Symmetric Cartan calculus

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Cartan calculus

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^r(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{\dim M}(M)$$

Cartan calculus

$$\Omega^0(M) \underset{\iota_X}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1(M) \underset{\iota_X}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots \underset{\iota_X}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^r(M) \underset{\iota_X}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots \underset{\iota_X}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^{\dim M}(M)$$

for $X \in \mathfrak{X}(M)$

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graded
Lie
algebra

Cartan calculus

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Lie derivative

Cartan calculus

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Lie bracket
of vector fields

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Symmetric forms = covariant completely symmetric tensor fields

$$\Upsilon^r(M) \coloneqq \Gamma(\mathrm{Sym}^r\, T^*M), \qquad \Upsilon^\bullet(M) \coloneqq \bigoplus_{r=0}^{\infty} \Upsilon^r(M)$$

Symmetric forms

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The map $(\,)^\vee \colon \Upsilon^\bullet(M) \to \mathcal{C}^\infty(TM)$ determined by

$$f^\vee \coloneqq \mathrm{pr}^* f, \qquad \varphi^\vee(u) \coloneqq \frac{1}{r!}\varphi(u, \dots, u)$$

$\mathrm{pr}: TM \longrightarrow M$ $\qquad$ $u \in TM$

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gives the unital graded algebra isomorphism:

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Lie algebra

Symmetric derivatives

d is the unique operator on $\Omega^\bullet(M)$ such that

- it is a degree-1 graded derivation of $(\Omega^\bullet(M), \wedge)$,
- $(\mathrm{d}f)(X) = Xf$
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$$\left\{\begin{array}{c}\text{affine}\\ \text{connections}\\ \text{on } M\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{degree-1 derivations of } (\Upsilon^\bullet(M), \odot)\\ \text{such that } (Df)(X) = Xf\end{array}\right\}$$

$$\nabla \longmapsto \nabla^s$$

Example: $\nabla^s\varphi := (r+1)\,\mathrm{sym}(\nabla\varphi)$

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Under the identification $\Upsilon^\bullet(M) \overset{\sim}{\longleftrightarrow} \mathcal{C}^\infty_{\text{hom}}(TM)$, we have

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$\hookrightarrow \in \mathfrak{X}(TM)$ is the geodesic spray

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Symmetric Lie derivatives and symmetric brackets

$$[\iota_X, \iota_Y] = 0, \qquad [\nabla^s, \nabla^s] = 0, \qquad [\iota_X, \nabla^s] =: L^s_X$$

Symmetric Lie derivatives and symmetric brackets

symmetric Lie derivative

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trivially satisfied

Symmetric Lie derivatives and symmetric brackets

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Parallel transport along $\gamma(\varepsilon) := \underbrace{\Psi^X_\varepsilon(m)}_{\text{the flow generated by } X}$

$$(L^s_X \varphi)_m = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} P^\gamma_{2\varepsilon,0} (\Psi^X_{-\varepsilon})^*_{\Psi^X_{2\varepsilon}(m)} \varphi$$

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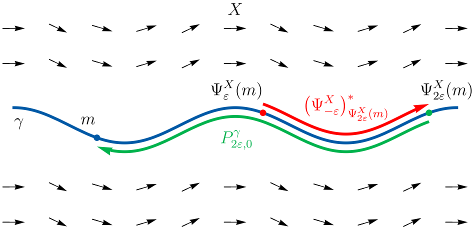

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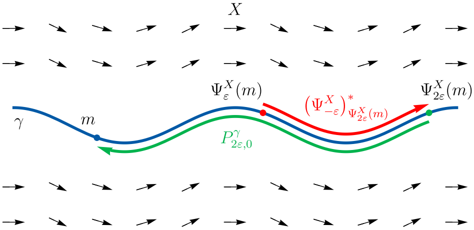

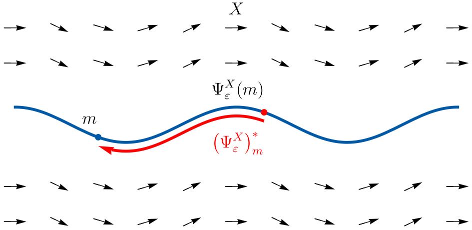

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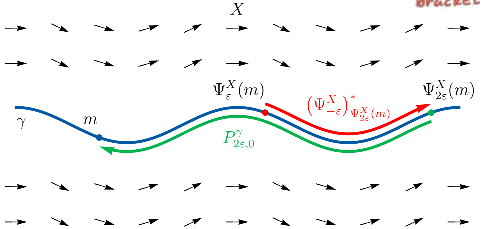
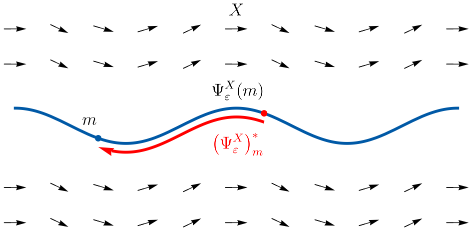
↳ symmetric bracket

$$(L_X^s \varphi)_m = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} P^{\gamma}_{2\varepsilon,0} (\Psi^X_{-\varepsilon})^*_{\Psi^X_{2\varepsilon}(m)} \varphi$$





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Symmetric Lie derivatives and symmetric brackets

$$[\iota_X, \iota_Y] = 0, \qquad [\nabla^s, \nabla^s] = 0, \qquad [\iota_X, \nabla^s] =: L^s_X \qquad [L^s_X, \iota_Y] =: \iota_{[X,Y]_s}$$

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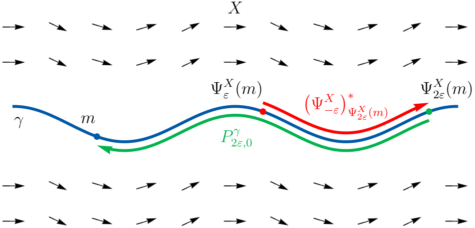
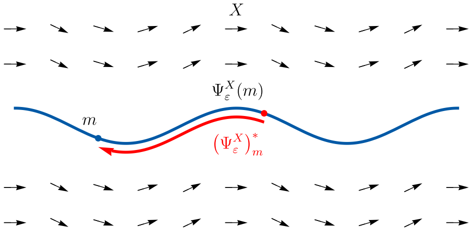


$$[X, Y]_s = \nabla_X Y + \nabla_Y X = L^s_X Y$$



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↳ works only on $\mathfrak{X}^\bullet(M)$

works on any tensor field

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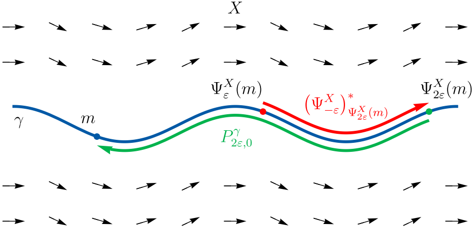

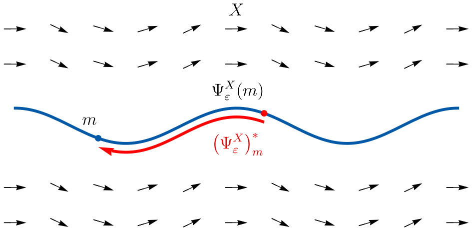

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Commutators

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$$[L^s_X, L^s_Y] = L^s_{[X,Y]} + \iota^s_{2(\nabla_{\nabla X} Y - \nabla_{\nabla Y} X - R(X,Y))},$$
$$[\nabla^s, L^s_X] = 2[\iota^s_{\nabla X}, \nabla^s] + \iota^s_{2\,\mathrm{sym}\,\iota_X R + R^s X}$$

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For $\sigma \in \Gamma(\mathrm{Sym}^r T^* M \otimes TM)$, the derivation ι^s_σ is determined by

$$\iota^s_\sigma f := 0, \qquad (\iota^s_\sigma \alpha)(X_1, \ldots, X_r) := \alpha(\sigma(X_1, \ldots, X_r)).$$

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$R^s \colon \mathfrak{X}(M) \to \Gamma(\mathrm{Sym}^2\, T^*M \otimes TM)$ is given by

$$(R^s X)(Y, Z) := \nabla_Y \nabla_Z X + \nabla_Z \nabla_Y X - \nabla_{[Y,Z]_s} X.$$

Theorem: All of the six relations

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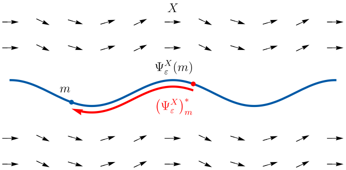

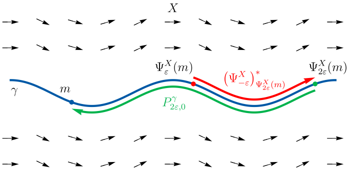

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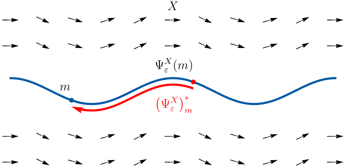
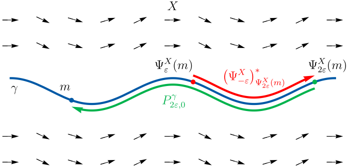
$$\Longleftrightarrow$$



$$=$$



Group of affine diffeomorphisms

$\phi \in \mathrm{Diff}(M)$ is called an **affine diffeomorphism** for an affine connection ∇ if

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- ↳ ∇ is Levi-Civita for g $\Rightarrow$ $\mathrm{Iso}(M,g) \leq \mathrm{Aff}(M,\nabla)$

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1. $\phi^* \circ \nabla^s = \nabla^s \circ \phi^*$,
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γ is a geodesic $\Leftrightarrow$ $\phi \circ \gamma$ is a geodesic

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$$\mathrm{aff}_0(M,\nabla) \coloneqq \{X \in \mathrm{aff}(M,\nabla) \,|\, \nabla X = 0\} \leq \mathrm{aff}(M,\nabla)$$

abelian
Lie
subalgebra

The Patterson-Walker metric

$\mathrm{pr}: T^*M \longrightarrow M$

A torsion-free connection ∇ on $M \rightsquigarrow T(T^*M) = H_\nabla \oplus V \cong \mathrm{pr}^*(TM \oplus T^*M)$.

$$H_\nabla \cong \mathrm{pr}^* TM \qquad V \cong \mathrm{pr}^* T^*M$$

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$$\text{the horizontal lift via } \nabla\text{: } (\ )^{\mathsf{h}} \colon \mathfrak{X}(M) \to \mathfrak{X}(T^*M),$$

The Patterson-Walker metric

A torsion-free connection ∇ on $M \rightsquigarrow T(T^*M) = H_\nabla \oplus V \cong \mathrm{pr}^*(TM \oplus T^*M)$.

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a split signature metric on T^*M

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→ a split signature metric on T^*M

→ In natural coordinates: $g_\nabla|_{T^*U} = dp_i \odot dx^i - p_k(\mathrm{pr}^* \Gamma^k{}_{ij}) dx^i \odot dx^j$

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Proposition: A connection ∇ on M lifts to the connection on T^*M:

$$\hat{\nabla}_{X^{\mathsf{h}}} Y^{\mathsf{h}} := (\nabla_X Y)^{\mathsf{h}}, \quad \hat{\nabla}_{X^{\mathsf{h}}} \alpha^{\mathsf{v}} := (\nabla_X \alpha)^{\mathsf{v}}, \quad \hat{\nabla}_{\alpha^{\mathsf{v}}} X^{\mathsf{h}} := 0, \quad \hat{\nabla}_{\alpha^{\mathsf{v}}} \beta^{\mathsf{v}} := 0.$$

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The Patterson-Walker metric can be then expressed by

$$g_\nabla = \hat{\nabla}^s \underbrace{\alpha_{\mathsf{can}}}_{\text{the canonical 1-form on } T^*M}.$$

Killing vector fields for the Patterson-Walker metric

Every vector field $Q \in \mathfrak{X}(T^*M)$ is fully determined by its values on

$$\{\text{degree-}1 \text{ smooth homogeneous functions}\} \subseteq C^\infty(T^*M)$$

Killing vector fields for the Patterson-Walker metric

Every vector field $Q \in \mathfrak{X}(T^*M)$ is fully determined by its values on

$$\{\text{degree-}1 \text{ smooth homogeneous functions}\} = \{Z^\vee : \zeta \mapsto \zeta(Z_{\mathrm{pr}(\zeta)}) \mid Z \in \mathfrak{X}(M)\}.$$

T^*M

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Killing tensors: $\mathrm{Kill}^\bullet(M,\nabla) := \ker \nabla^S \subseteq \Upsilon^\bullet(M)$

$\mathrm{Kill}^r(M,\nabla) := \mathrm{Kill}^\bullet(M,\nabla) \cap \Upsilon^r(M)$

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$$X^c Z^\vee := [X, Z]^\vee.$$

Theorem: Killing vector fields for g_∇ admit the natural decomposition:

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$R_\pi \in \Gamma(\mathrm{Sym}^2 T^*M \otimes \mathrm{Sym}^2 TM)$

$R_\pi(X,Y) := \mathrm{sym}\left(R(X,\pi(\cdot))Y + R(Y,\pi(\cdot))X\right)$

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with the Lie algebra structure:

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Lie algebra actions

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Lie algebra actions

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(-1,0,1)-grading

Relation to symmetric Cartan calculus

Theorem: All of the six relations

$$[\iota_X, \iota_Y] = 0, \qquad [\nabla^s, \nabla^s] = 0, \qquad [\iota_X, \nabla^s] =: L^s_X,$$
$$[L^s_X, \iota_Y] =: \iota_{[X,Y]_s}, \qquad [L^s_X, L^s_Y] = L^s_{[X,Y]_s}, \qquad [\nabla^s, L^s_X] = 0$$

are satisfied if and only if, for every $(\varepsilon, m) \in \mathbb{R} \times M$ such that $\Psi^X_\varepsilon(m)$ is defined, we have

$$P^\gamma_{\varepsilon,0} = (\Psi^X_\varepsilon)^*_m.$$

Relation to symmetric Cartan calculus

Theorem: All of the six relations

$$
\begin{aligned}
&[\iota_X, \iota_Y] = 0, && [\nabla^s, \nabla^s] = 0, && [\iota_X, \nabla^s] =: L^s_X, \\
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\end{aligned}
$$

are satisfied if and only if $X \in \mathrm{aff}_0(M, \nabla)$.

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Cartan calculus	Symmetric Cartan calculus

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Cartan calculus

The canonical symplectic structure:

$$\omega_{\mathsf{can}} = \mathrm{d}\alpha_{\mathsf{can}}$$

Symmetric Cartan calculus

The Patterson-Walker metric:

$$g_\nabla = \hat{\nabla}^s \alpha_{\mathsf{can}}$$

Relation to symmetric Cartan calculus

Theorem: All of the six relations

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The canonical symplectic structure:

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Symplectic vector fields

Symmetric Cartan calculus

The Patterson-Walker metric:

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Relation to symmetric Cartan calculus

Theorem: All of the six relations

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are satisfied if and only if X^c is a gradient Killing vector field for $g_\triangledown$.

Cartan calculus

The canonical symplectic structure:

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Symplectic vector fields

Hamiltonian vector fields

Symmetric Cartan calculus

The Patterson-Walker metric:

$$g_\triangledown = \hat{\nabla}^s \alpha_{\mathsf{can}}$$

Killing vector fields

gradient Killing vector fields

Relation to symmetric Cartan calculus

Theorem: All of the six relations

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are satisfied if and only if X^c is a gradient Killing vector field for $g_\triangledown$.

Cartan calculus

The canonical symplectic structure:

$$\omega_{\mathsf{can}} = \mathrm{d}\alpha_{\mathsf{can}}$$

Symplectic vector fields

Hamiltonian vector fields

X^c is always Hamiltonian

Symmetric Cartan calculus

The Patterson-Walker metric:

$$g_\triangledown = \hat{\nabla}^s \alpha_{\mathsf{can}}$$

Killing vector fields

gradient Killing vector fields

X^c is gradient Killing $\Leftrightarrow X \in \mathrm{aff}_0(M, \nabla)$

M., Rubio

Symmetric Cartan calculus, the Patterson-Walker metric and Killing vectorfields

arXiv:2501.12442

M., Rubio

*Symmetric Cartan calculus,
the Patterson-Walker metric
and Killing vectorfields*

arXiv:2501.12442

Thank you for your attention!