Finite generation of singular distributions

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Most of this is joint work with Lance Drager (Texas Tech University), Jeff Lee (Texas Tech University), and Efton Park (TCU). I will also mention some work of other mathematicians.

Outline

1. Introduction to generalized distributions
   - Generalized distributions
   - Smooth general distributions
   - Cosmooth general distributions
   - Examples
   - Smooth vs. cosmooth distributions
2. Are general distributions finitely generated?
   - Pointwise version of being finitely generated
   - Finitely generated spaces of sections
   - Results
3. Sketches of Proofs
4. Related problems

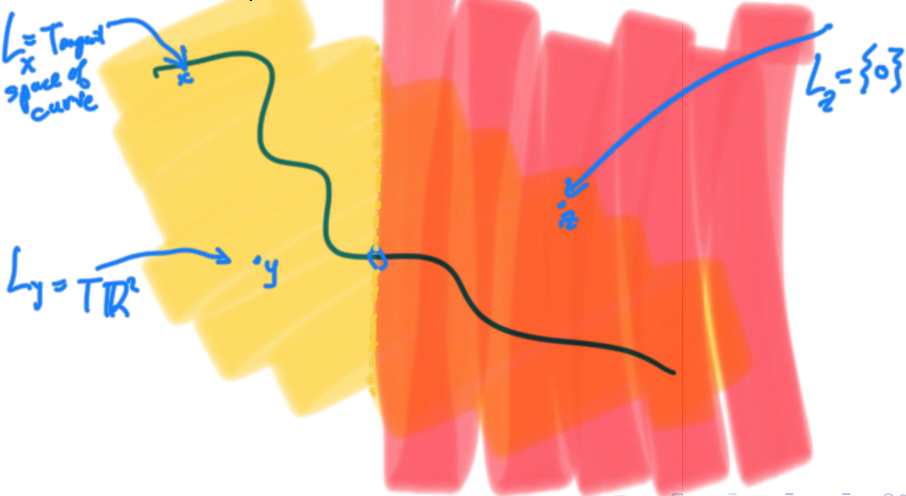
Let

- M be a smooth manifold
- L be a distribution on M (subbundle of the tangent bundle TM)
- The rank of L is allowed to vary.
- Examples include:
  - Tangent bundle of a singular foliation
  - Poisson distributions
  - Distributions of control theory
  - Exterior differential systems: kernels of families of one-forms.
- What is the appropriate notion of smoothness?

- First considered by Sussman and Stefan independently
- The well-known Frobenius Theorem states that a smooth (constant rank) distribution L is the tangent bundle of a foliation if and only if the distribution is involutive (i.e. $[\Gamma(L), \Gamma(L)] \subset \Gamma[L]$)
- Sussmann (TAMS, 1973) and Stefan (JLMS, 1980) extended the Frobenius theorem to generalized distributions.
- Sussmann and Stefan considered a distribution L to be smooth if for each $p \in M$, there are locally defined (smooth) vector fields that are sections of L whose values at p span L_p.
- One can define a smooth generalized subbundle of a general vector bundle in a similar way.

- As in the theory of exterior differential systems, one may a distribution whose rank is not necessarily constant as the kernel of a family of one-forms.
- If near each point of the manifold, there exist families of (smooth) one-forms whose pointwise kernel is the given distribution, then we say that the distribution is *cosmooth*.
- Also, in this situation, the subspaces of the cotangent spaces spanned by these one-forms determine a generalized subbundle of the cotangent bundle.
- Similarly, one may define cosmooth distributions of a general vector bundle.

- Any ordinary distribution is both smooth and cosmooth.
- Other examples



- Tame examples: the image of a smooth bundle map is a smooth generalized subbundle, and the kernel of a smooth bundle map is a cosmooth generalized subbundle.
- More examples

Let $\{V_\alpha\}_{\alpha\in I}$ be an uncountable collection of v. fields on a manifold M. Define $L \subseteq TM$ by

$L_x = \mathrm{span}\{V_\alpha(x) : \alpha \in I\}$ } yield smooth distributions

Let $\{\theta_\alpha\}_{\alpha\in I}$ be a collection of 1-forms on M

Let $K_x = \bigcap_\alpha \ker \theta_\alpha$ ← yields a cosmooth distribution

Many examples: Take any general distribution of TM (doesn't even have to be "continuous") L

- Smoothification, cosmoothification

↑ union of smooth vector fields tangent to distr.

↑ pointwise span of these → $\underset{\text{smooth}}{T} \subseteq L$ not nec. smooth.

Alternately, take union of all one-forms whose kernel contains L at each pt.

cosmoothification = pointwise intersection of kernels of these 1-forms.

Proposition

A generalized distribution L is cosmooth if and only if L is complementary to a smooth distribution.

Proposition

If L is a smooth generalized distribution, then the function $p \mapsto \dim L_p$ is a lower semicontinuous function. If L is a cosmooth generalized distribution, then the function $p \mapsto \dim L_p$ is an upper semicontinuous function.

- We say that a smooth generalized distribution L is **finitely generated** over an open set $U \subset M$ if there exists a finite set of smooth sections $s_1, s_2, \ldots, s_k$ of L such that the values $s_1(p), \ldots, s_k(p)$ span L_p for all $p \in U$. One can ask if such generators exist, either locally or globally.
- It was conjectured that not every smooth generalized distribution is finitely generated, even locally.
- One may define the notion of a cosmooth distribution being finitely generated in a similar way, with the vector fields being replaced by one-forms.

- Consider the space of sections of a smooth generalized subbundle L restricted to an open set U as a module over the ring of smooth functions. We ask whether there exists a finite set $s_1, \ldots, s_k$ of smooth sections of L over U that form a set of generators for the module. This question can be asked locally and globally.
- Bullo and Lewis (Geometric Control of Mechanical Systems, 2005) showed that if the smooth generalized distribution is real analytic, then the set of sections is locally finitely generated. This follows from the fact that the ring of convergent power series is Noetherian.

Theorem

(Drager, Lee, Park, R., AGAG 2012) Every smooth generalized distribution over a smooth manifold is finitely generated. Every cosmooth generalized distribution over a smooth manifold is finitely generated.

Remark

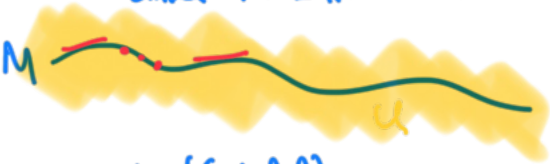
Sussman proved this theorem independently in the smooth case, with a different proof (Isidori Control Theory Conference Proceedings 2008), but the math review only appeared in 2012.

Remark

Many examples of smooth nonregular distributions have the property that the smooth sections are not even locally finitely generated as modules over the smooth functions. In some simple cases, the smooth sections are finitely generated as modules over the smooth functions.

- Sketch of proof that every smooth distribution is finitely generated: M smooth mfld.

• Embed $M \subseteq \mathbb{R}^N$ L distribution over M



U open in $\mathbb{R}^N$

$\Rightarrow$ L(extended) is a gen. distr. on $\mathbb{R}^N$.

• $\mathbb{R}^N = \bigcup_d \Sigma_d$, where $\Sigma_d = \{x \in \mathbb{R}^N : \dim L_x = d\}$.

• $\forall p \in \Sigma_d$, choose $s_1, \ldots, s_k \in \Gamma(L)$ s.t.
$\mathrm{span}\{s_1, \ldots, s_k|_{x \in \Sigma_d \cap V}\} = L_x$.

V open nbhd of p

- Sketch of proof, continued ...

Then Gramm-Schmidt $s_1, \dots, s_k$ so they form an o-n basis of $L|_{\Sigma_d \cap V}$ $\Rightarrow$ create a projection

$P_V : T\mathbb{R}^N \to T\mathbb{R}^N$ s.t. $P_V(W_x) \in L_x$ $\forall x \in \Sigma_d \cap V$

$\mathrm{Image}(P_V)_x = L_x$ $\forall x \in \Sigma_d \cap V$

$\Rightarrow$ if $e_1, \dots, e_N$ are standard basis vectors of $\mathbb{R}^N$,

then $P_V(e_1), \dots, P_V(e_N)$ are local v. fields on V

s.t. span(") $= L$ when restricted to $\Sigma_d \cap V$.

partition of unity $\{\psi\}$ sub. to cover of Σ_d by such V.

$\psi P_V(e_j) \overset{\text{sum}}{\rightsquigarrow} V_j$ global v. field on U.

N global v. fields for each d $\longrightarrow$ Total span$|_M = L$.

Example of smooth general distribution for which the space of sections is not finitely generated:

- Let M be the interval $(-1,1)$.
- Let $L \subset TM$ be the smooth generalized distribution defined by

$$L_x = \begin{cases} \mathbb{R} & \text{if } x > 0 \\ \{0\} & \text{if } x \leq 0 \end{cases}$$

Note: smooth since $L_x = \text{span}\{\psi(x)\partial_x\}$

$\Gamma(L)$ is not f.g. as module over $C^\infty(\mathbb{R})$.

Proof: Sp. $\{v_1\partial_x, \ldots, v_k\partial_x\}$ generates $\Gamma(L)$ over $C^\infty(\mathbb{R})$.

Let $W = \left(\sum v_j^2\right)^{1/4}\partial_x$. Can prove W is a smooth v.field.

Sp. $\left(\sum v_j^2\right)^{1/4} = \left|\sum b_j v_j\right| \leq \left(\sum b_j^2\right)^{1/2}\left(\sum v_j^2\right)^{1/2} \Rightarrow \left(\sum b_j^2\right)^{1/2} \geq \frac{1}{|W|}$

Example of smooth general distribution for which the space of sections is finitely generated:

- Let M be the interval $(-1, 1)$.
- Let $K \subset TM$ be the smooth generalized distribution defined by

$$K_x = \begin{cases} \mathbb{R} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

$$\Gamma(K) = \underset{C^\infty(\mathbb{R})}{\text{span}} \{x\partial_x\}$$

finitely generated.

- Singular Riemannian foliations (SRFs) (including orbit spaces of compact Lie group actions) have tangent bundles that are smooth distributions and normal bundles that are cosmooth distributions. Transversal differential operators (like the transversal Laplacian) are naturally defined on cosmooth distributions. Do they have well-behaved eigenvalue asymptotics and heat kernel asymptotics? Does the basic L^2-orthogonal projection preserve the smooth sections? Does the mean curvature one-form have similar properties to that corresponding to Riemannian foliations?
- Partial answers: it turns out that when the mean curvature one form is a basic form (i.e. depending only on transverse variables), then some of these things are true. Unlike the case of regular Riemannian foliations, it is not known whether an SRF always admits an SRF metric that has basic mean curvature.

- A consequence of Ricardo A. E. Mendes, Marco Radeschi paper: *Laplacian algebras, manifold submetries and the Inverse Invariant Theory Problem* (GAFA, 2020): Given a singular Riemannian foliations (SRFs) of round spheres in Euclidean space, there exists a finite set of basic polynomials in the Euclidean variables such that the intersections of level sets of these polynomials equals the SRF.

- Singular foliations with geometric structures (eg metric for which the leaves are totally geodesic, or a bundle-like metric) may be coaxed to have cosmooth tangent bundles for which tangential Dirac operators and a tangential Laplacian is defined. Does the leafwise heat flow preserve the smoothness of differential forms?

- Applications to subRiemannian geometry. For a cosmooth generalized distribution that is bracket-generating, is it true that any two points can be connected by a horizontal curve? Is the corresponding subLaplacian hypoelliptic? How do the eigenvalues behave? Heat kernels?