Higher General Linear Groupoids and Higher Holonomy

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Program

Higher General Linear Groupoids and Higher Holonomy

- Overture.
- Act I. Non-abelian Stokes's theorem.
- Act II. Higher general linear groupoids.
- Act III. Higher parallel transport and higher holonomy.

Overture

Scene 1. Lie algebroid holonomy

Lie algebroid holonomy

Let $A \to M$ be a Lie algebroid and $E \to M$ a vector bundle.

- An A-**connection** on E is a degree 1 derivation
$$d^\nabla : \Omega(A, E) \to \Omega(A, E)$$
with symbol d_A, i.e., satisfying the Leibniz rule
$$d^\nabla(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d^\nabla \beta.$$
- **Curvature** $R^\nabla := (d^\nabla)^2 \in \Omega^2(A, \mathrm{End}(E))$.
- **Holonomy**: $\mathrm{Hol}(\nabla) := \{\tau_a : a \text{ is an } A\text{-loop at } x\}$.

Theorem (Fernandes 02)

$$\begin{aligned}\mathfrak{hol}(\nabla) = \mathrm{span}\{\tau_e^{-1} R^\nabla(a, b)\tau_e &: \textit{e is an A-path with } \gamma_e(0) = x \\ &\quad \textit{and } a, b \in A_{\gamma_e(1)}\} \\ + \mathrm{span}\{\tau_e^{-1}\nabla_c\tau_e &: \textit{e is an A-path with } \gamma_e(0) = x \\ &\quad \textit{and } c \in \ker \rho_{\gamma_e(1)}\}\end{aligned}$$

Lie algebroid holonomy

We can define

$$\mathfrak{curv}(\nabla) := \mathrm{span}\{\tau_e^{-1} R^{\nabla}(a,b)\tau_e : e \text{ is an } A\text{-path with } \gamma_e(0) = x \\ \text{and } a, b \in A_{\gamma_e(1)}\}.$$

- It is an ideal of $\mathfrak{hol}(\nabla)$.
- There is a normal subgroup $\mathrm{Curv}(\nabla) \subseteq \mathrm{Hol}^0(\nabla)$ integrating it.

Conclusion: there is an inclusion of Lie groups $\mathrm{Curv}(\nabla) \hookrightarrow \mathrm{Hol}(\nabla)$.

- For a Lie theorist, this is just an inclusion of Lie groups.
- For a higher Lie theorist, this is the easiest example of a strict 2-group.

Question

Does the inclusion $\mathrm{Curv}(\nabla) \hookrightarrow \mathrm{Hol}(\nabla)$ have a higher life? Is it geometric?

Overture

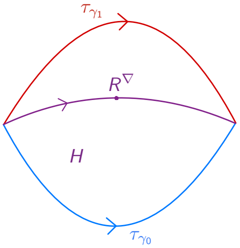
Scene 2. Non-flat Riemann–Hilbert correspondence

Non-flat Riemann–Hilbert correspondence

Lemma

Let ∇ be a connection on $E \to M$ and H a homotopy between paths γ_0 and γ_1. Then

$$\tau_{\gamma_1} - \tau_{\gamma_0} = \int_H R^\nabla.$$



Non-flat Riemann–Hilbert correspondence

Lemma

Let ∇ be a connection on $E \to M$ and H a homotopy between paths γ_0 and γ_1. Then

$$\tau_{\gamma_1} - \tau_{\gamma_0} = \int_H R^\nabla.$$

Corollary

If ∇ is flat and $\gamma_0 \sim \gamma_1$, then $\tau_{\gamma_0} = \tau_{\gamma_1}$. In particular, ∇ defines a Lie groupoid morphism $\tau : \Pi_1(M) \to \mathrm{GL}(E)$.

Recall:

- $\Pi_1(M) \rightrightarrows M$ with $\Pi_1(M) = \{[\gamma] : \gamma \text{ path}\}$,
- $\mathrm{GL}(E) \rightrightarrows M$ with $\mathrm{GL}(E) = \{(y, A, x) : x, y \in M, A : E_x \xrightarrow{\cong} E_y\}$.

Non-flat Riemann–Hilbert correspondence

Theorem (Riemann–Hilbert correspondence)

There is an equivalence of categories

$$\mathrm{Flat}_M(E) \to [\Pi_1(M), \mathrm{GL}(E)]^\infty_*.$$

Question

What if ∇ is not flat?

- We still have a functor $\tau : \mathcal{P}_1(M) \to \mathrm{GL}(E)$. *Anything else?*
- Idea: if H_0 and H_1 are two homotopies between γ_0 and γ_1 and L is a homotopy from H_0 to H_1 , then a non-abelian Stokes's theorem (??) would give, by the Bianchi identity,

$$\int_{H_1} R^\nabla - \int_{H_0} R^\nabla = \int_L d^\nabla R^\nabla = 0.$$

Question

Do we get a functor $\Pi_2(M) \to$?? ?

Overture

Scene 3. Higher connections

Higher connections

Let $A \to M$ be a Lie algebroid and $E = \bigoplus_k E^k \to M$ a graded vector bundle. The space $\Omega(A, E)$ is bigraded:

$$\Omega(A, E) = \bigoplus_{p,q} \Omega^p(A, E^q)$$

Definition (Arias Abad–Crainic 07)

A **higher A-connection** on E is a degree 1 derivation

$$D : \Omega(A, E) \to \Omega(A, E)$$

with symbol d_A, i.e., satisfying the Leibniz rule

$$D(\alpha \wedge \beta) = d_A \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge D\beta.$$

The **(higher) curvature** of D is $D^2 \in \mathrm{End}^2_{\Omega(A)}(\Omega(A, E))$. If $D^2 = 0$ we say that D is **flat** or a **representation up to homotopy** (or **ruth**).

Higher connections

Decompose D into operators $D_i : \Omega(A, E) \to \Omega(A, E)$ of bidegree $(i, 1 - i)$.

Proposition

Higher A-connections on E are equivalently given by tuples (δ, ∇, ω), where

- $\delta \in \mathrm{End}^1(E)$,
- ∇ *is a degree-preserving A-connection on E,*
- $\omega = \sum_{i \geq 2} \omega_i$, *with* $\omega_i \in \Omega^i(A, \mathrm{End}^{1-i}(E))$.

Its curvature is represented by $K = \sum_{i \geq 0} K_i$, *with* $K_i \in \Omega^i(A, \mathrm{End}^{2-i}(E))$ *given by*

- $K_0 := \delta^2$,
- $K_1 := \nabla \delta$,
- $K_2 := R^\nabla + \omega_2 \wedge \delta + \delta \wedge \omega_2$,
- $K_i := d^\nabla \omega_{i-1} + \omega_i \wedge \delta + \delta \wedge \omega_i + \sum_j \omega_j \wedge \omega_{i-j}$, *for* $i > 2$.

Higher connections

Example

If $E = E^0 \oplus E^1$, then a higher A-connection is given by

$$\delta \in \mathrm{Hom}(E^0, E^1), \quad \nabla^i \in \mathrm{Conn}_A(E^i), \quad \omega_2 \in \Omega^2(A, \mathrm{Hom}(E^1, E^0))$$

Then $K_0 = 0$ and

- $K_1 \in \Omega^1(A, \mathrm{Hom}(E^0, E^1))$ is given by
$$K_1(a)e_0 = \nabla^1_a(\delta e_0) - \delta(\nabla^0_a e_0),$$
- $K_2 \in \Omega^2(A, \mathrm{End}(E^0) \oplus \mathrm{End}(E^1))$ is given by
$$K_2(a,b)e_0 = R^{\nabla^0}(a,b)e_0 + \omega_2(a,b)\delta e_0,$$
$$K_2(a,b)e_1 = R^{\nabla^1}(a,b)e_1 + \delta\omega_2(a,b)e_1.$$

If $(V \to M, \nabla)$ is a vector bundle with A-connection, then

$$E^0 = E^1 = V, \quad \delta = \mathbf{1}, \quad \nabla^i = \nabla, \quad \omega_2 = -R^{\nabla} \oplus -R^{\nabla}$$

defines a ruth.

Higher parallel transport

Idea. Integrate ω_i along appropriate surfaces.

Question. How do we integrate $\mathrm{End}(E)$-valued forms? Do they satisfy some form of non-abelian Stokes's theorem?

Act I

Non-abelian Stokes's theorem

Scene 1. Naïve integrals

Naïve integrals

Let $E \to M$ be a vector bundle.

Definition

A **parallel pinched k-cube** in E is a pair (ϕ, ∇), where $\phi : I^k \to M$ is smooth with

$$\phi(m, x_1, \ldots, x_{k-1}) = \phi(m, 0, \ldots 0) =: w_m, \qquad m = 0, 1$$

and ∇ is a TI^k-connection on $\phi^* E \to I^k$.

If $\eta \in \Omega^k(I^k, \mathrm{End}(\phi^* E))$, the **integral of** η is

$$\int_\phi \eta := \int_{I^k} \hat{\eta} \in \mathrm{Hom}(E_{w_1}, E_{w_0}),$$

where $\hat{\eta} \in \Omega^k(I^k, \mathrm{Hom}(E_{w_1}, E_{w_0}))$ is

$$\hat{\eta}(\partial_0, \ldots, \partial_{k-1})(x_0, x) := \tau_{0 \leftarrow x_0, x} \eta(\partial_0, \ldots, \partial_{k-1})(x_0, x) \tau_{x_0 \leftarrow 1, x}.$$

Naïve integrals

Stokes?

$$\int_{\partial\phi} \eta = \int_{\phi} \left(d^{\nabla}\eta + R^{\nabla} \otimes \eta - (-1)^k \eta \otimes R^{\nabla} \right),$$

where

$$\int_{\phi} R^{\nabla} \otimes \eta = \sum_{i=1}^{k-1} (-1)^i \int_{I^k} \int_0^1 \tau_{0 \leftarrow u,x} R^{\nabla}(\partial_0, \partial_i)(u,x) \tau_{u \leftarrow x_0, x}$$
$$\eta(\partial_0, \partial_1, \ldots, \widehat{\partial}_i, \ldots, \partial_{k-1})(x_0, x) \tau_{x_0 \leftarrow 1, x} \, du \, d(x_0, x)$$

Question. Is there a differential D (with $D^2 = 0$) such that

$$\int_{\partial\phi} \eta = \int_{\phi} D\eta \ ?$$

Act I

Non-abelian Stokes's theorem

Scene 2. Bar construction

Bar construction for curved dgas

Definition

A **curved dga** is a tuple $(\mathscr{A}, \wedge, d, R)$, where $(\mathscr{A}, \wedge)$ is a graded associative algebra, d is a degree 1 derivation and $R \in \mathscr{A}^2$ is such that

$$d^2 = [R, \cdot], \qquad dR = 0.$$

Example

If $(E \to M, \nabla)$ is a vector bundle with A-connection, then

$$(\Omega(A, \mathrm{End}(E)), \wedge, d^\nabla, R^\nabla)$$

is a curved dga.

Bar construction for curved dgas

Definition

The **bar complex** of the curved dga $(\mathscr{A}, \wedge, d, R)$ is $(B\mathscr{A}, D)$, where $B\mathscr{A} := T(\mathscr{A}[1]) = \bigoplus_{k\geq 0} \mathscr{A}[1]^{\otimes k}$ is the tensor algebra and

$$
\begin{aligned}
D(a_1 \otimes \cdots \otimes a_k) := & \sum_{i=1}^{k} (-1)^{|a_1|+\cdots+|a_{i-1}|-i+1} a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_k \\
& + \sum_{i=1}^{k-1} (-1)^{|a_1|+\cdots+|a_i|-i} a_1 \otimes \cdots \otimes a_i \wedge a_{i+1} \otimes \cdots \otimes a_k \\
& + \sum_{i=1}^{k+1} (-1)^{|a_1|+\cdots+|a_{i-1}|-i+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes R \otimes a_i \otimes \cdots \otimes a_k.
\end{aligned}
$$

Theorem

$D^2 = 0$.

Curved A_∞-algebras [Kontsevich–Soibelman, Keller–Lowen–Nicolás, Positselski...]

Act I

Non-abelian Stokes's theorem

Scene 3. Iterated integrals

Iterated integrals

Question. Can we define integrals on the elements of $B\Omega(A, \mathrm{End}(E))$?

Definition

Let (ϕ, ∇) be a parallel pinched k-cube. We define the **horizontal integral** over ϕ as the degree -1 linear operator

$$\fint_{\phi} : \Omega(I^k, \mathrm{End}(\phi^* E)) \to \Omega(I^k, \mathrm{End}(\phi^* E))$$

defined as follows: if $\eta \in \Omega^{\ell+1}(I^k, \mathrm{End}(\phi^* E))$ and $(x_0, x) \in I^k$, then

$$\left(\fint_{\phi} \eta\right)(\partial_{i_1}, \ldots, \partial_{i_\ell})(x_0, x) := \int_{x_0}^{1} \tau_{x_0 \leftarrow t, x}\, \eta(\partial_0, \partial_{i_1}, \ldots, \partial_{i_\ell})(t, x)\, \tau_{t \leftarrow x_0, x}\, dt.$$

Iterated integrals

Definition

Let (ϕ, ∇) be a parallel pinched k-cube. Let $\eta_i \in \Omega(I^k, \mathrm{End}(\phi^* E))$ be such that $|\eta_1| + \cdots + |\eta_\ell| - \ell + 1 = k$. Then we define the **(iterated) integral of** $\eta_1 \otimes \cdots \otimes \eta_\ell$ as

$$\int_\phi (\eta_1 \otimes \cdots \otimes \eta_\ell) :=$$

$$= \int_\phi \left(\eta_1 \wedge \fint_\phi \left(\eta_2 \wedge \fint_\phi \left(\eta_3 \wedge \ldots \fint_\phi \left(\eta_{\ell-1} \wedge \fint_\phi \eta_\ell \right) \ldots \right) \right) \right).$$

Different approaches: [Chen 77, Gugenheim 77, Arias Abad–Schaetz 12, Block–Smith 14] and [Igusa 09, Antweiler 24]

Iterated integrals

Example

Let $k = 1$ and $E = \underline{\mathbb{R}^r} \to I$ be the trivial rank r vector bundle over I. Let $A = A(t)dt \in \Omega^1(I, \mathfrak{gl}_r(\mathbb{R}))$, then

$$\left(\fint_I A \right)(t) = \int_t^1 A(s)ds$$

and

$$\int_I A^{\otimes \ell} = \int_0^1 A(t_1) \int_{t_1}^1 A(t_2) \ldots \int_{t_{\ell-2}}^1 A(t_{\ell-1}) \int_{t_{\ell-1}}^1 A(t_\ell) dt_\ell \ldots dt_1$$

The solution to the matrix linear equation $\dot{H} = -AH$ with $H(1) = \mathbf{1}$ is given by

$$H(t) = \mathbf{1} + \sum_{\ell \geq 1} \int_t^1 A^{\otimes \ell}.$$

Act I
Non-abelian Stokes's theorem

Scene 4. Non-abelian Stokes's theorem

Non-abelian Stokes's theorem

Theorem (P., Holtrop)

Let $E \to M$ be a vector bundle, (ϕ, ∇) a parallel pinched $(k+1)$-cube and $\eta_i \in \phi^*\Omega(M, \mathrm{End}(E))$ such that $|\eta_1| + \cdots + |\eta_\ell| - \ell + 1 = k$. Then

$$\int_{\partial\phi} (\eta_1 \otimes \cdots \otimes \eta_\ell) = \int_\phi D(\eta_1 \otimes \cdots \otimes \eta_\ell).$$

Example

If $\ell = 1$ (and $|\eta_1| = k$), then

$$\int_{\partial\phi} \eta_1 = \int_\phi \left(d^\nabla \eta_1 + R^\nabla \otimes \eta_1 - (-1)^{|\eta_1|} \eta_1 \otimes R^\nabla \right).$$

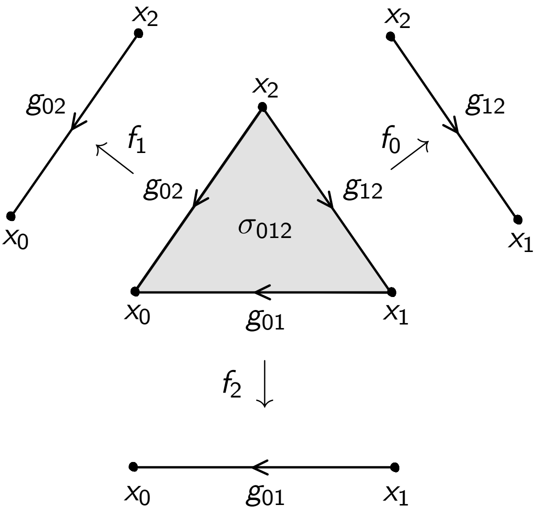
If $\ell = 2$ (and $|\eta_1| + |\eta_2| = k+1$), then

$$\int_{\partial\phi} (\eta_1 \otimes \eta_2) = \int_\phi \Big(d^\nabla \eta_1 \otimes \eta_2 - (-1)^{|\eta_1|} \eta_1 \otimes d^\nabla \eta_2 - (-1)^{|\eta_1|} \eta_1 \wedge \eta_2$$
$$+ R^\nabla \otimes \eta_1 \otimes \eta_2 - (-1)^{|\eta_1|} \eta_1 \otimes R^\nabla \otimes \eta_2 + (-1)^{|\eta_1| + |\eta_2|} \eta_1 \otimes \eta_2 \otimes R^\nabla \Big)$$

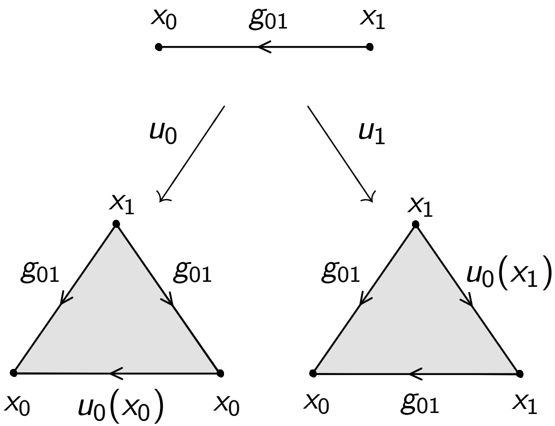
Act II
Higher general linear groupoids

Scene 1. Higher Lie groupoids

Higher Lie groupoids



Higher Lie groupoids



Higher Lie groupoids

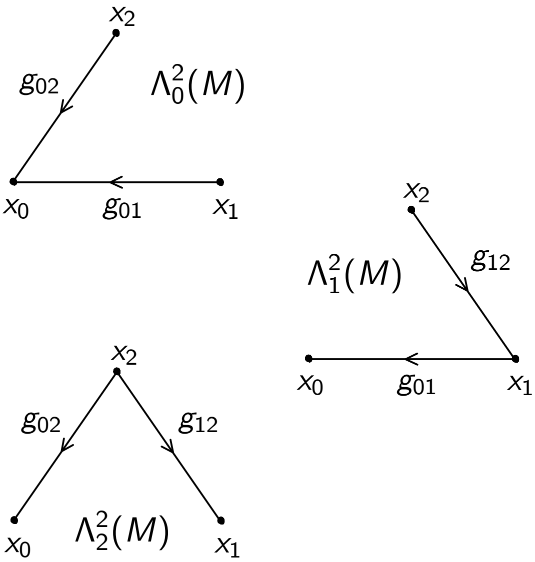
Definition

A **simplicial manifold** M is given by

- manifolds $\{M_n\}_n$, where M_n is called the space of n-**simplices**,
- surjective submersions $f_i : M_n \to M_{n-1}$, for $i = 0, \ldots, n$, called the **face maps**,
- smooth maps $u_i : M_{n-1} \to M_n$, for $i = 0, \ldots, n-1$, called the **units** or **degeneracy maps**,

satisfying the **simplicial relations**.

Higher Lie groupoids



Higher Lie groupoids

Definition

An (n, k)-**horn** is an n-simplex with the interior and the kth face removed. The space of (n, k)-horns of M is denoted by $\Lambda^n_k(M)$.

There are natural **horn maps** $\lambda^n_k : M_n \to \Lambda^n_k(M)$.

Definition (Henriques, Zhu 06)

A simplicial manifold G is

- a **Lie ∞-groupoid** if all the horn maps are surjective submersions,
- a **Lie n-groupoid** if it is a Lie ∞-groupoid and the horn maps λ^m_k are diffeomorphisms for $m > n$.

Higher Lie groupoids

Example

If $G \rightrightarrows M$ is a Lie groupoid, its **nerve** $\mathrm{N}(G)$ is the Lie 1-groupoid defined by

$$\mathrm{N}(G)_k := G \times_M \overset{(k)}{\cdots} \times_M G.$$

Example

If $A \to M$ is a Lie algebroid, its **homotopy ∞-groupoid** $\Pi_\infty(A)$ is the ∞-groupoid defined by

$$\Pi_\infty(A)_k := \mathrm{Hom}_{\mathrm{LA}}(T\Delta^k, A).$$

Its **homotopy n-groupoid** $\Pi_n(A)$ we define by $\Pi_n(A)_k = \Pi_\infty(A)_k$ for $k < n$, and

$$\Pi_n(A)_n := \mathrm{Hom}_{\mathrm{LA}}(T\Delta^n, A)/\mathrm{LA}\text{ homotopy}.$$

Act II

Higher general linear groupoids

Scene 2. Higher general linear groupoids

Desiderata

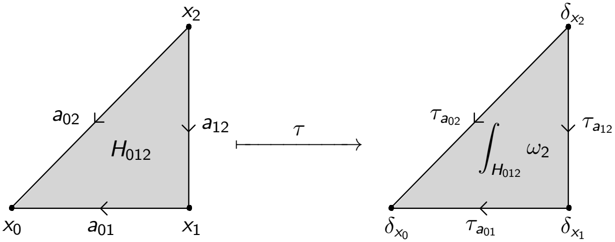
$$\begin{tikzcd} x_0 \arrow[r, mapsto, "\tau"] & \delta_{x_0} \end{tikzcd}$$

$$\delta_{x_0}^2 = K_0$$

$$\begin{tikzcd} x_0 & x_1 \arrow[l, "a_{01}"'] \arrow[r, mapsto, "\tau"] & \delta_{x_0} & \delta_{x_1} \arrow[l, "\tau_{a_{01}}"'] \end{tikzcd}$$

$$\tau_{a_{01}} \delta_{x_1} - \delta_{x_0} \tau_{a_{01}} = \int_{a_{01}} \nabla \delta = \int_{a_{01}} K_1$$

Desiderata



$$
\begin{aligned}
\tau_{a_{02}} - \tau_{a_{01}}\tau_{a_{12}} &= \int_{H_{012}} R^{\nabla} = \int_{H_{012}} (K_2 - \omega_2 \wedge \delta - \delta \wedge \omega_2) \\
&= \int_{H_{012}} K_2 - \left(\int_{H_{012}} \omega_2 \right) \delta_{x_2} + \int_{H_{012}} (\omega_2 \otimes \nabla \delta) \\
&\qquad - \delta_{x_0} \int_{H_{012}} \omega_2 - \int_{H_{012}} (\nabla \delta \otimes \omega_2),
\end{aligned}
$$

equivalently,

$$
\tau_{a_{02}} - \tau_{a_{01}}\tau_{a_{12}} + \left(\int_{H_{012}} \omega_2 \right) \delta_{x_2} + \delta_{x_0} \int_{H_{012}} \omega_2 = \int_{H_{012}} (K_2 + \omega_2 \otimes K_1 - K_1 \otimes \omega_2).
$$

Guess

We would like:

- $\mathrm{GL}_\infty(E)_0 = \mathrm{End}^1(E)$,
- $\mathrm{GL}_\infty(E)_1 = \mathrm{End}^1(\pi_1^* E) \oplus \mathrm{GL}^0(\pi_1^* E, \pi_0^* E) \oplus \mathrm{End}^1(\pi_0^* E)$,
- $$\mathrm{GL}_\infty(E)_2 = \bigoplus_{i=0}^{2} \mathrm{End}^1(\pi_i^* E) \oplus \bigoplus_{i<j} \mathrm{GL}^0(\pi_j^* E, \pi_i^* E) \oplus \mathrm{Hom}^{-1}(\pi_2^* E, \pi_0^* E)$$
- ...

Higher general linear groupoids

Definition

We define the **higher general linear groupoid** of E as the simplicial set $\mathrm{GL}_\infty(E)$ defined as follows: for each k, consider the vector bundle

$$V_k := \bigoplus_{\alpha:[i]\hookrightarrow[k]} \mathrm{Hom}^{1-i}(\pi^*_{\alpha(i)}E, \pi^*_{\alpha(0)}E) \to M^{k+1}$$

and set

$$\mathrm{GL}_\infty(E)_k := \{B \in V_k : B_\alpha \text{ is an isomorphism for all } \alpha : [1] \hookrightarrow [k]\},$$

and appropriate face and degeneracy maps.

"free" dg-nerve [Lurie, kerodon]

Proposition

If $E = \bigoplus_{k=0}^{n-1} E^k$, then $\mathrm{GL}_\infty(E)$ is a Lie $(n+1)$-groupoid, in which case we also denote it by $\mathrm{GL}_{n+1}(E)$. Moreover the natural map

$$q : \mathrm{GL}_{n+1}(E) \to \mathrm{N}(M \times M)$$

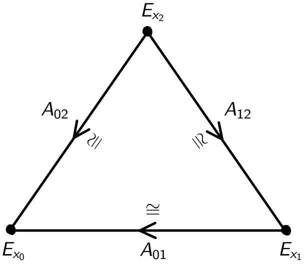
is a morphism of Lie $(n+1)$-groupoids.

One term case

Example

If E only has one term, then

- $\mathrm{GL}_2(E)_0 = M$,
- $\mathrm{GL}_2(E)_1 = \{(x_0, x_1, A_{01}) : x_i \in M \text{ and } A_{01} : E_{x_1} \xrightarrow{\cong} E_{x_0}\} = \mathrm{GL}(E)_1$,
- $\mathrm{GL}_2(E)_2 = \{(x_0, x_1, x_2, A_{01}, A_{02}, A_{12}) : x_i \in M \text{ and } A_{ij} : E_{x_j} \xrightarrow{\cong} E_{x_i}\}$.



Very important: we do not impose

$$A_{01}A_{12} - A_{02} = 0.$$

Act III

Higher parallel transport and higher holonomy

Scene 1. Higher parallel transport

Higher parallel transport

Definition

Let (δ, ∇, ω) be a higher A-connection on E. We define the **higher parallel transport** as the higher functor $\tau : \Pi_\infty(A) \to \mathrm{GL}_\infty(E)$ acting on $\sigma \in \Pi_\infty(A)_k$ by

$$\tau(\sigma)_{\alpha:[0]\hookrightarrow[k]} = \delta_{\alpha^*\sigma},$$
$$\tau(\sigma)_{\alpha:[1]\hookrightarrow[k]} = \tau_{\alpha^*\sigma},$$
$$\tau(\sigma)_{\alpha:[i]\hookrightarrow[k]} = \int_{\alpha^*\sigma} \sum_{\ell\geq 1} \omega^{\otimes\ell}.$$

Example

- $\tau(\sigma)_{\alpha:[2]\hookrightarrow[k]} = \int_{\alpha^*\sigma} \omega_2,$
- $\tau(\sigma)_{\alpha:[3]\hookrightarrow[k]} = \int_{\alpha^*\sigma} (\omega_3 + \omega_2 \otimes \omega_2).$

Higher parallel transport

Definition

Let (δ, ∇, ω) be a higher A-connection on E. We define the **higher parallel transport** as the higher functor $\tau : \Pi_\infty(A) \to \mathsf{GL}_\infty(E)$ acting on $\sigma \in \Pi_\infty(A)_k$ by

$$\begin{aligned}
\tau(\sigma)_{\alpha:[0]\hookrightarrow[k]} &= \delta_{\alpha^*\sigma}, \\
\tau(\sigma)_{\alpha:[1]\hookrightarrow[k]} &= \tau_{\alpha^*\sigma}, \\
\tau(\sigma)_{\alpha:[i]\hookrightarrow[k]} &= \int_{\alpha^*\sigma} \sum_{\ell \geq 1} \omega^{\otimes \ell}.
\end{aligned}$$

Remark. If $E = \bigoplus_{k=0}^{n-1} E^k$, then τ descends to a functor

$$\tau : \Pi_{n+1}(A) \to \mathsf{GL}_{n+1}(E).$$

Act III
Higher parallel transport and higher holonomy

Scene 2. Curvature

Curvature bundle and curvature section

Definition

We define the **curvature bundle** $\mathrm{Curv}_\infty(E) \to \mathrm{N}(M \times M)$ as the simplicial vector bundle defined by

$$\mathrm{Curv}_\infty(E)_k := \bigoplus_{\alpha:[i]\hookrightarrow[k]} \mathrm{Hom}^{2-i}(\pi^*_{\alpha(i)}E, \pi^*_{\alpha(0)}E) \to M^{k+1}$$

and appropriate face and degeneracy maps.
We define the **curvature section** $K_E \in \Gamma(q^*\mathrm{Curv}_\infty(E))$ given by

$$K_E(B)_{\alpha:[i]\hookrightarrow[k]} := -\sum_{j=1}^{i-1}(-1)^j B_{\alpha\circ\delta^j} + \sum_{j=0}^{i}(-1)^j B_{\alpha\circ\alpha^j} B_{\alpha\circ\beta^{i-j}}.$$

Proposition

If $E = \bigoplus_{k=0}^{n-1} E^k$, then $\mathrm{Curv}_\infty(E)$ is a Lie $(n+1)$-groupoid, in which case we also denote it by $\mathrm{Curv}_{n+1}(E)$. Moreover the curvature section K_E is a simplicial section.

One term case

If E has one term, then

$$\begin{aligned} \mathrm{Curv}_2(E)_0 &= \mathrm{Curv}_2(E)_1 = 0, \\ \mathrm{Curv}_2(E)_2 &= \mathrm{Hom}^0(\pi_2^* E, \pi_0^* E), \end{aligned}$$

and

$$K_E(x_i, A_{ij}) = A_{02} - A_{01} A_{12}.$$

Hence,

$$K_E^{-1}(0) = \mathrm{N}(\mathrm{GL}(E)) \subsetneq \mathrm{GL}_2(E).$$

Two terms case

If $E = E^0 \oplus E^1$ has two terms, then

$$\begin{aligned}
\mathrm{Curv}_2(E)_0 &= 0, \\
\mathrm{Curv}_1(E)_1 &= \mathrm{Hom}^1(\pi_1^* E, \pi_0^* E), \\
\mathrm{Curv}_2(E)_2 &= \bigoplus_{i<j} \mathrm{Hom}^1(\pi_j^* E, \pi_i^* E) \oplus \mathrm{Hom}^0(\pi_2^* E, \pi_0^* E), \\
\mathrm{Curv}_2(E)_3 &= \bigoplus_{i<j} \mathrm{Hom}^1(\pi_j^* E, \pi_i^* E) \oplus \bigoplus_{i<j<k} \mathrm{Hom}^0(\pi_k^* E, \pi_i^* E) \\
&\qquad \oplus \mathrm{Hom}^{-1}(\pi_3^* E, \pi_0^* E)
\end{aligned}$$

and

$$\begin{aligned}
K_E(\delta_i, A_{01}) &= \delta_0 A_{01} - A_{01}\delta_1 \\
K_E(\delta_i, A_{ij}, H_{012}) &= (\dots, A_{02} + \delta_0 H_{012} - A_{01}A_{12} + H_{012}\delta_2) \\
K_E(\delta_i, A_{ij}, H_{ijk}, L_{0123}) &= (\dots, H_{023} - H_{013} \\
&\qquad + \delta_0 L_{0123} - A_{01}H_{123} + H_{012}A_{23} - L_{0123}\delta_3)
\end{aligned}$$

Higher parallel transport revisited

Theorem (P., Holtrop)

Higher parallel transport $\tau : \Pi_\infty(A) \to \mathrm{GL}_\infty(E)$ satisfies the following Stokes formula: if $\sigma \in \Pi_\infty(A)_k$, then

$$K_E(\tau(\sigma))_{\alpha:[0]\hookrightarrow[k]} = \delta^2_{\alpha^*\sigma},$$

$$K_E(\tau(\sigma))_{\alpha:[i]\hookrightarrow[k]} = \int_{\alpha^*\sigma} \sum_{\ell_1,\ell_2\geq 0} (-1)^{\ell_2}\omega^{\otimes\ell_1} \otimes K_{\geq 1} \otimes \omega^{\otimes \ell_2}.$$

$$\begin{tikzcd}
\tau^* q^* \mathrm{Curv}_\infty(E) \arrow[d] & q^*\mathrm{Curv}_\infty(E) \arrow[d] & \mathrm{Curv}_\infty(E) \arrow[d] \\
\Pi_\infty(A) \arrow[r, "\tau"'] \arrow[u, bend left, "\tau^* K_E"] & \mathrm{GL}_\infty(E) \arrow[r, "q"'] \arrow[u, bend right, "K_E"'] & \mathrm{N}(M\times M)
\end{tikzcd}$$

$$\tau^* K_E = \int \sum_{\ell_1,\ell_2\geq 0} (-1)^{\ell_2}\omega^{\otimes\ell_1} \otimes K_{\geq 1} \otimes \omega^{\otimes \ell_2}$$

Higher gauge groupoids

Definition

We define the **higher gauge groupoid** of E as

$$\mathsf{G}_\infty(E) := K_E^{-1}(0) \subseteq \mathsf{GL}_\infty(E).$$

dg-nerve [Lurie, kerodon]

Proposition

If $E = \bigoplus_{k=0}^{n-1} E^k$, then $\mathsf{G}_\infty(E)$ is a n-groupoid, in which case we also denote it by $\mathsf{G}_n(E)$. It is Lie if $n \leq 2$ and quasi-smooth if $n > 2$.

[Getzler–Behrend 17], $n = 2$ case: [Ortiz–Brahic 19, del Hoyo–Stefani 19]

Corollary (of the theorem)

If $E = \bigoplus_{k=0}^{n-1} E^k$ and the higher connection is flat, then higher parallel transport descends to a functor $\tau : \Pi_n(A) \to \mathsf{G}_n(E)$.

Act III

Higher parallel transport and higher holonomy

Scene 3. Higher Riemann–Hilbert correspondence
and higher holonomy

Higher Riemann–Hilbert correspondence and holonomy

Theorem (P., Holtrop)

If $E = \bigoplus_{i=0}^{n-1} E^i$, there is a commutative diagram of A_∞-functors between curved dg-categories

$$\begin{tikzcd}
\mathsf{HConn}_A(E) \arrow[r] & {[\Pi_{n+1}(A), \mathsf{GL}_{n+1}(E)]^\infty_*} \\
\mathsf{HFlat}_A(E) \arrow[r] \arrow[u, hook] & {[\Pi_n(A), \mathsf{G}_n(E)]^\infty_*} \arrow[u, hook]
\end{tikzcd}$$

where the vertical arrows are inclusions of full subcategories.

flat case for Lie algebroids: [Arias Abad–Schaetz 12],

quasi-equivalence for tangent bundles: [Block–Smith 14]

Definition

We define the **higher holonomy group** of (δ, ∇, ω) at $x \in M$ as

$$\mathsf{Hol}_x := \tau(\Pi_\infty(A))_x.$$

The end

Thanks