Filling Dehn surfaces and Knot Theory

Pablo Serrano

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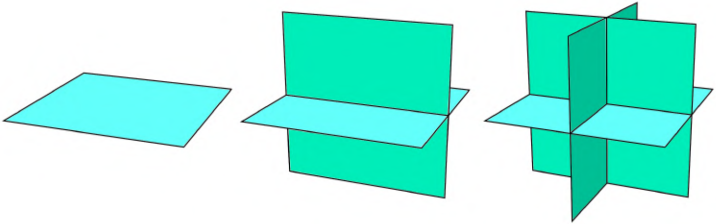
Definition

We fix a compact, orientable surface S and a closed, connected and orientable 3-manifold M.

A **Dehn surface** Σ is the image of an immersion $f : S \longrightarrow M$ such that there are no self-intersections in the boundary and the local configuration around each point is one of the following:



Which we call simple, double and triple points, respectively.

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We say that a Dehn surface is **filling** if it induces a celular decomposition in M, whose vertices are triple points, its edges are double curves and its faces are made up of simple points.

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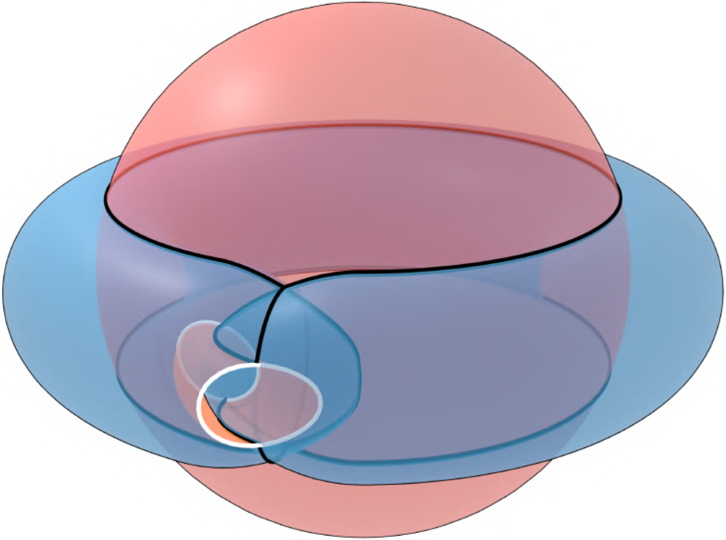
Theorem 1 (Montesinos)

For any M, there exists a filling Dehn sphere (surface) $\Sigma \subset M$.

Example



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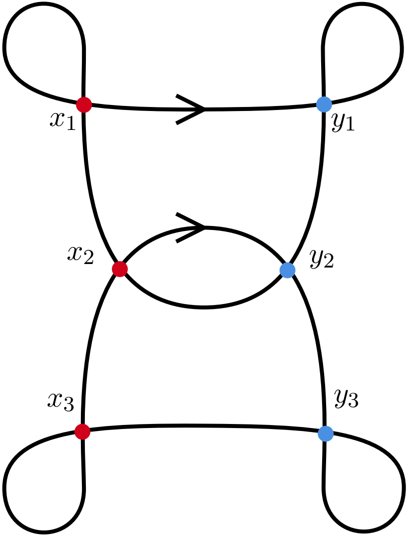
- We also have a pairing between them, τ, which assigns each curve to the other lift of the same double curve.
- This is what we call the **Johansson diagram** of Σ. Moreover, if Σ is filling, the diagram determines completely the 3-manifold in which it is realized as a filling Dehn surface.

Example of a Johansson diagram



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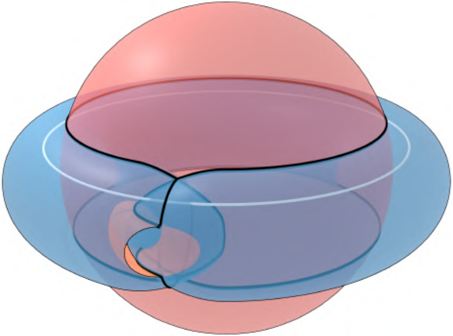
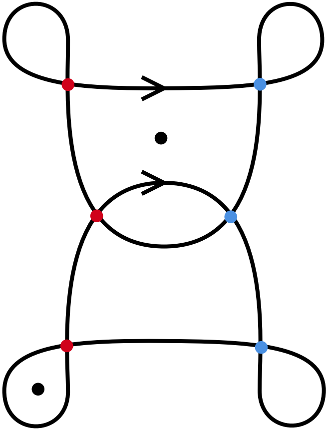
If we include in the diagram the intersections of K and Σ, we can completely recover the knot in the 3-manifold.

Example: the unknot





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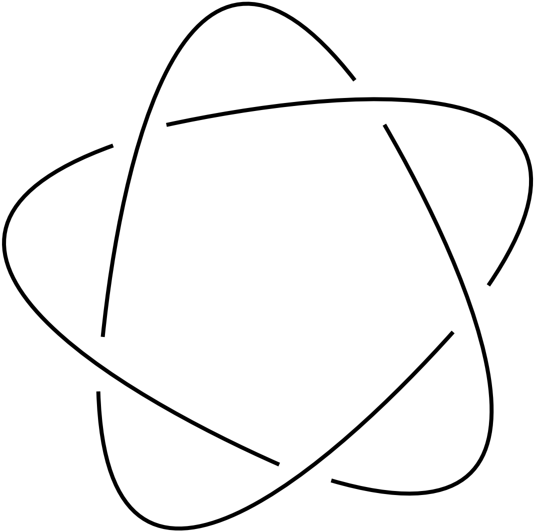
- We would like have a practical method for finding filling Dehn spheres that split an arbitrary knot in $\mathbb{S}^3$.
- Moreover, we also need to be able to obtain the diagram easily from the surface, as it can get quite confusing to do so from a drawing.

Sphere in diagram position



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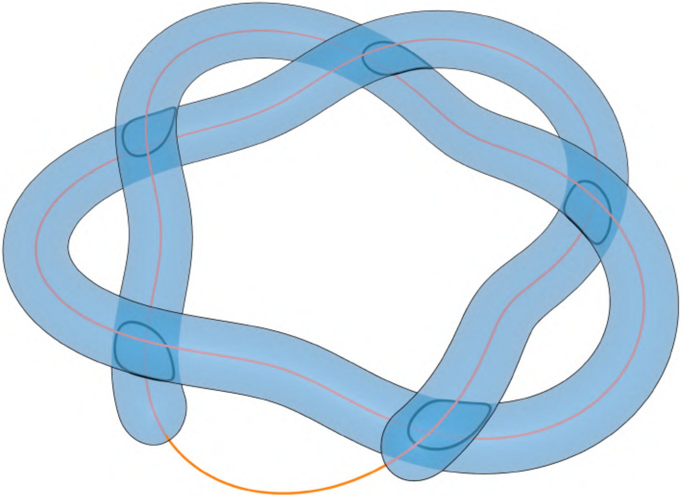
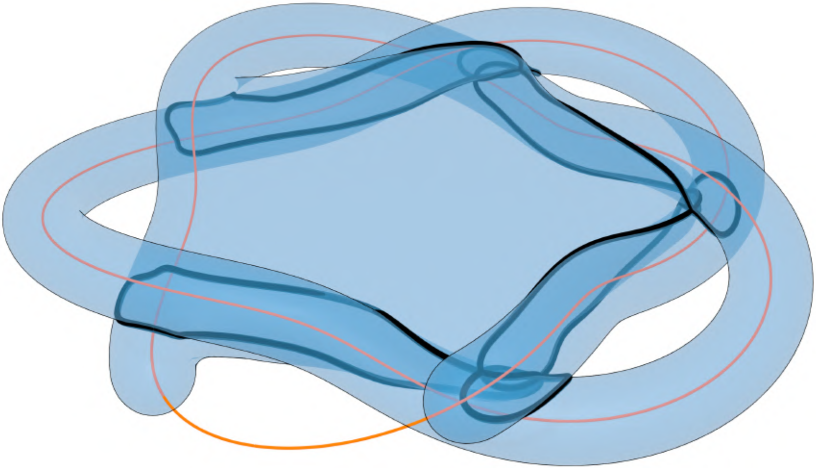
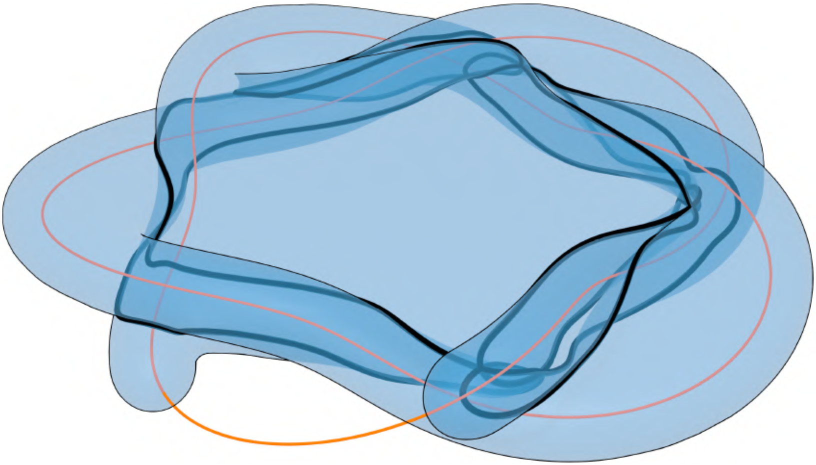


n-gon operation



2-gon operation



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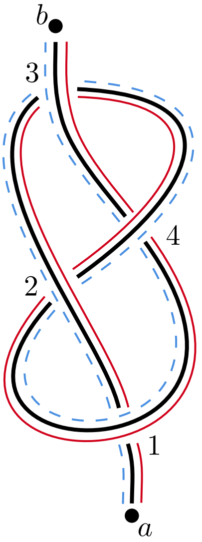

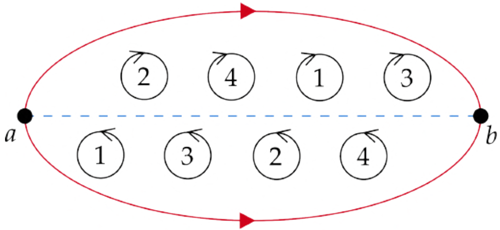

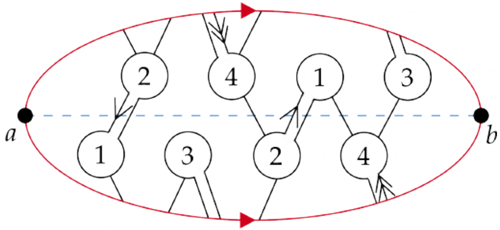

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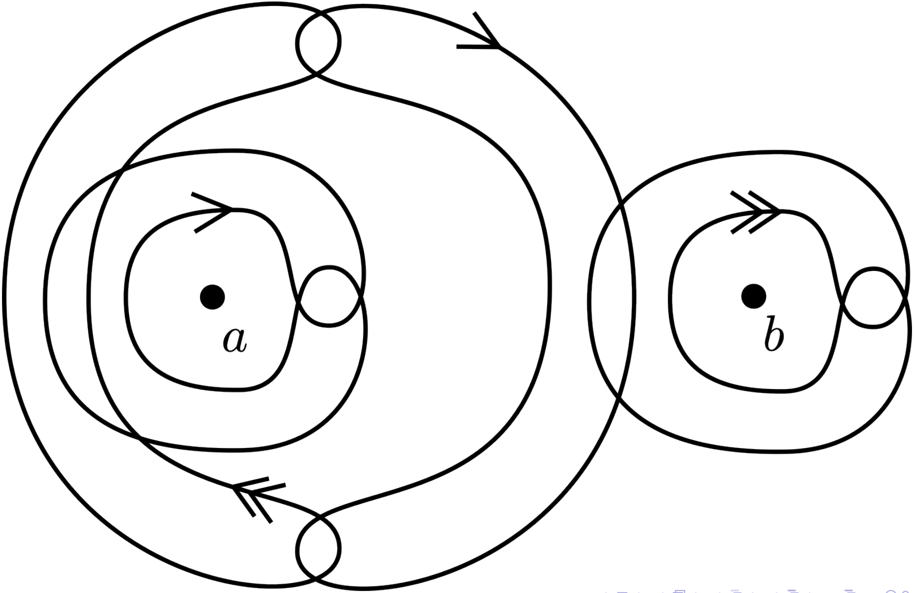

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- Ramified coverings over a knot can be used to lift Dehn surfaces that split said knot. Orbifolds can also be studied in this context.
- Is every 2-knot K in $M \times I$ isotopic to $\tilde{K}$, such that $p(K)$ is a filling Dehn surface in M?
- Generalize the theory to Dehn 3-manifolds in 4-manifolds.

Thanks!