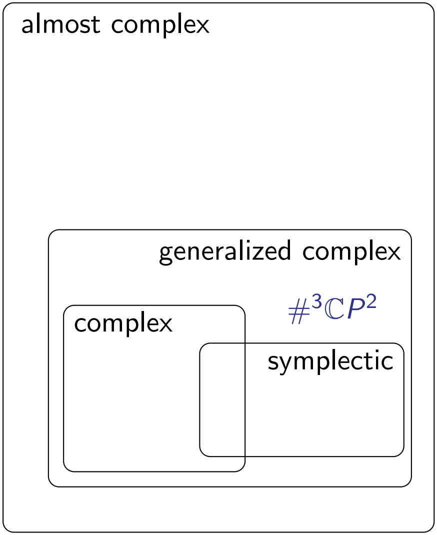
- **Type-change example**: on $\mathbb{C} \times \mathbb{R}$ with coordinates (z, t),

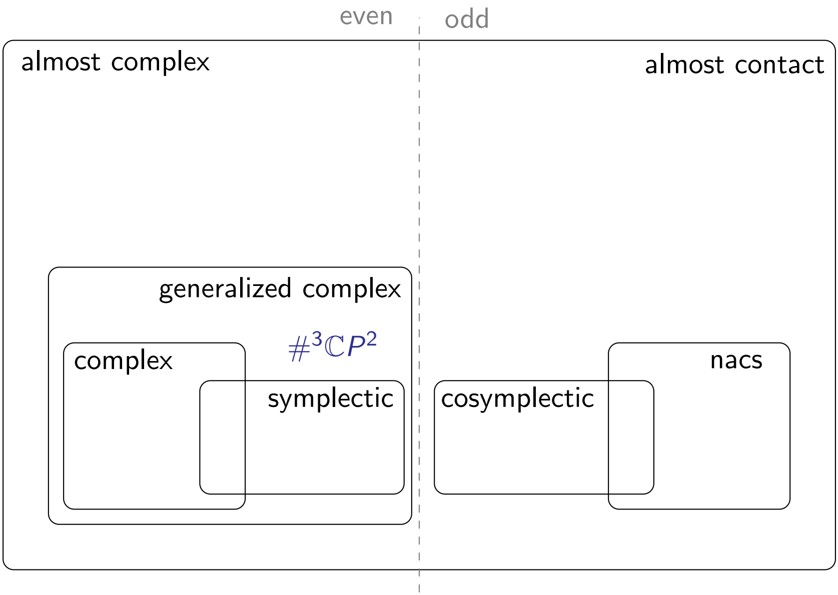
$$\rho = z + \lambda dz + i\,\mu dz \wedge dt$$

Local invariants (related to topology) and type change in $\dim \geq 2$.

What we had


almost complex
generalized complex
$\#^3\mathbb{C}P^2$
complex
symplectic

What we are getting



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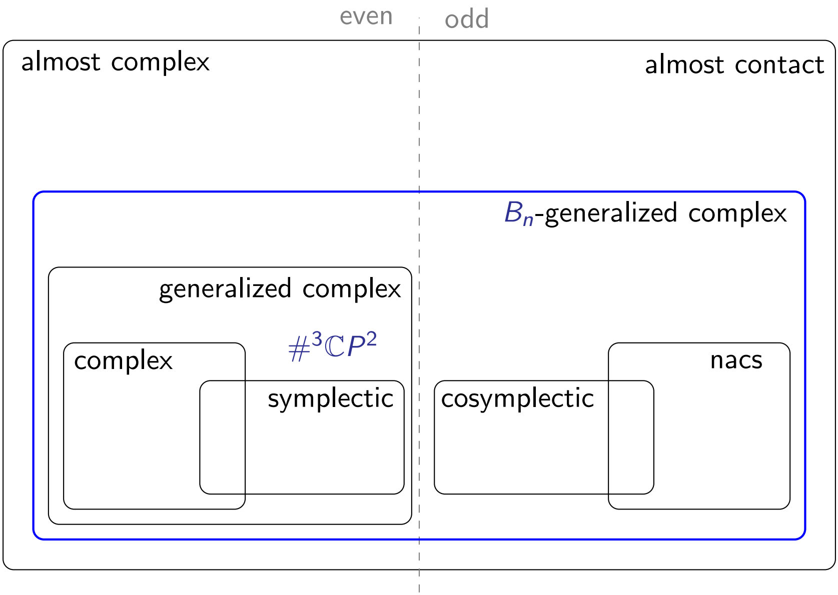

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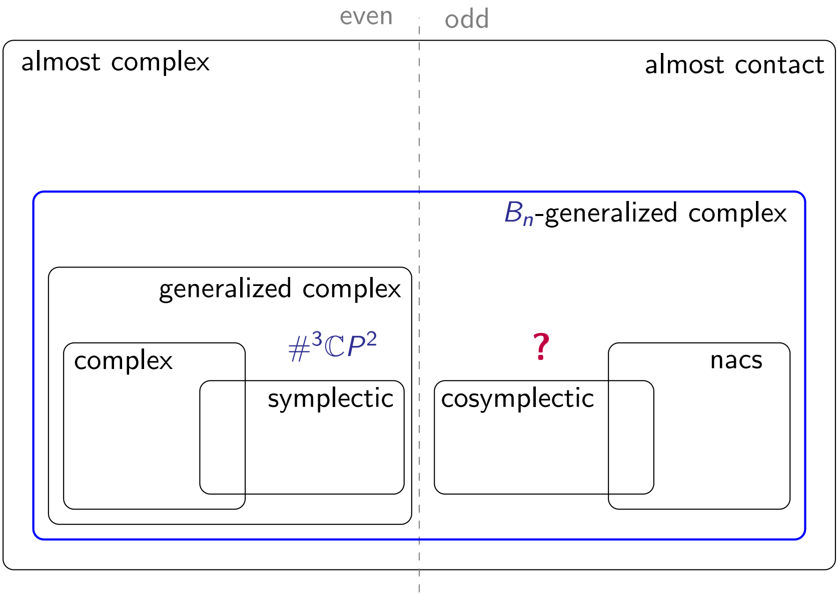
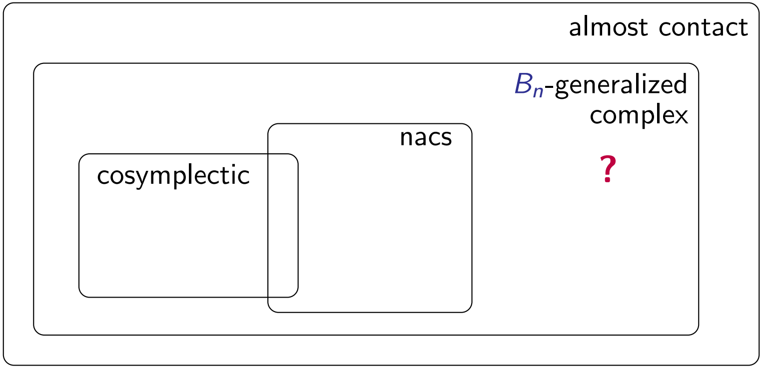


Zooming in



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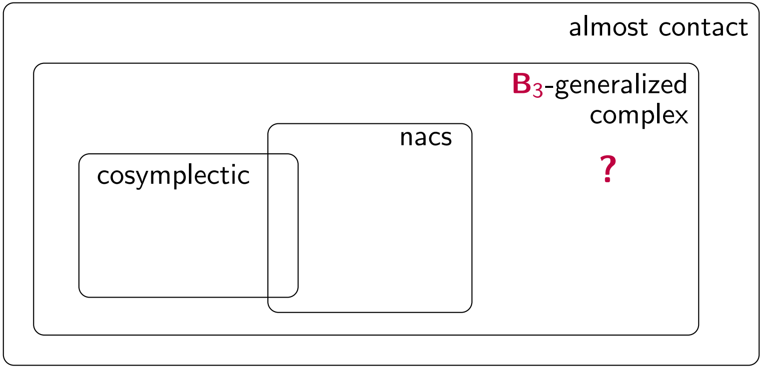

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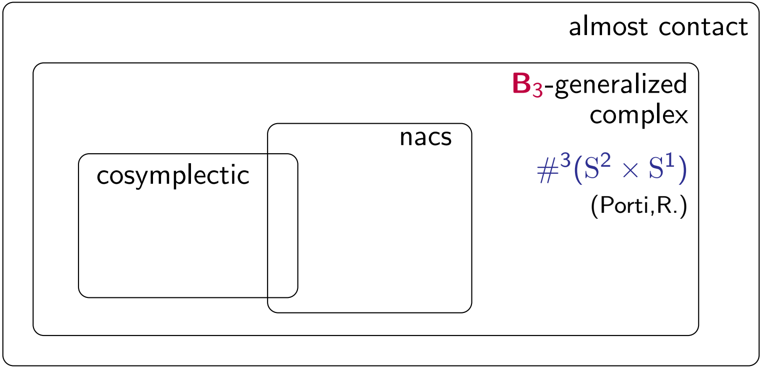
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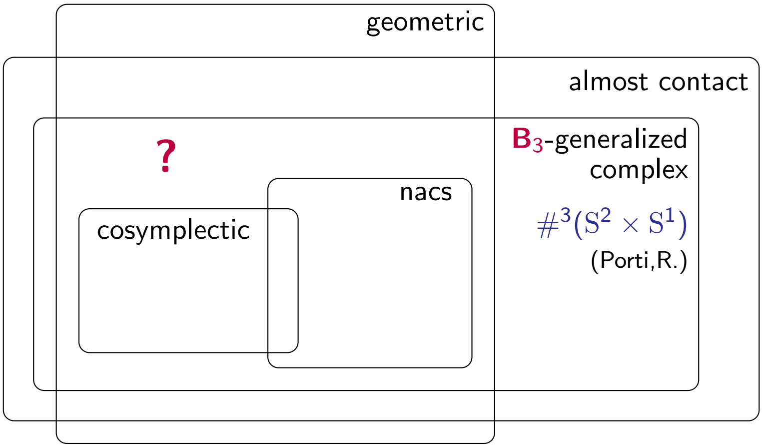
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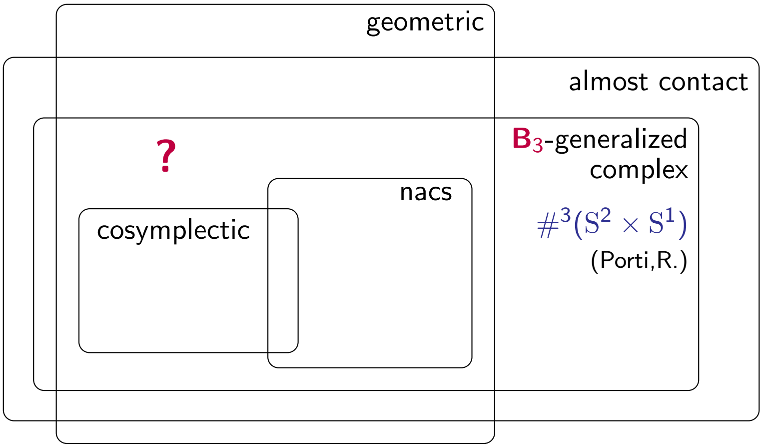

Theorem (Porti, R.; arXiv:2402.12471)

Any closed oriented 3-manifold admits a B_3-generalized complex structure, which is moreover stable.

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...I think this was the beginning of a wonderful mathematical story.

Part III

C

(joint work with **Filip Moučka**)

The C_n-generalized meaning of Lagrangian

Consider $TM \oplus T^*M$ with the skew-symmetric pairing

$$\langle X + \alpha, Y + \beta \rangle_- := \alpha(Y) - \beta(X).$$

Lagrangian now captures symmetry!

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$$\mathcal{J}_g := \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}, \qquad \mathcal{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

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Do we have ‘spinors’? What does integrability mean?

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Dorfman-involutive C_n-generalized almost complex structures are...

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(Moučka, R. *Symmetric Cartan calculus, the Patterson-Walker metric and Killing vector fields*)

For a torsion-free connection ∇, just as $d\rho = (|\rho|+1)\,\mathrm{Skew}(\nabla\rho)$, define

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$[[(X+\alpha)\cdot, \nabla^s], (Y+\beta)\cdot] \in \mathrm{End}(\mathrm{Sym}^\bullet\, T^*M)$ corresponds to

$$[X+\alpha, Y+\beta]_s := [X,Y]_s + L^s_X\beta + \iota_Y\nabla^s\alpha,$$

where $[X,Y]_s := \nabla_X Y + \nabla_Y X$ and $L^s_X := [\iota_X, \nabla^s]$.

The world of C_n-generalized geometry

$$(TM \oplus T^*M, \langle\ ,\ \rangle_-, [\ ,\ ]_s, pr_{TM})$$

- C-fields e^C for C Killing 2-tensor replace B-fields.
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Let us look at it through the examples!

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Integrability means:

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- $N^s_J = 0$, or J is an anti-complex structure (∼Gray-Hervella, nearly Kähler), (for symmetric Nijenhuis $N^s_J(X,Y) = [JX,JY]_s - [X,Y]_s - J([JX,Y]_s + [X,JY]_s = 0)$.
  Some of them are complex, when $(\nabla_{JX} J)Y = J(\nabla_X J)Y$.

The yin and yang of C_n-generalized complex structures

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Good to see the
C_n-Dorfman bracket,
C_n-Dirac structures,
symmetric algebroids.

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More than the type, the signature!

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2	$(4,0)$	0	Riemannian metric
	$(2,2)$	0	Lorentzian metric
		1	complex structure
3	$(6,0)$	0	Riemannian metric
	$(4,2)$	0	Lorentzian metric
		1	1D Riemannian $\oplus$ 2D complex

Type-changing example

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4	$(8,0)$	0	Riemannian metric
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Example (Moučka, R.; to appear)

On $M = \mathbb{R}^4 \cong \mathbb{C}^2$, deform $L = TM^{0,1} \oplus T^*M^{1,0}$, by acting with the symmetric bivector $z_2\,\partial_{z_1} \otimes \partial_{z_1}$. We get:

- anti-complex structure (even parallel complex) on $z_2 = 0$,
- sum of 2D Lorentzian metric and 2D anti-complex structure on $z_2 \neq 0$.

Within C_n-generalized complex structures

Example (Moučka, R.; to appear)

Given an anti-Kähler manifold, (g, J) such that

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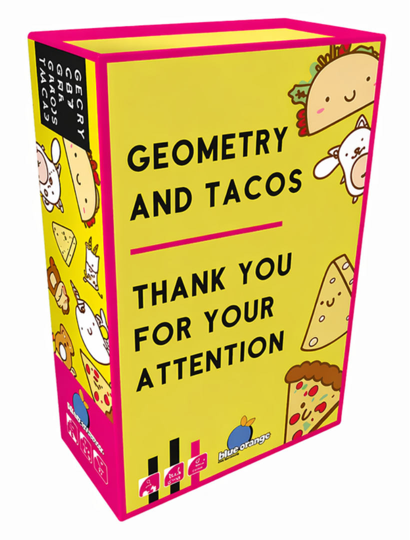
These are definitely **new directions in complex geometry**.



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