An introduction to generalized (complex) geometry

Roberto Rubio

UAB
Universitat Autònoma
de Barcelona

2th October 2026



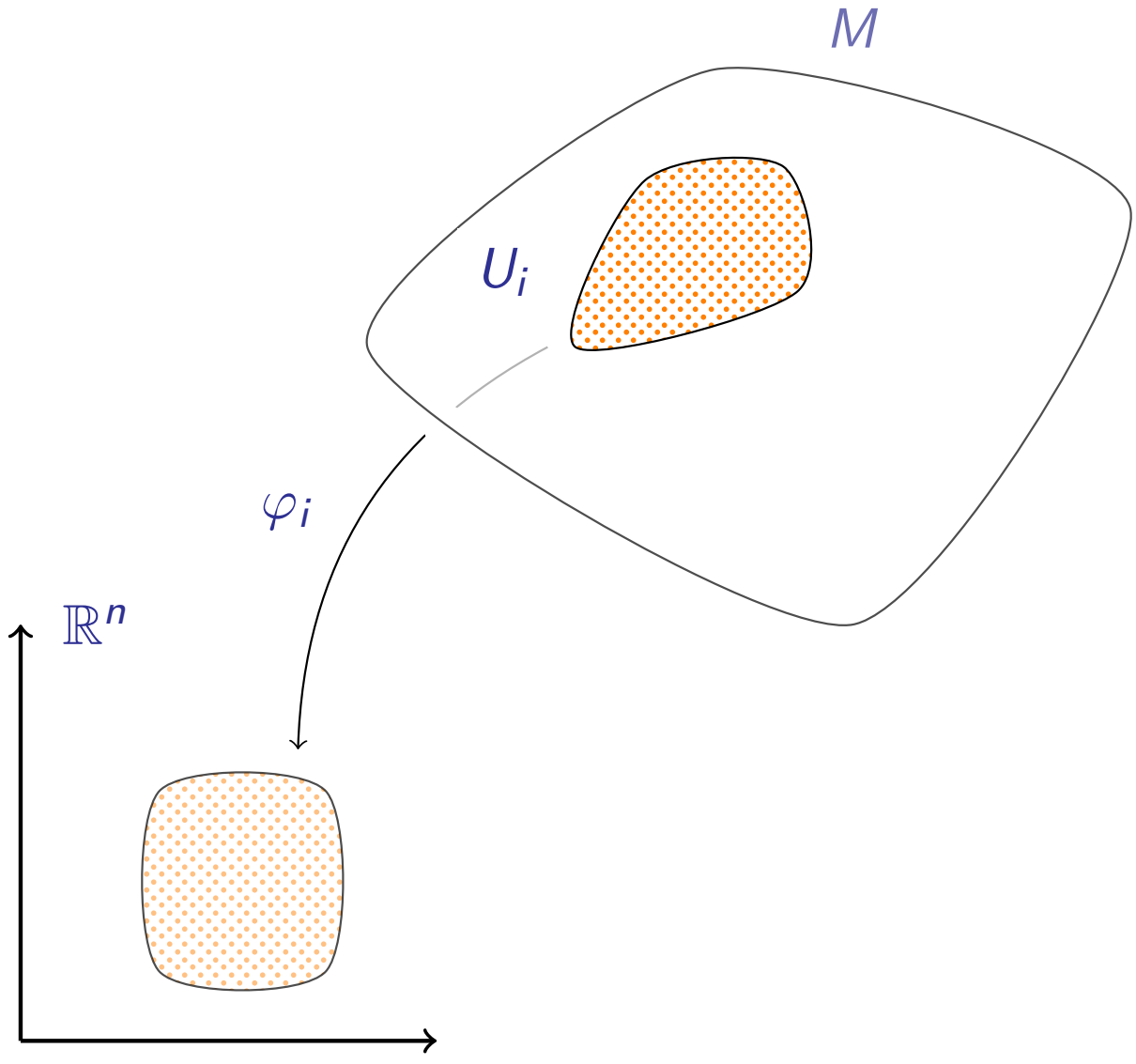
SEOUL
NATIONAL
UNIVERSITY

Intro (recap) $\rightarrow$ **Main ideas** $\rightarrow$ Recent directions

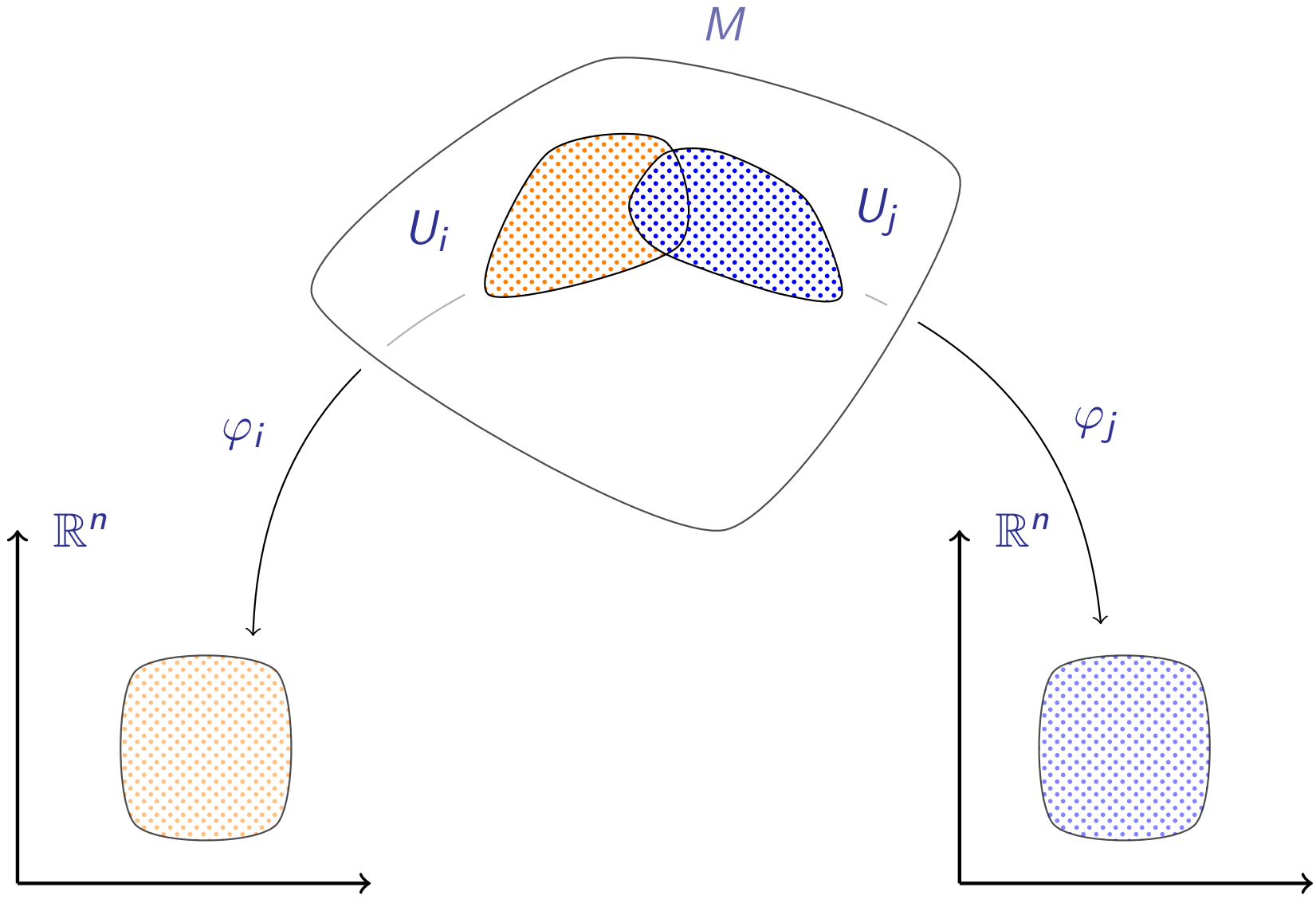
Intro (recap) $\rightarrow$ **Main ideas** $\rightarrow$ Recent directions

Intro: from manifolds to
(some) geometric structures.

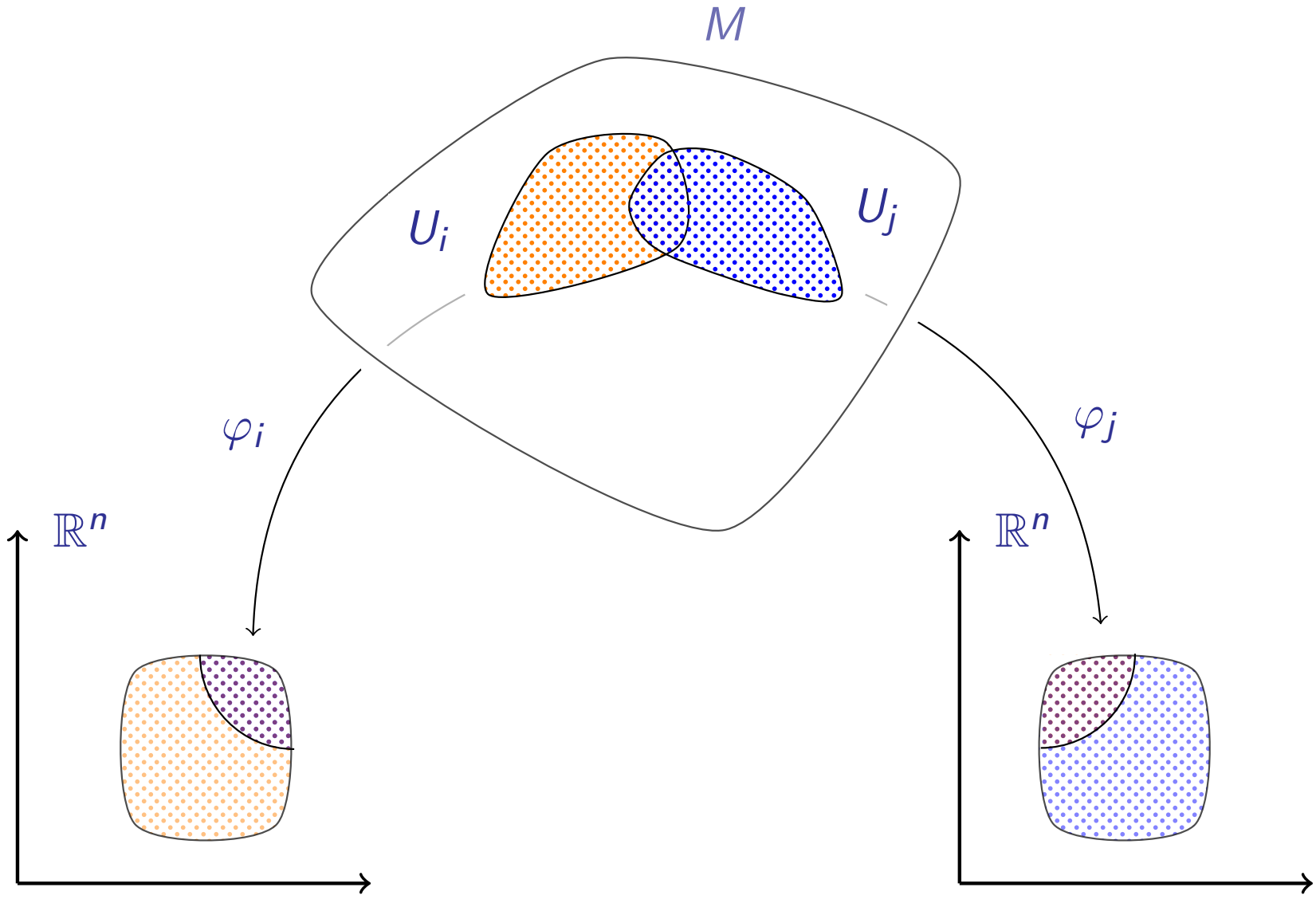
Recall smooth manifolds: on a set M



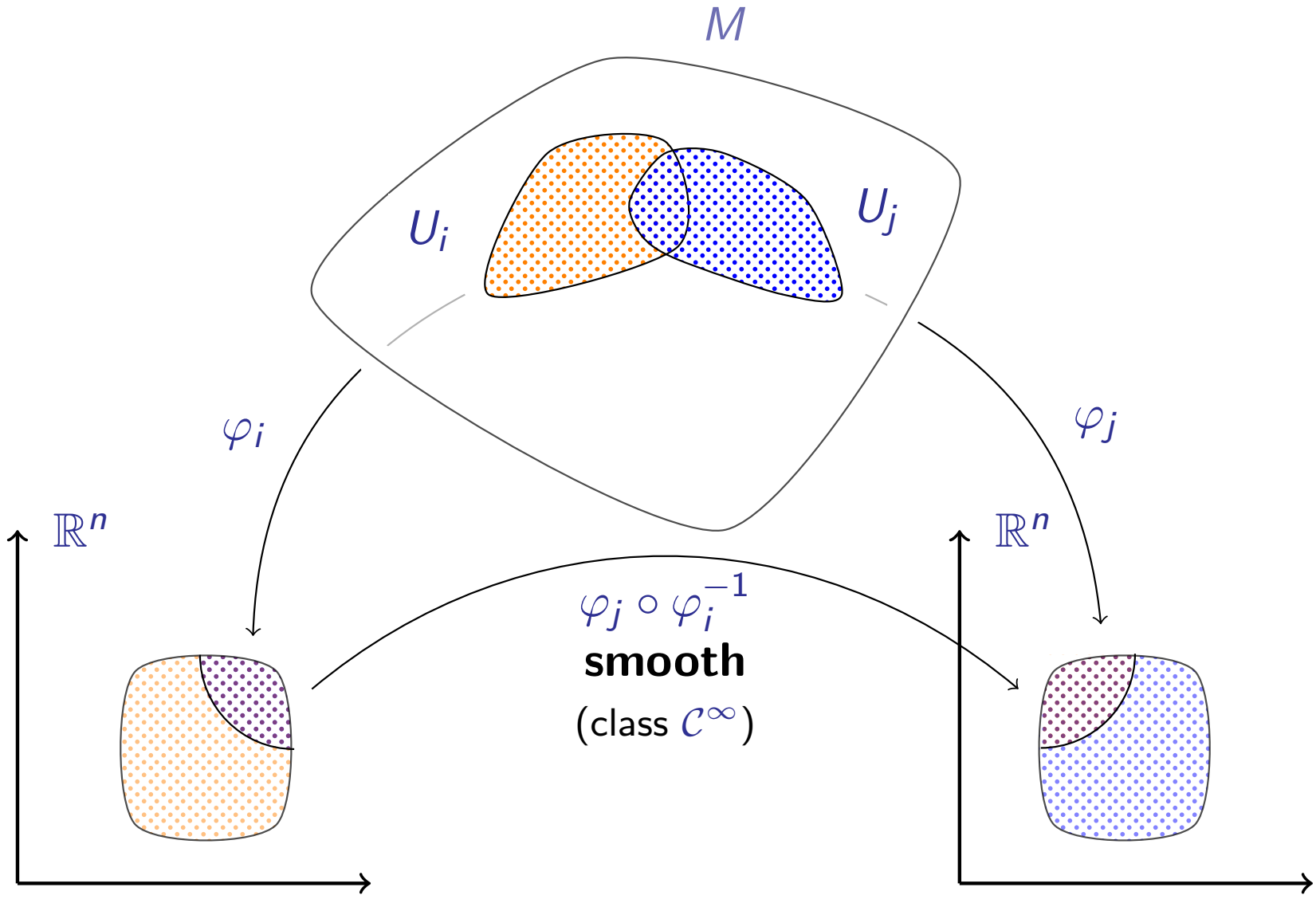
Recall smooth manifolds: on a set M



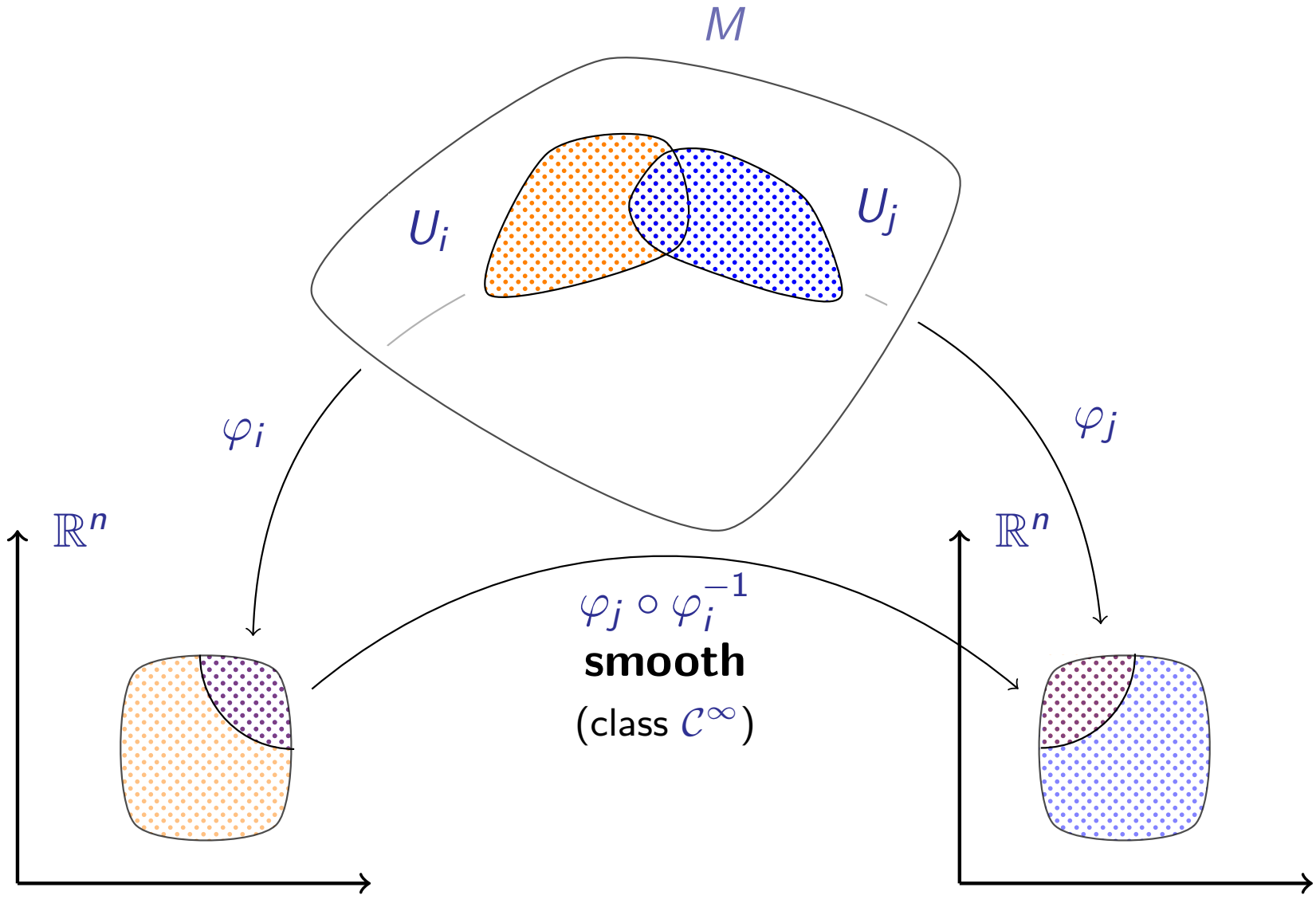
Recall smooth manifolds: on a set M



Recall smooth manifolds: on a set M



Recall smooth manifolds: on a set M

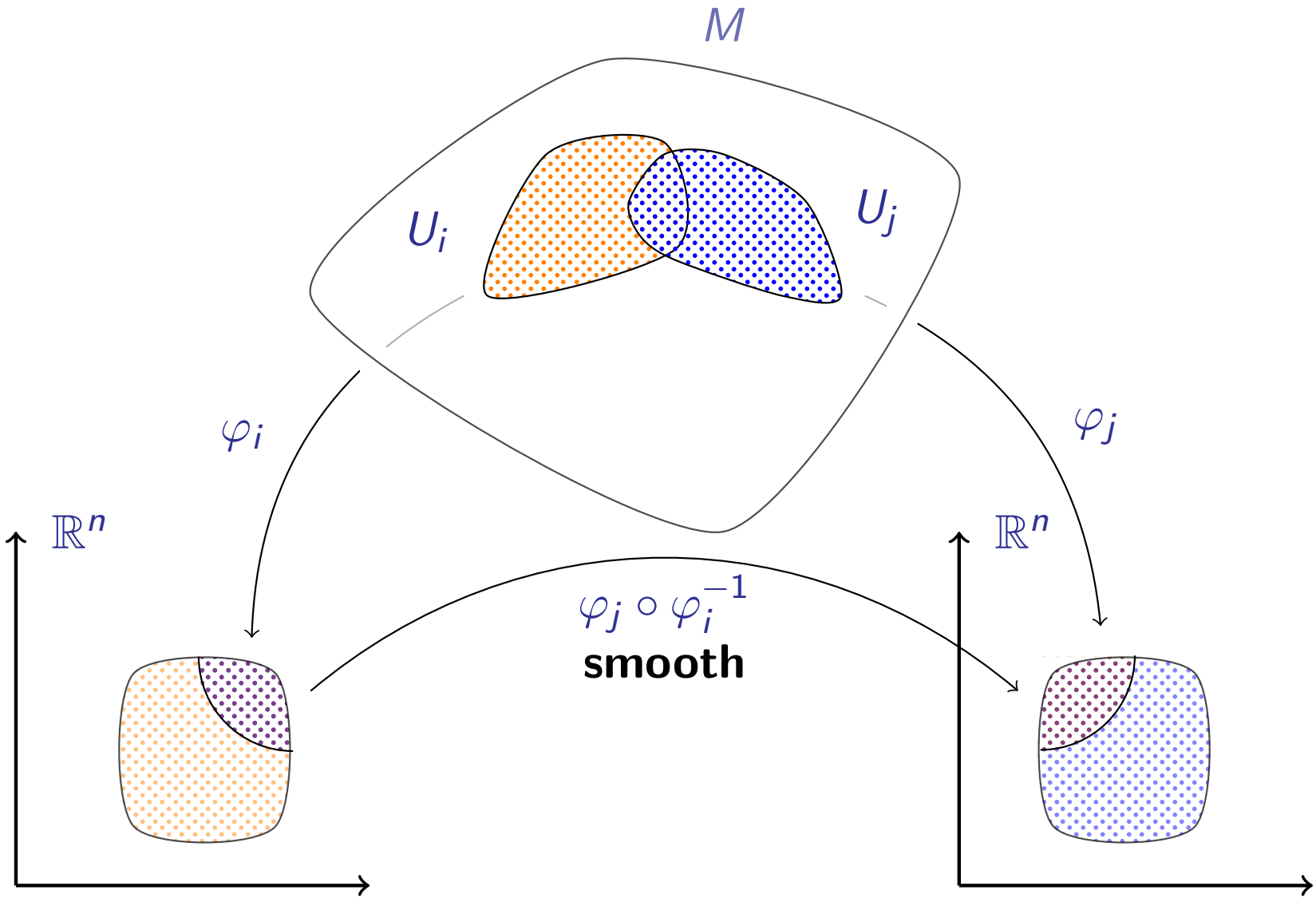


M gets a topology

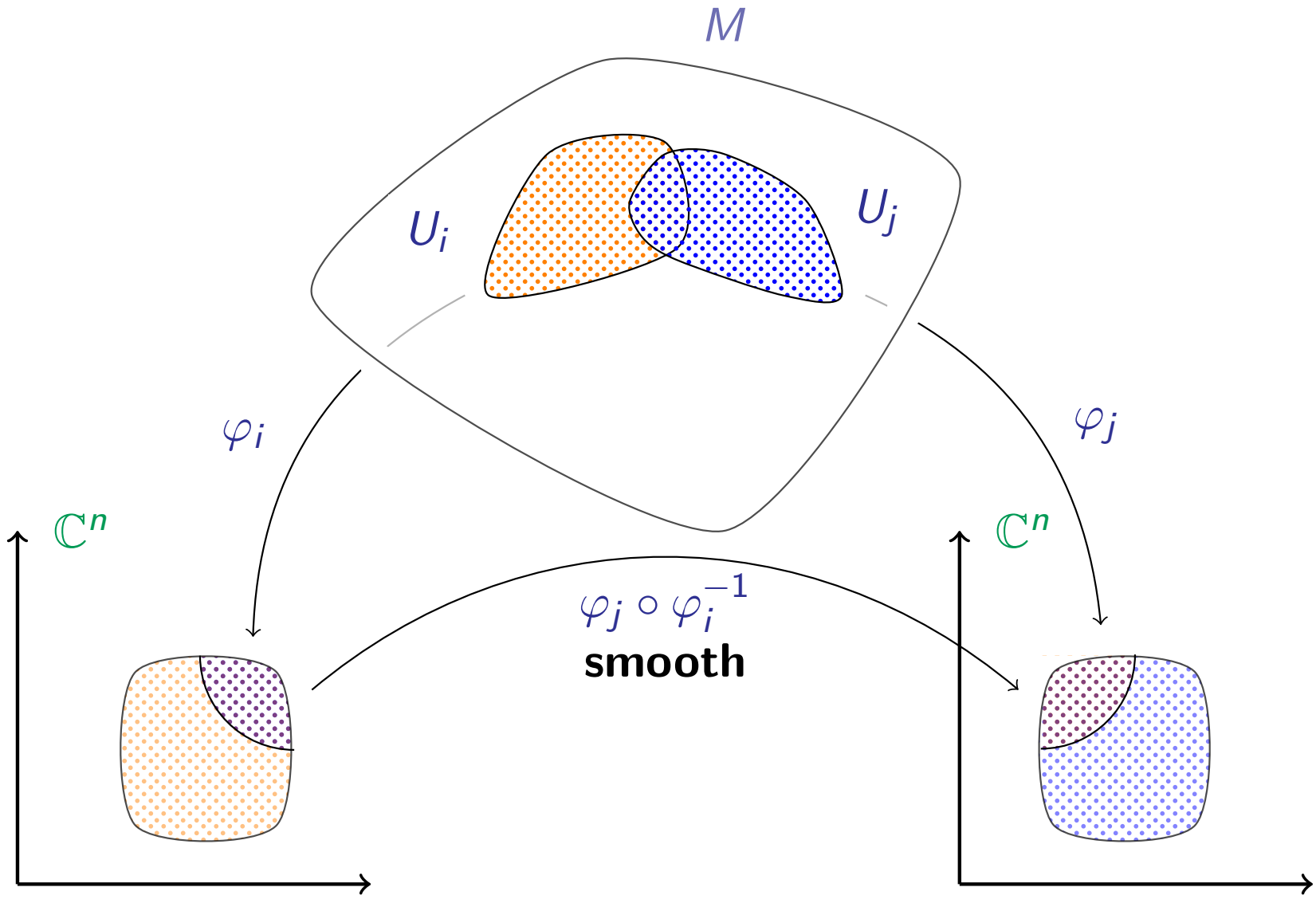
(usually required to be Hausdorff + countable basis)

Geometric structure: enrichment of local model and changes of chart

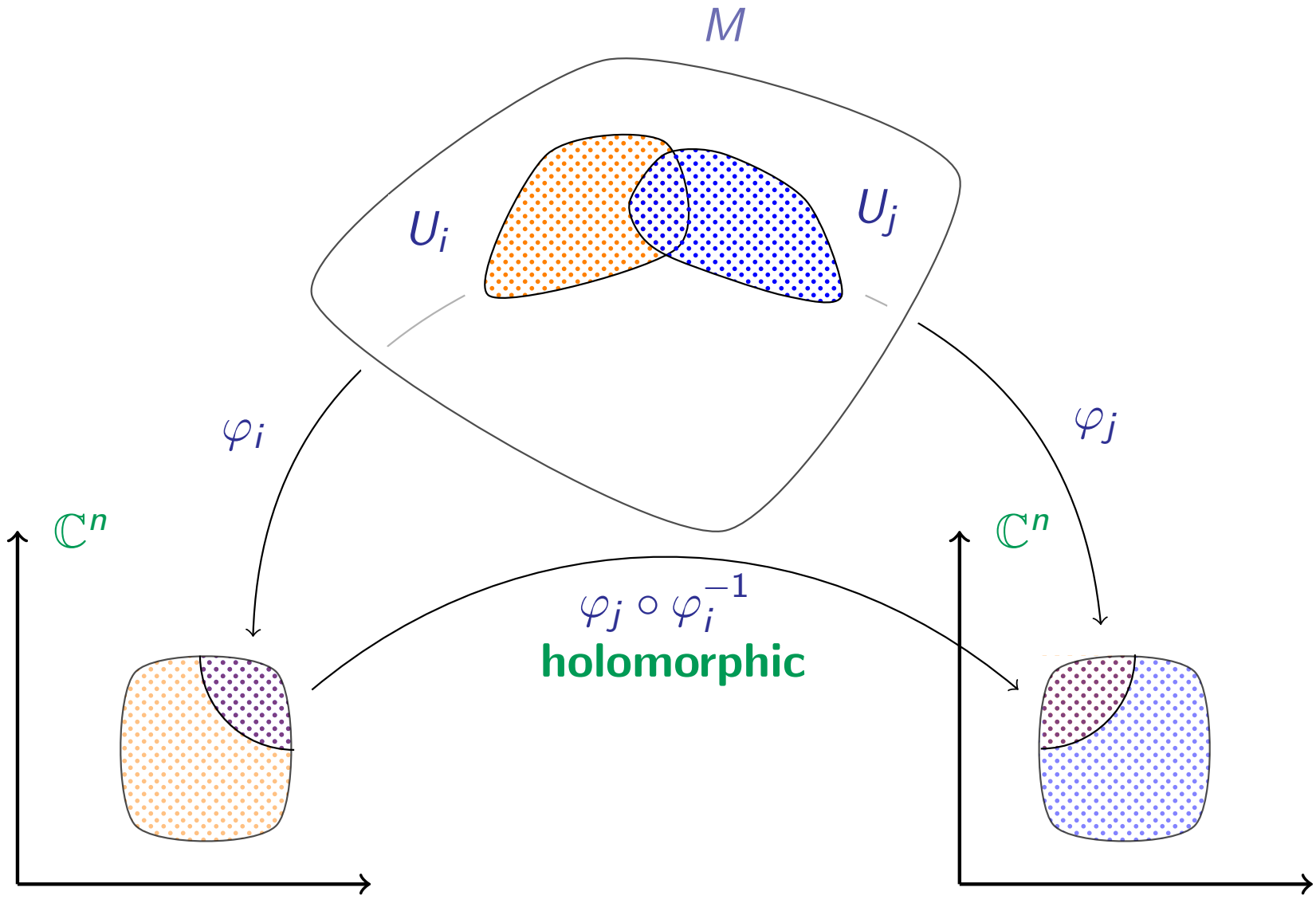
Geometric structure: enrichment of local model and changes of chart



Geometric structure: enrichment of local model and changes of chart

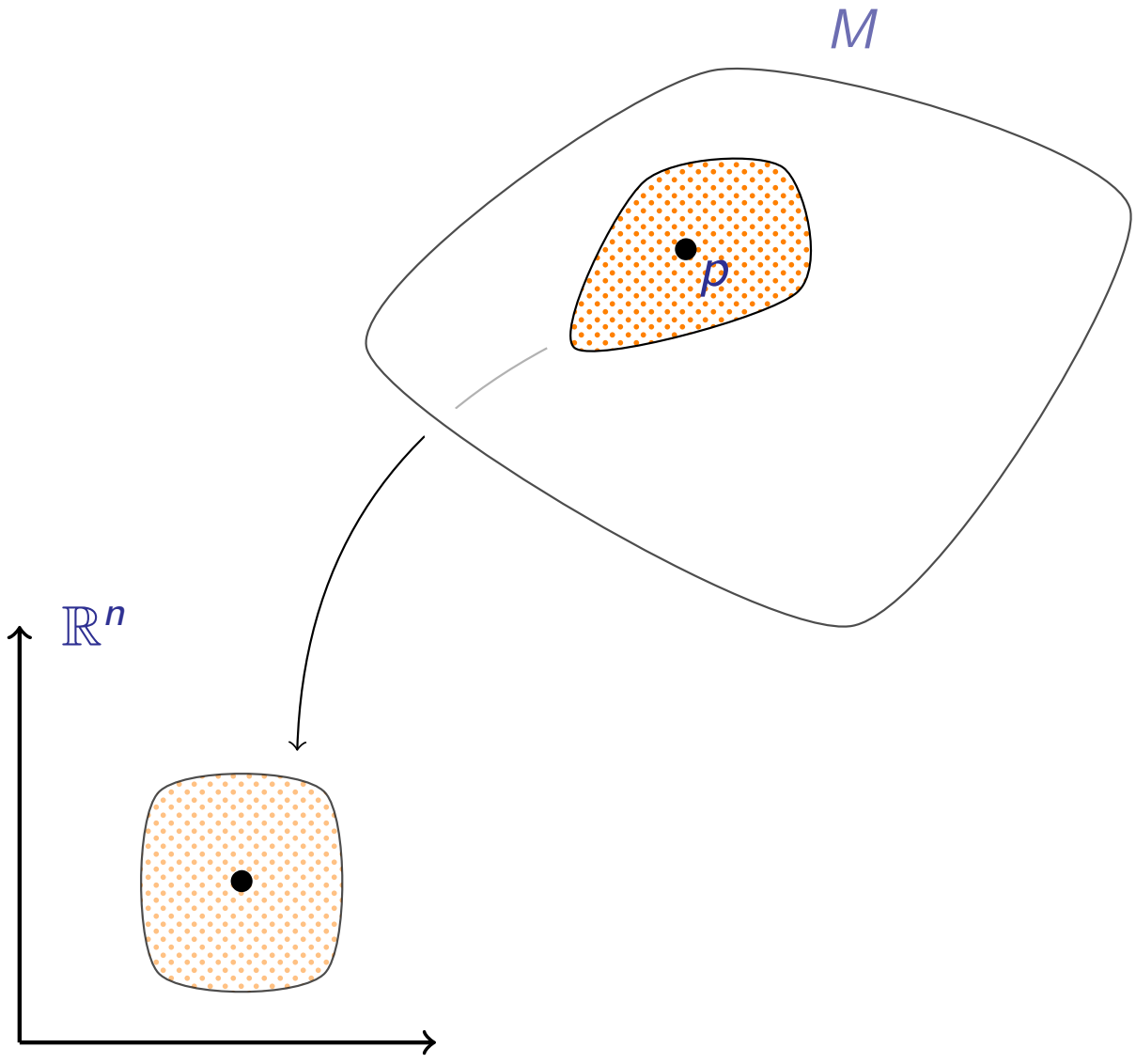


Geometric structure: enrichment of local model and changes of chart

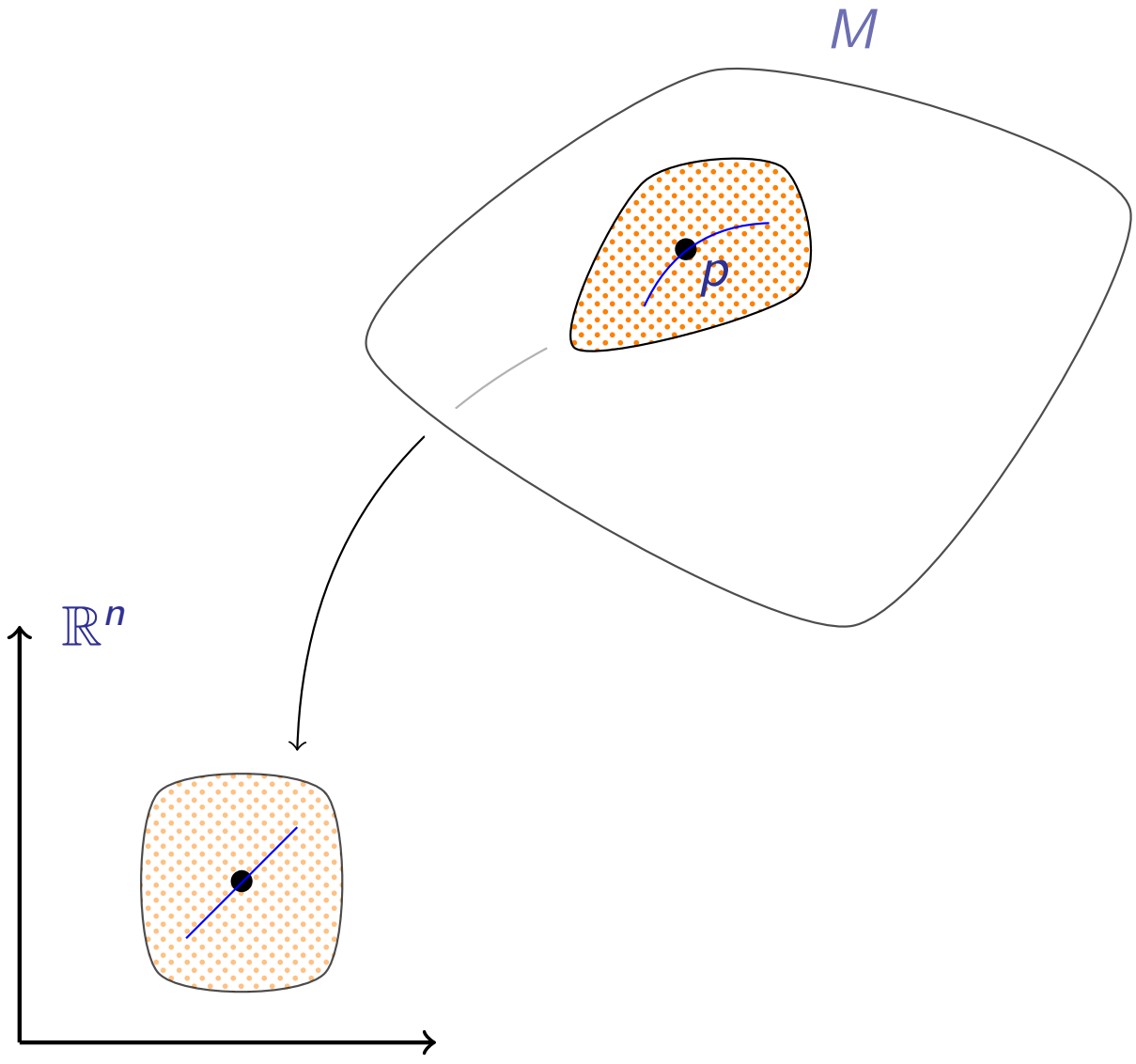


Charts are complicated... the tangent bundle comes to help.

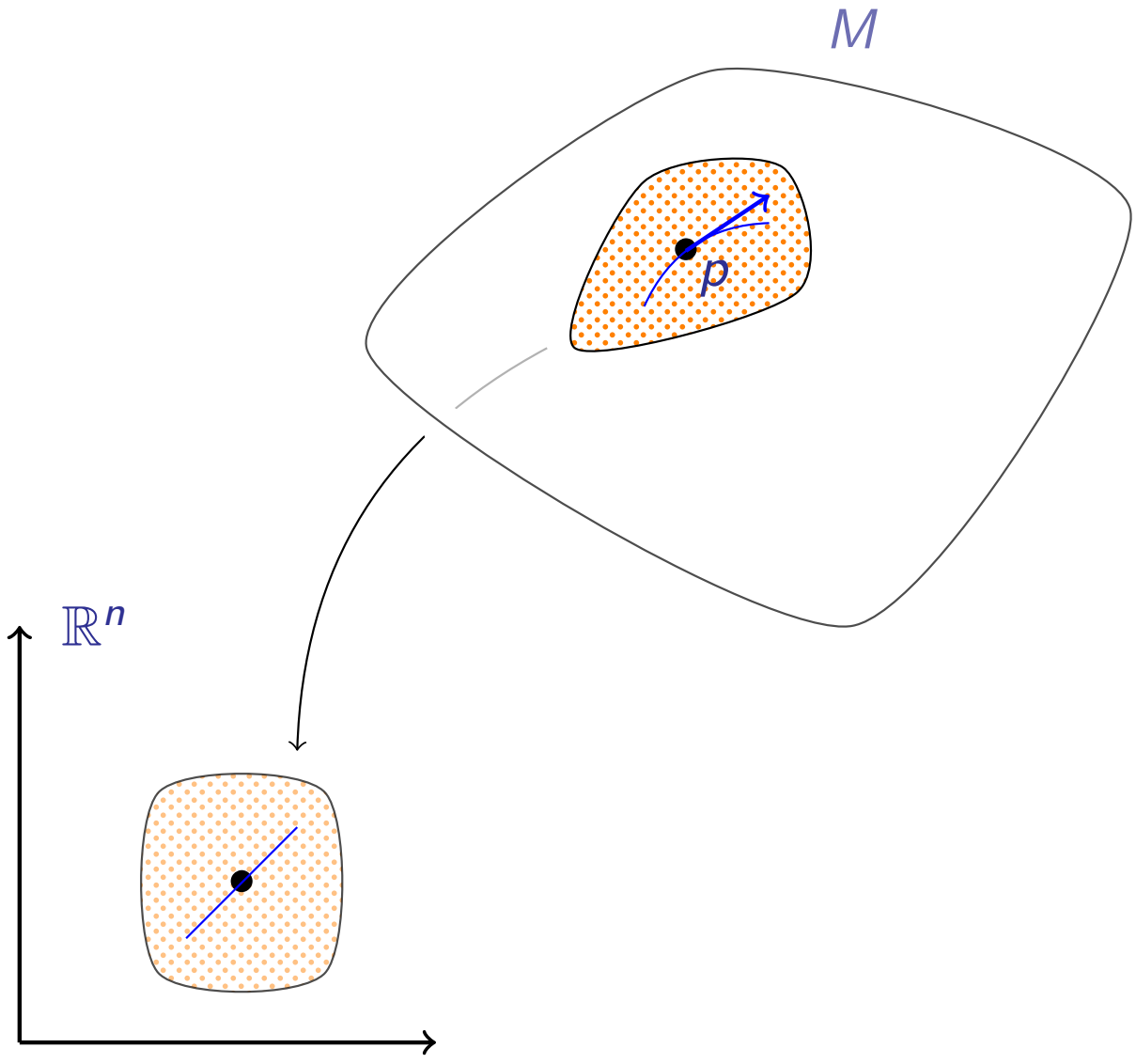
Charts are complicated... the tangent bundle comes to help.



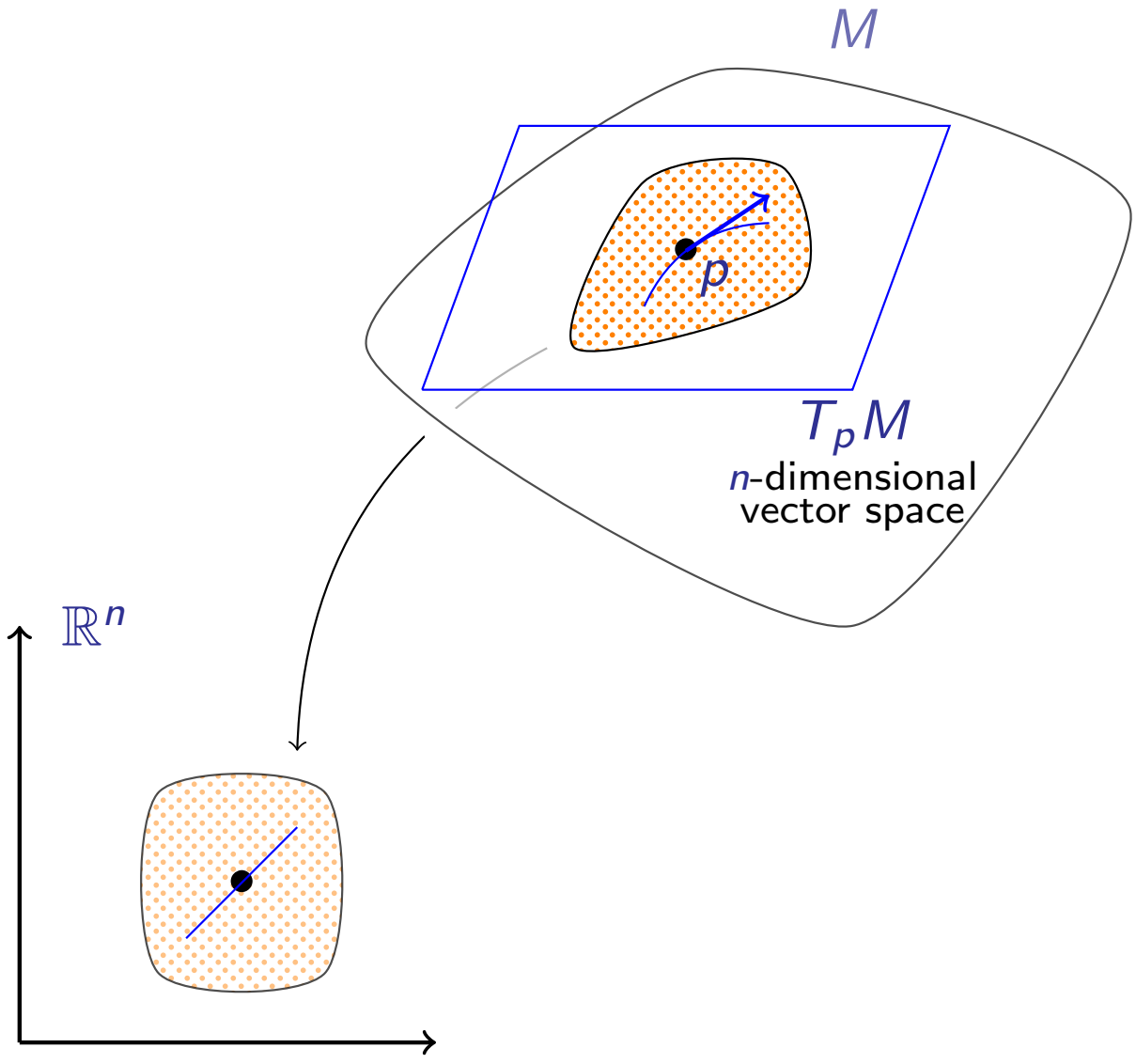
Charts are complicated... the tangent bundle comes to help.



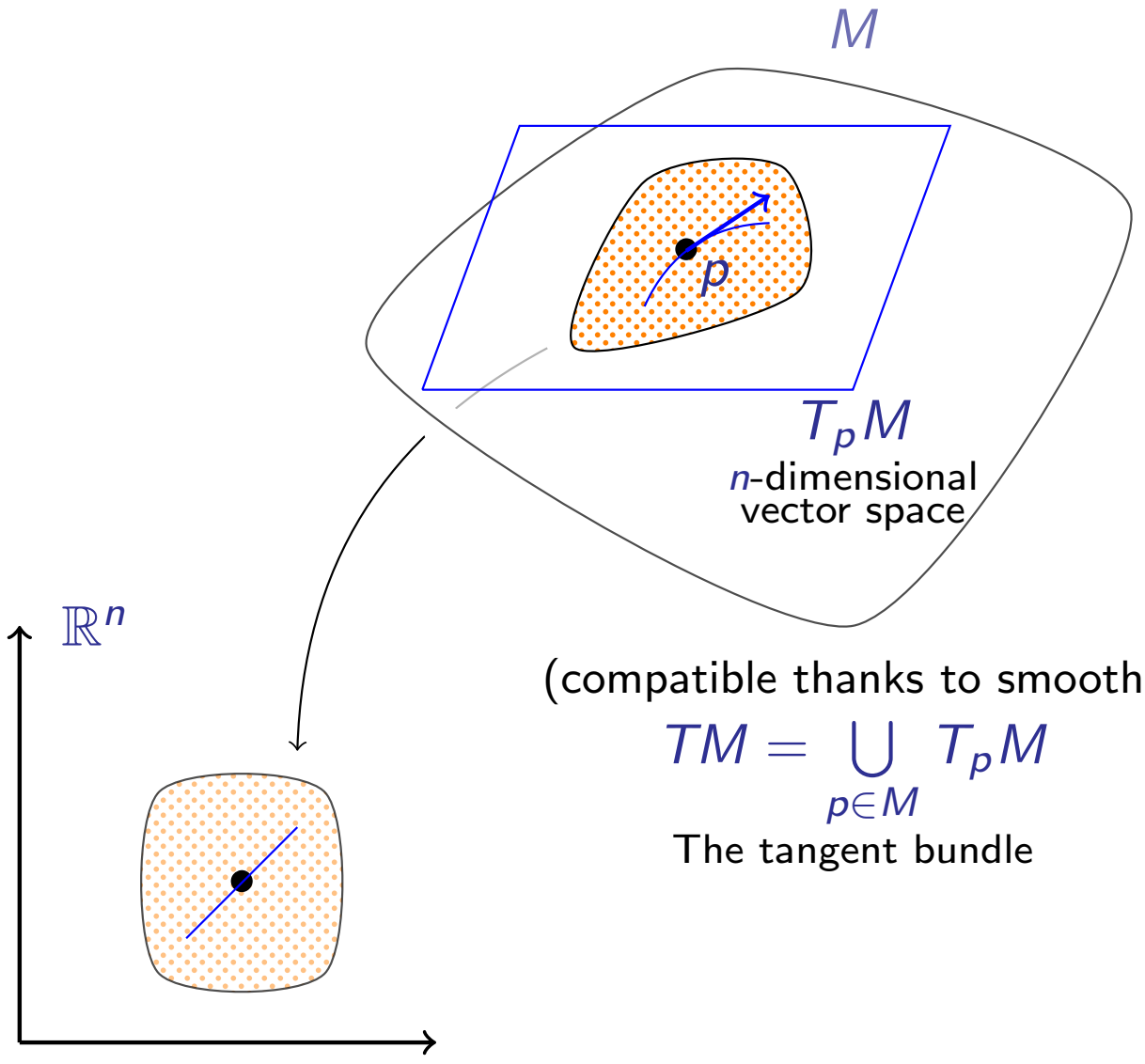
Charts are complicated... the tangent bundle comes to help.



Charts are complicated... the tangent bundle comes to help.



Charts are complicated... the tangent bundle comes to help.

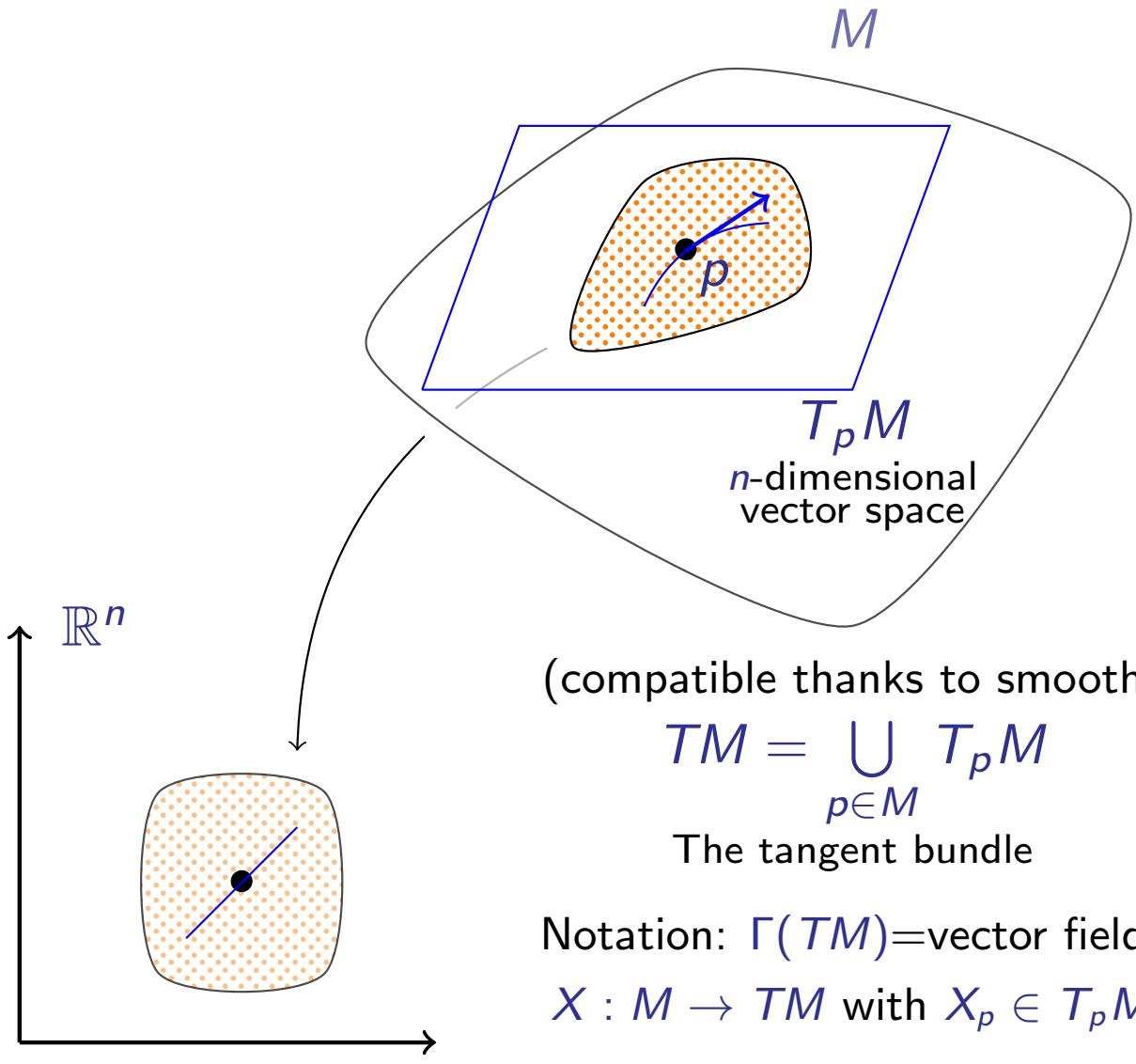


(compatible thanks to smooth)

$$TM = \bigcup_{p \in M} T_pM$$

The tangent bundle

Charts are complicated... the tangent bundle comes to help.



(compatible thanks to smooth)

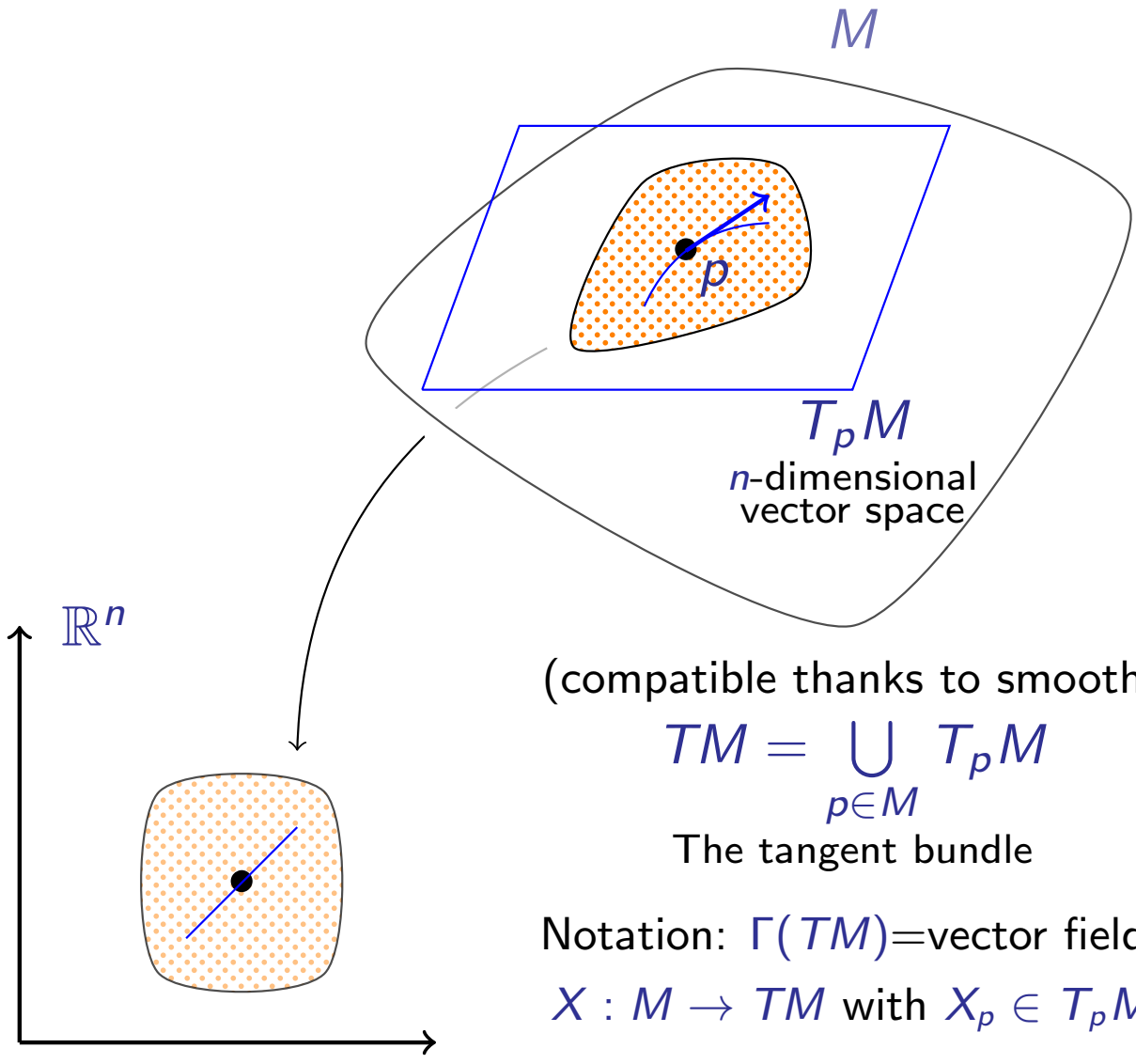
$$TM = \bigcup_{p \in M} T_pM$$

The tangent bundle

Notation: $\Gamma(TM)$=vector fields

$X : M \to TM$ with $X_p \in T_pM$

Charts are complicated... the tangent bundle comes to help.



(compatible thanks to smooth)

$$TM = \bigcup_{p \in M} T_pM$$

The tangent bundle

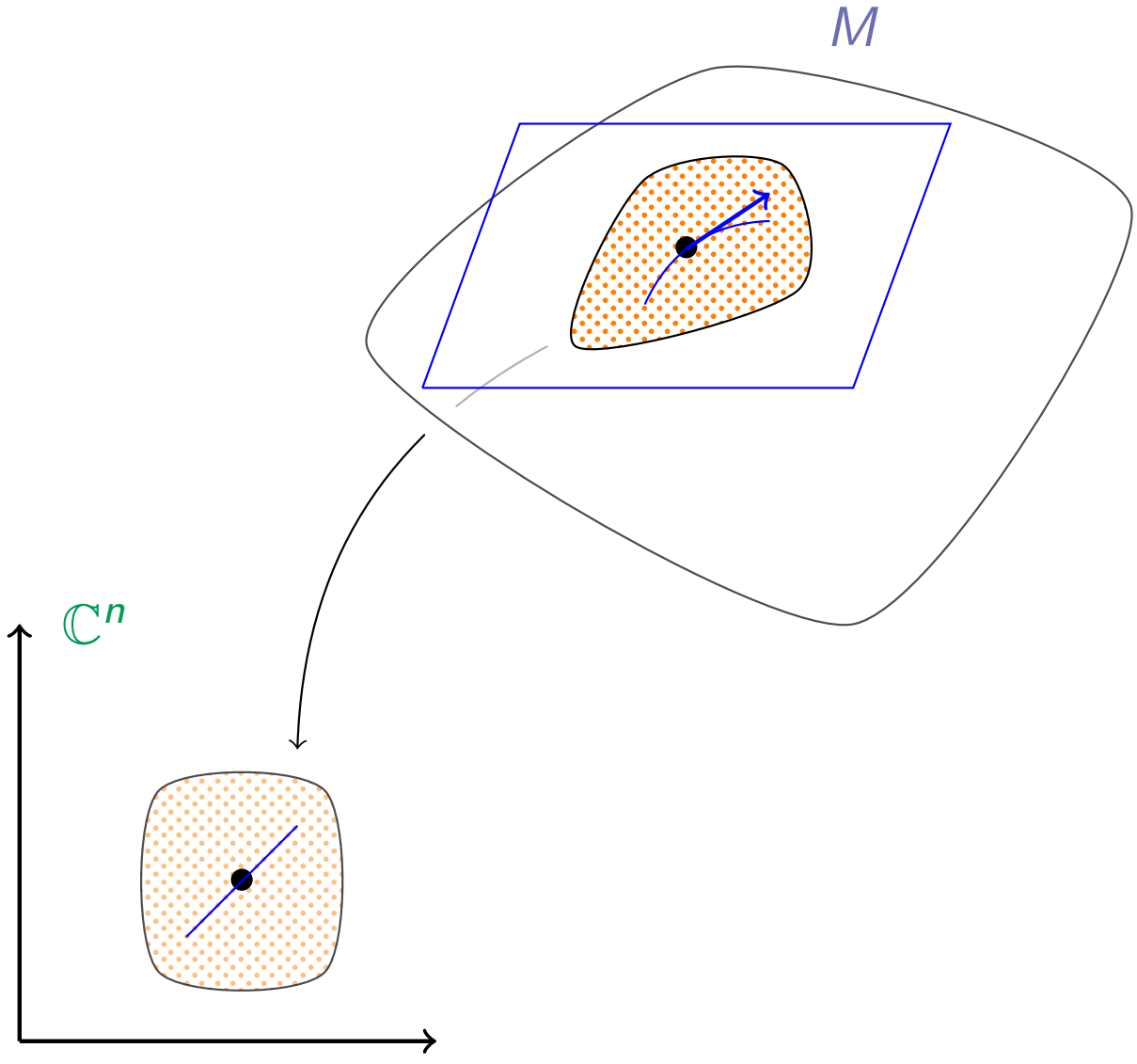
Notation: $\Gamma(TM)$=vector fields

$X : M \to TM$ with $X_p \in T_pM$

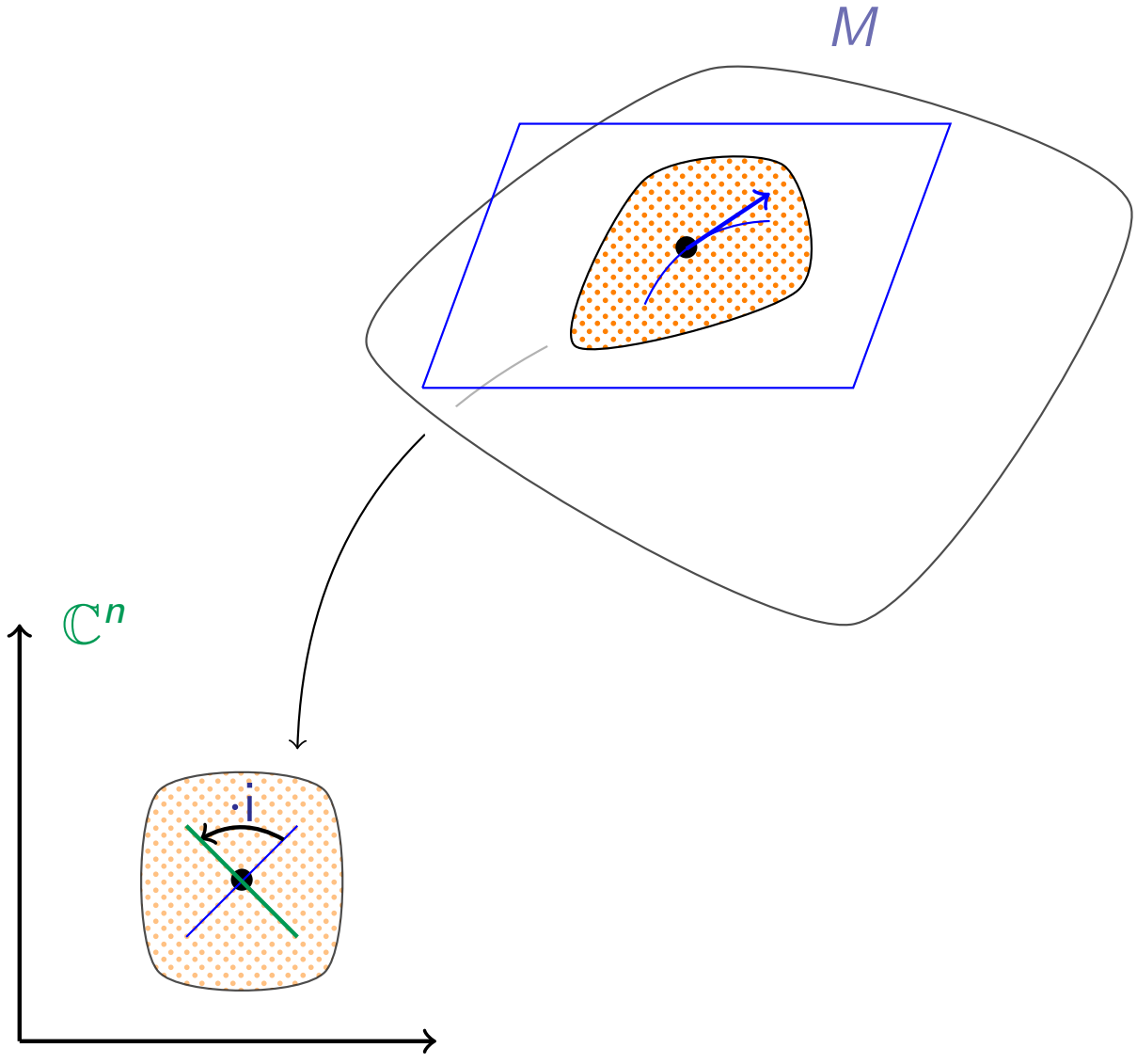
TM is more complicated than M, how does it help?

We can push the geometric structure to the tangent bundle:

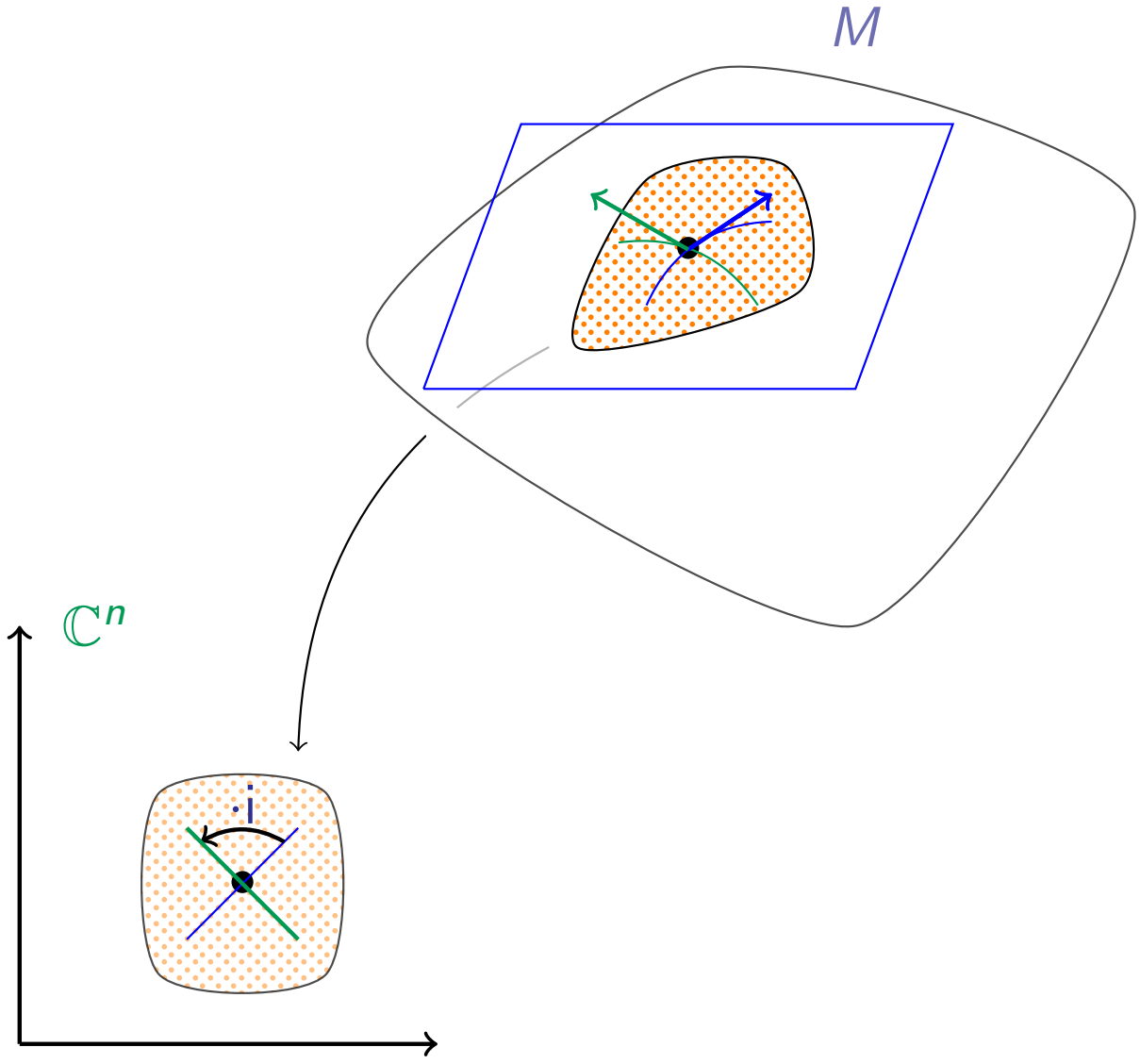
We can push the geometric structure to the tangent bundle:



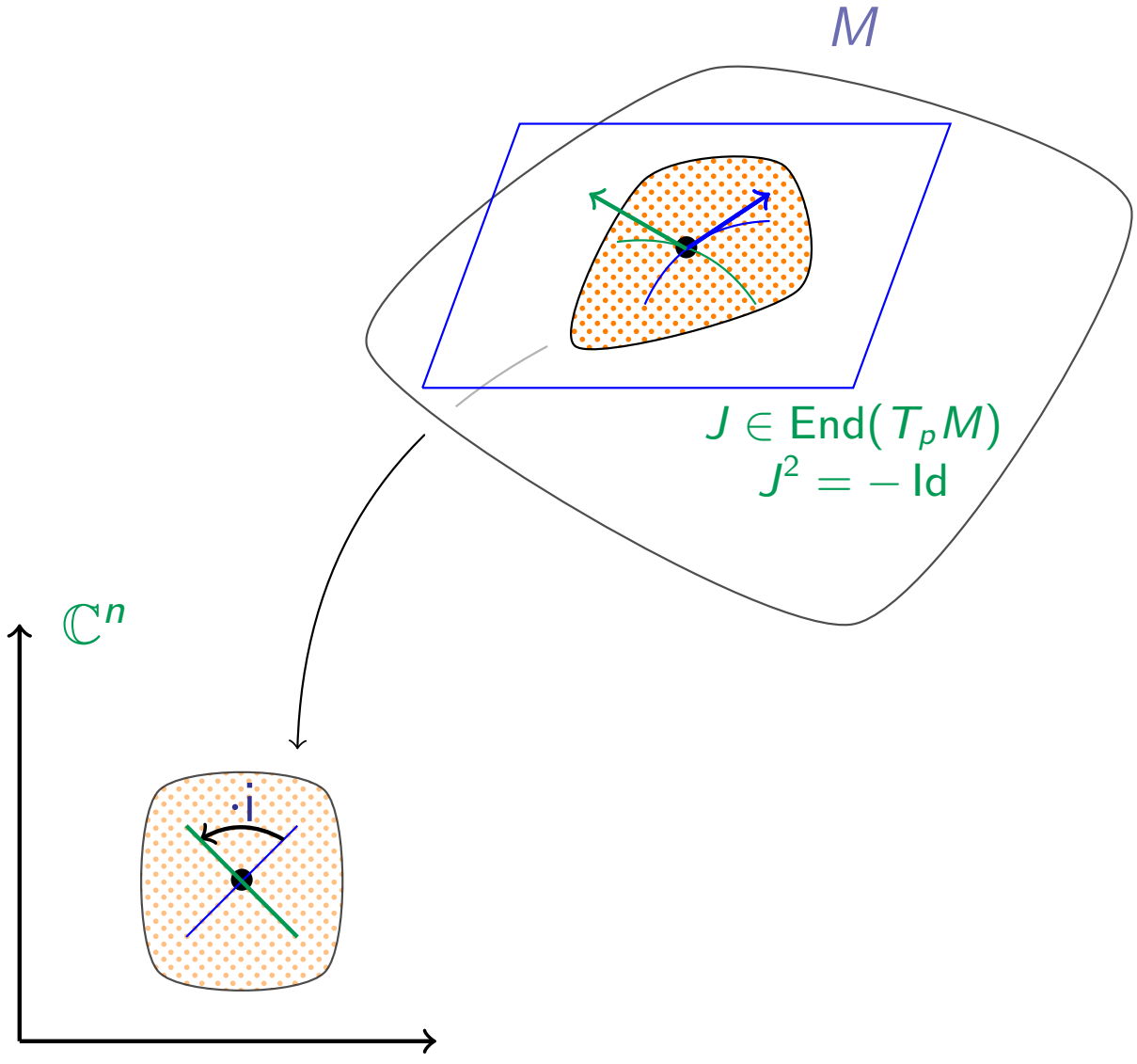
We can push the geometric structure to the tangent bundle:



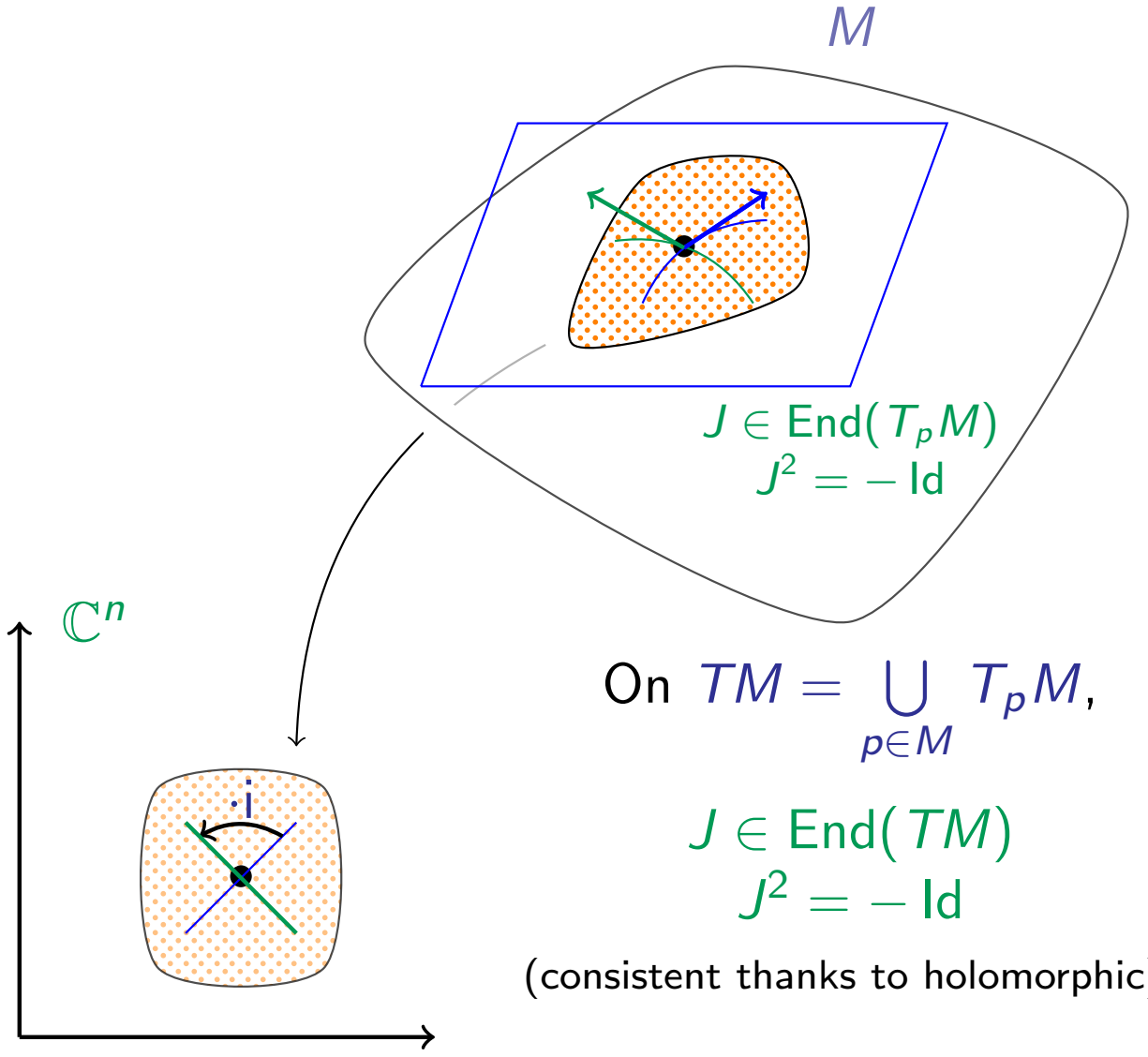
We can push the geometric structure to the tangent bundle:



We can push the geometric structure to the tangent bundle:



We can push the geometric structure to the tangent bundle:



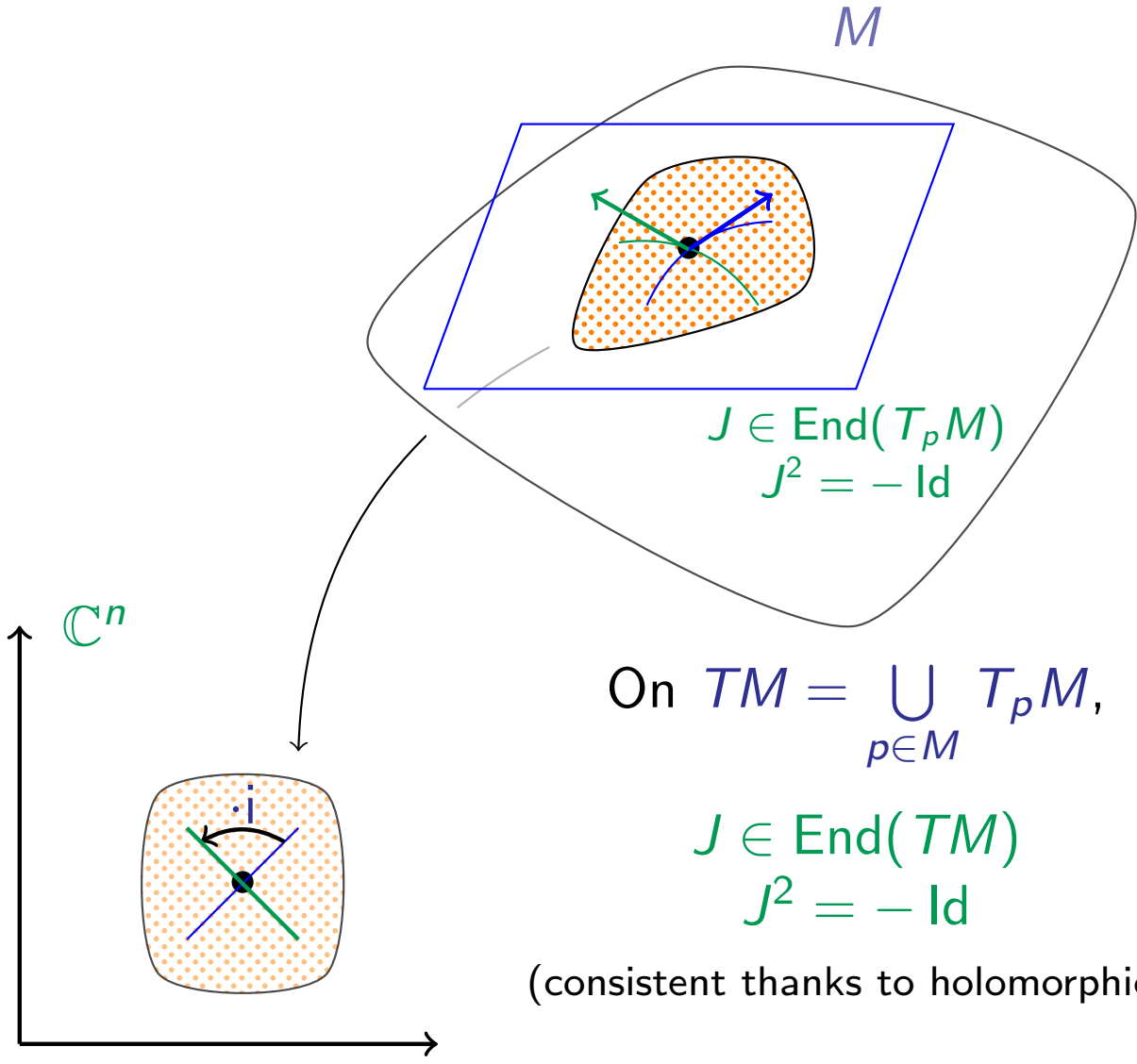
On $TM = \bigcup_{p \in M} T_pM$,

$J \in \mathrm{End}(TM)$

$J^2 = -\mathrm{Id}$

(consistent thanks to holomorphic)

We can push the geometric structure to the tangent bundle:



On $TM = \bigcup_{p \in M} T_pM$,

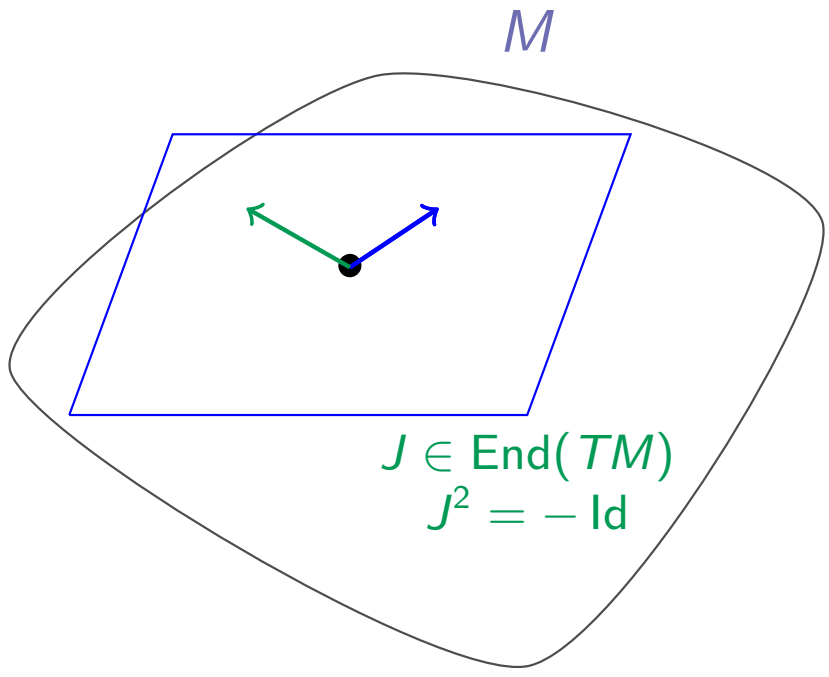
$J \in \mathrm{End}(TM)$

$J^2 = -\mathrm{Id}$

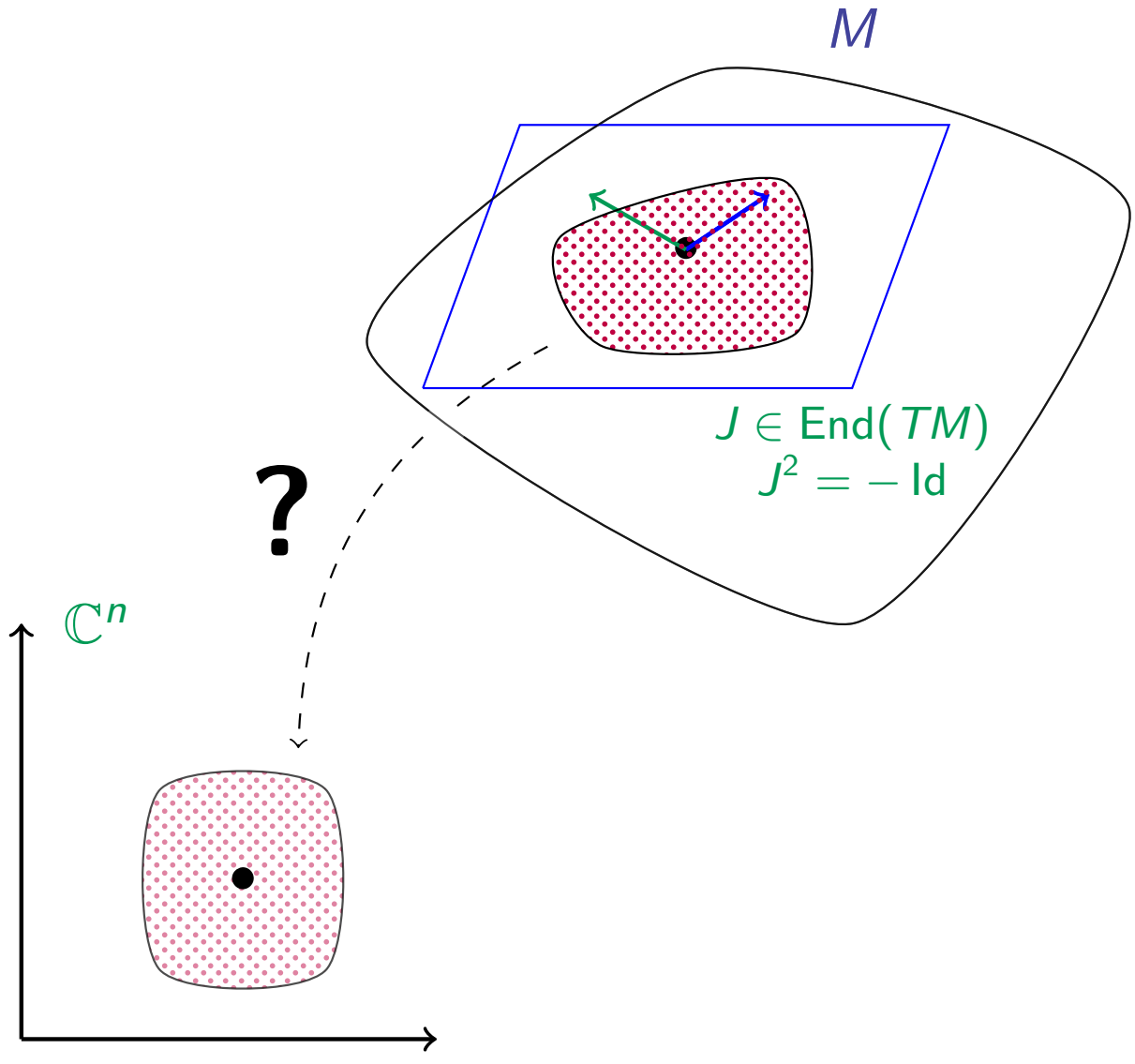
(consistent thanks to holomorphic)

...but we are losing information.

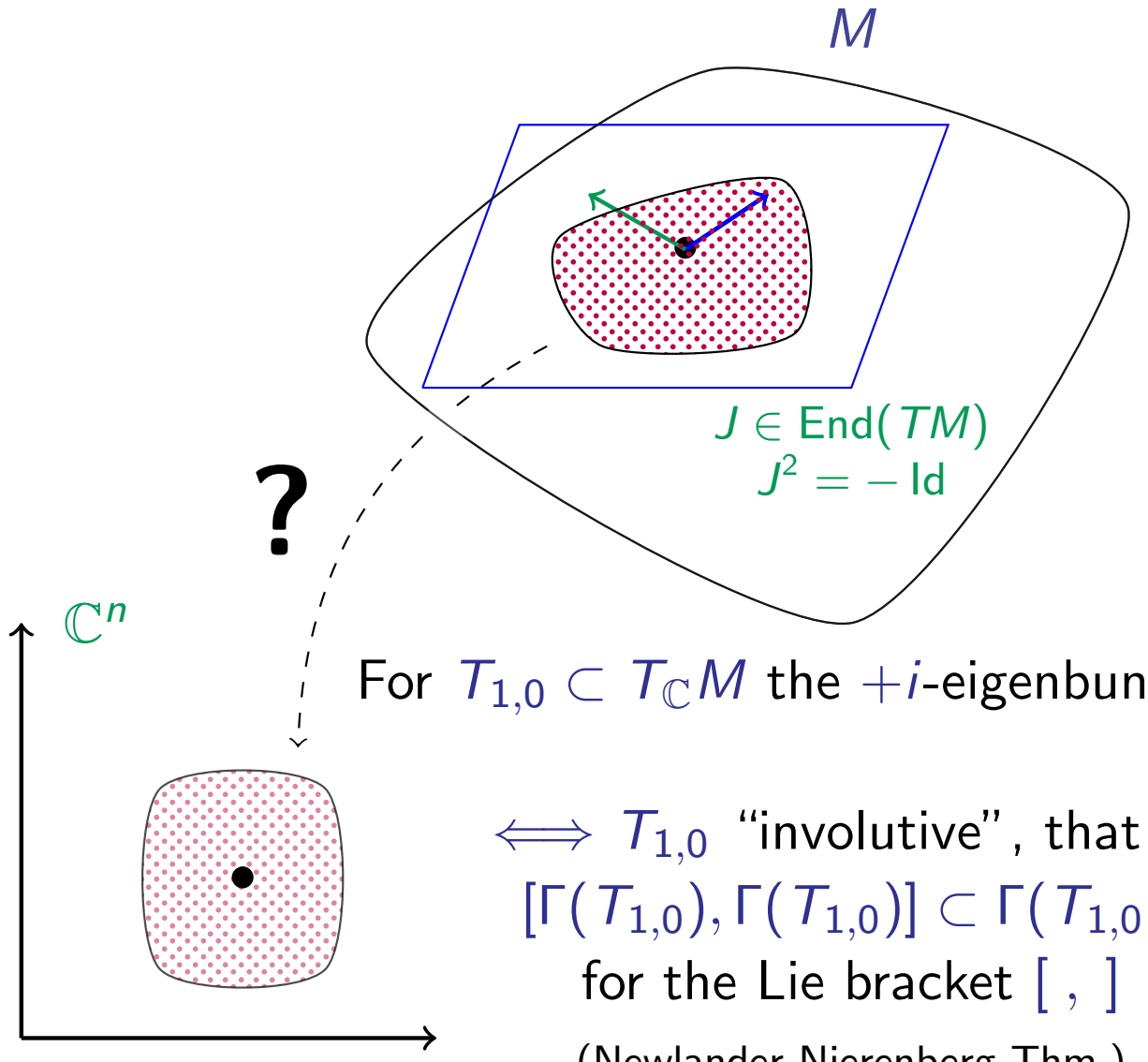
Starting with J (almost complex manifold):



Starting with J (almost complex manifold):



Starting with J (almost complex manifold):



For $T_{1,0} \subset T_{\mathbb{C}}M$ the $+i$-eigenbundle of J,

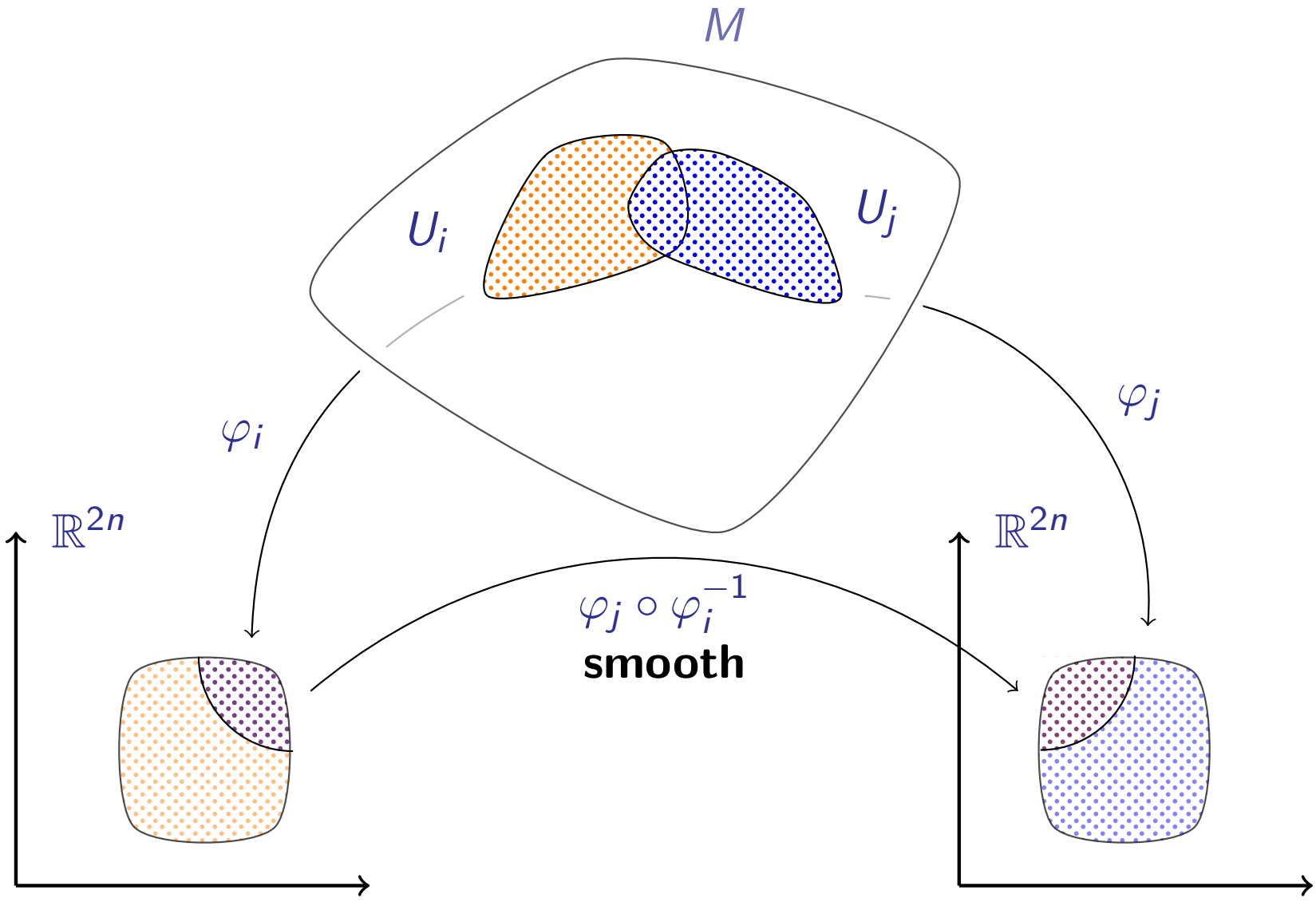
$\Longleftrightarrow$ $T_{1,0}$ "involutive", that is,

$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(T_{1,0})$,

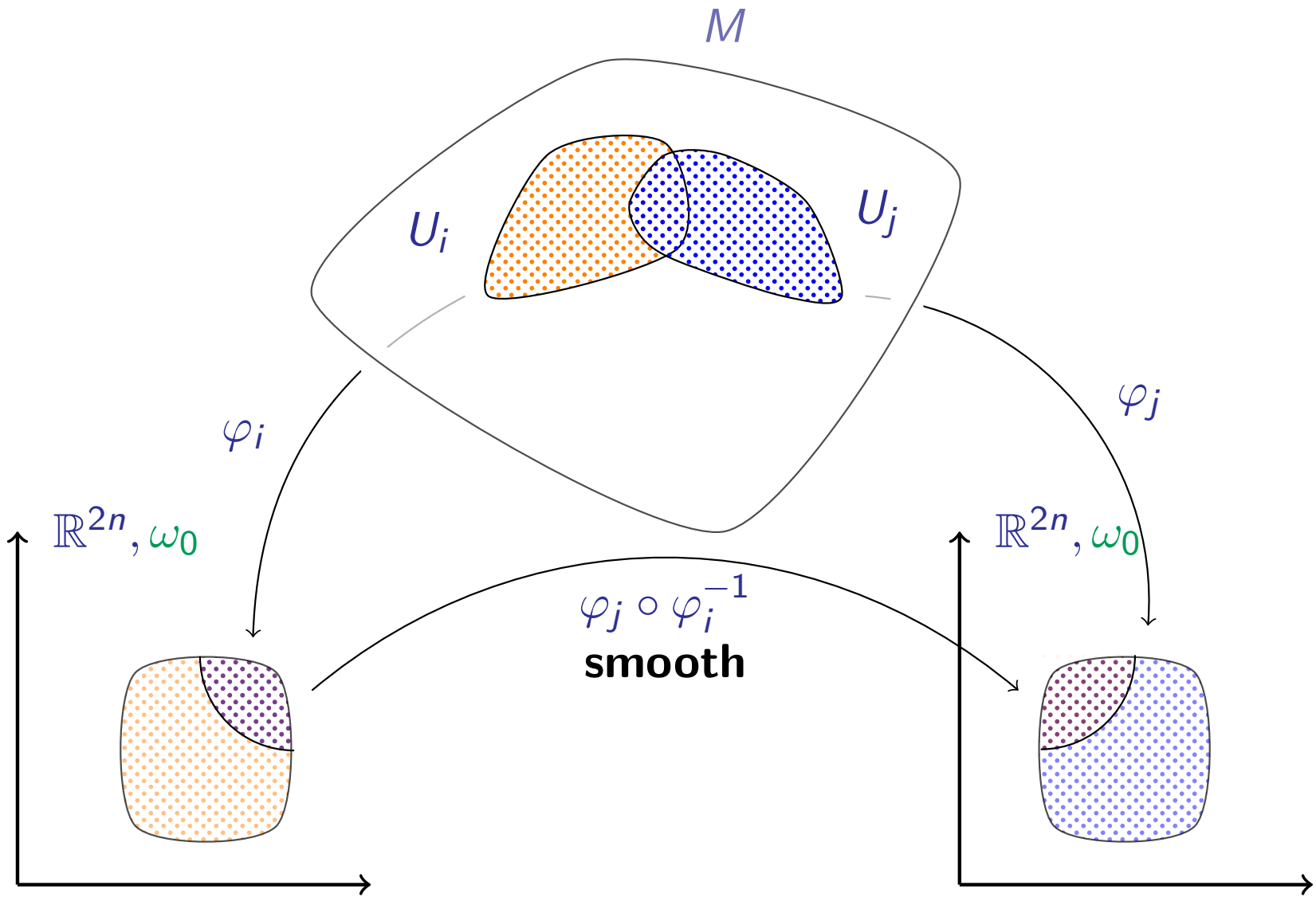
for the Lie bracket $[\ ,\ ]$

(Newlander-Nierenberg Thm.)

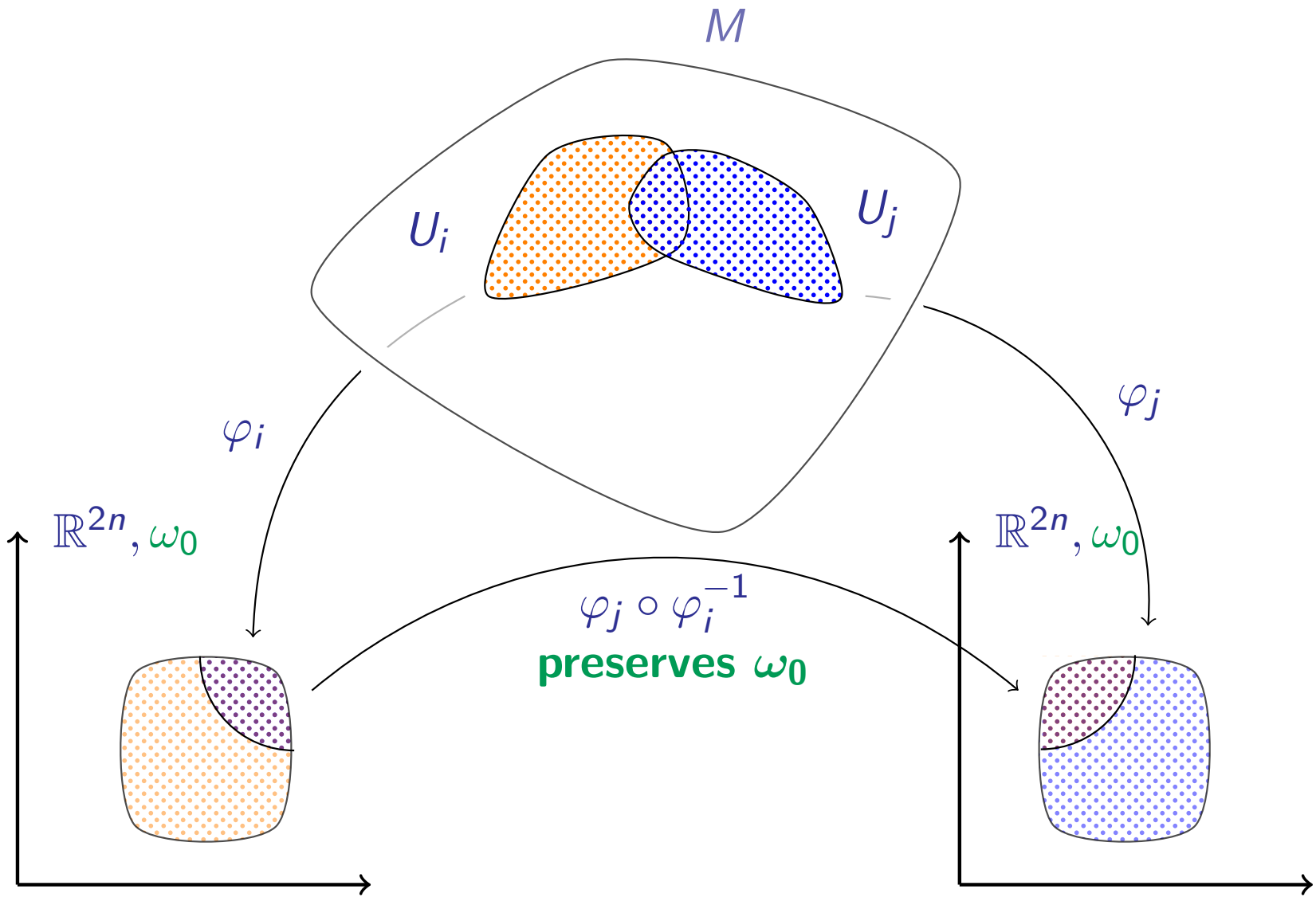
Another example:



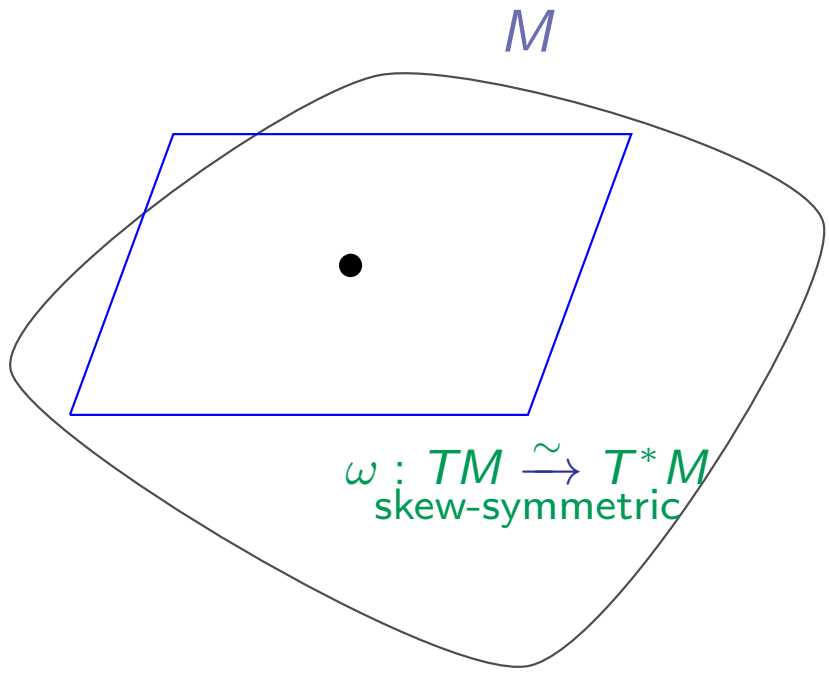
Another example: for $\omega_0 = dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n}$,



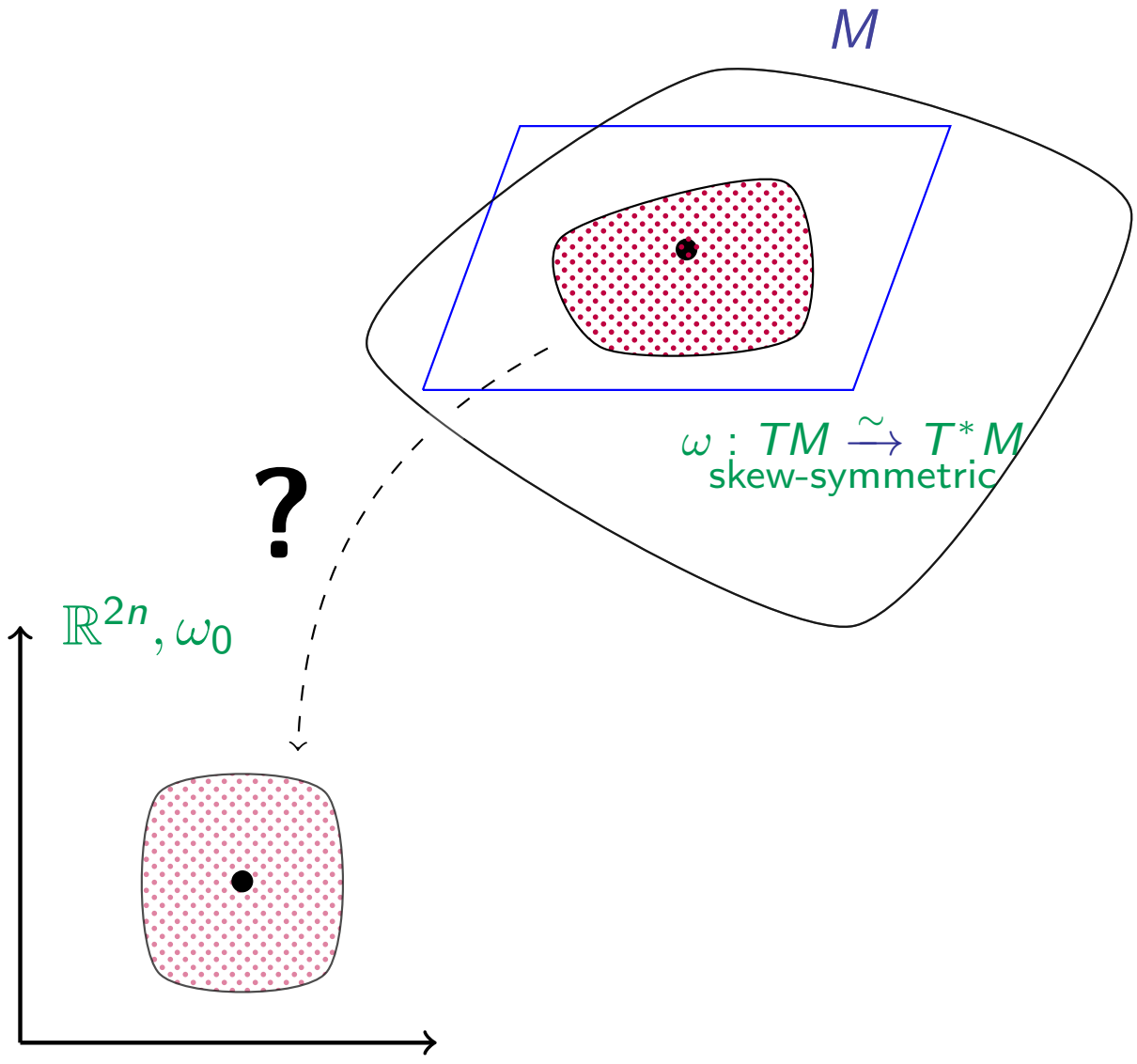
Another example: for $\omega_0 = dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n}$,



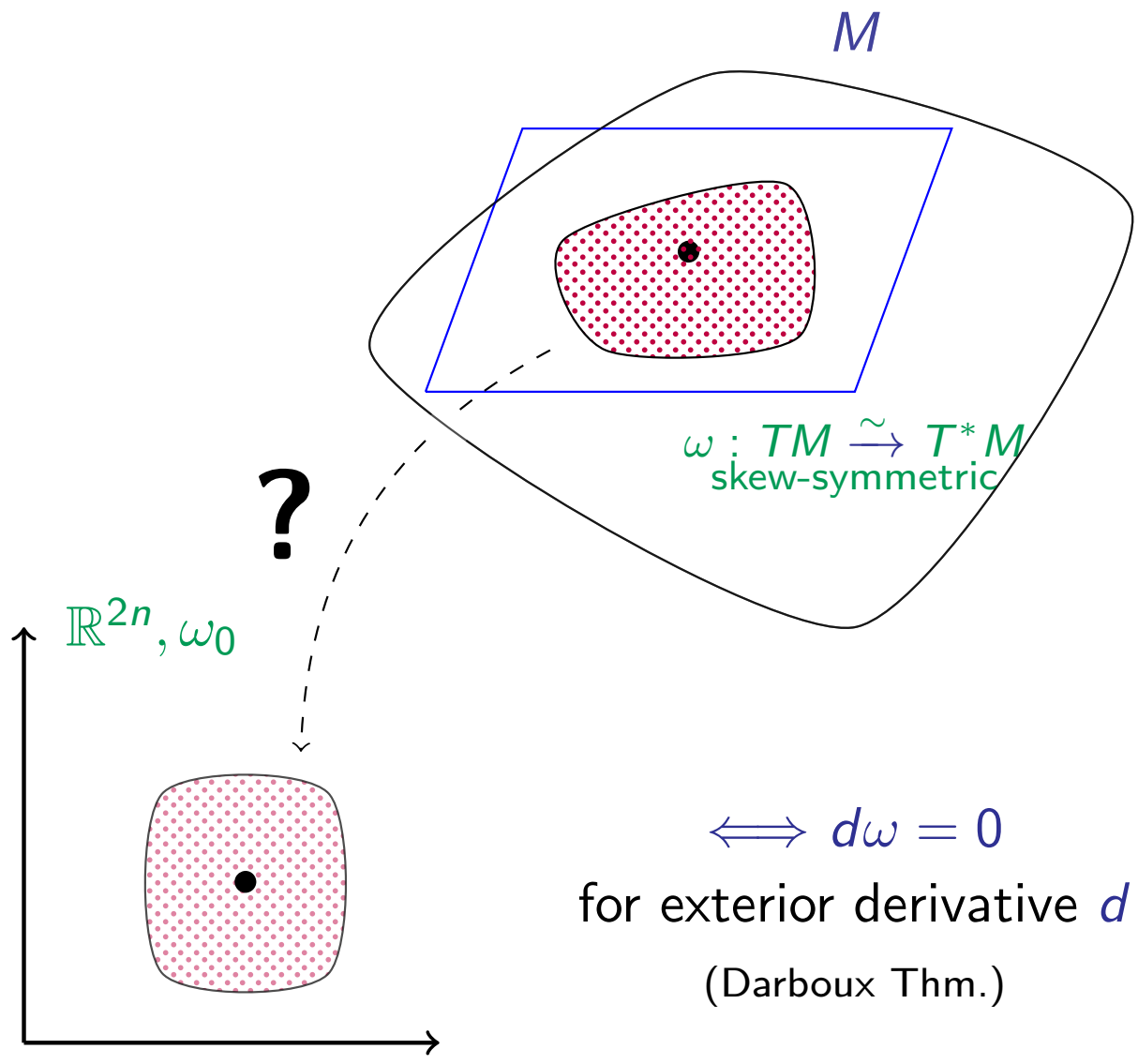
On each tangent space we get a non-degenerate 2-form, globally:



On each tangent space we get a non-degenerate 2-form, globally:



On each tangent space we get a non-degenerate 2-form, globally:



$\iff d\omega = 0$

for exterior derivative d

(Darboux Thm.)

Some observations:

Some observations:

- Geometric structures now have a 'linear part' (in terms of TM) plus a 'geometric' or integrability condition.

Some observations:

• Geometric structures now have a 'linear part' (in terms of TM) plus a 'geometric' or integrability condition.

• By removing the integrability condition: almost complex structures and almost symplectic structures (non-degenerate 2-forms).

Some observations:

• Geometric structures now have a 'linear part' (in terms of TM) plus a 'geometric' or integrability condition.

• By removing the integrability condition: almost complex structures and almost symplectic structures (non-degenerate 2-forms).

• A manifold admits an almost complex structure if and only if it admits an almost symplectic structure.

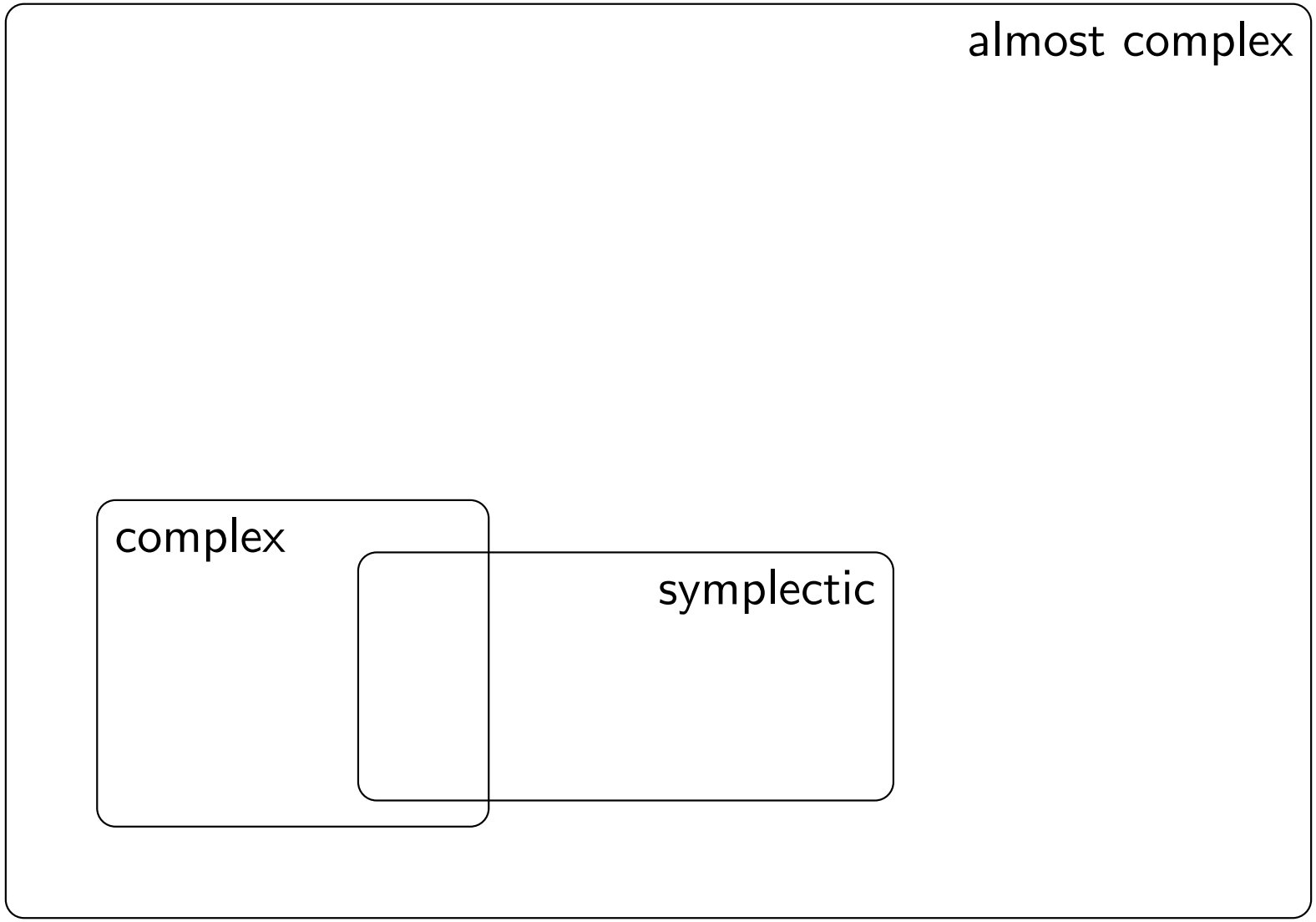
Some observations:

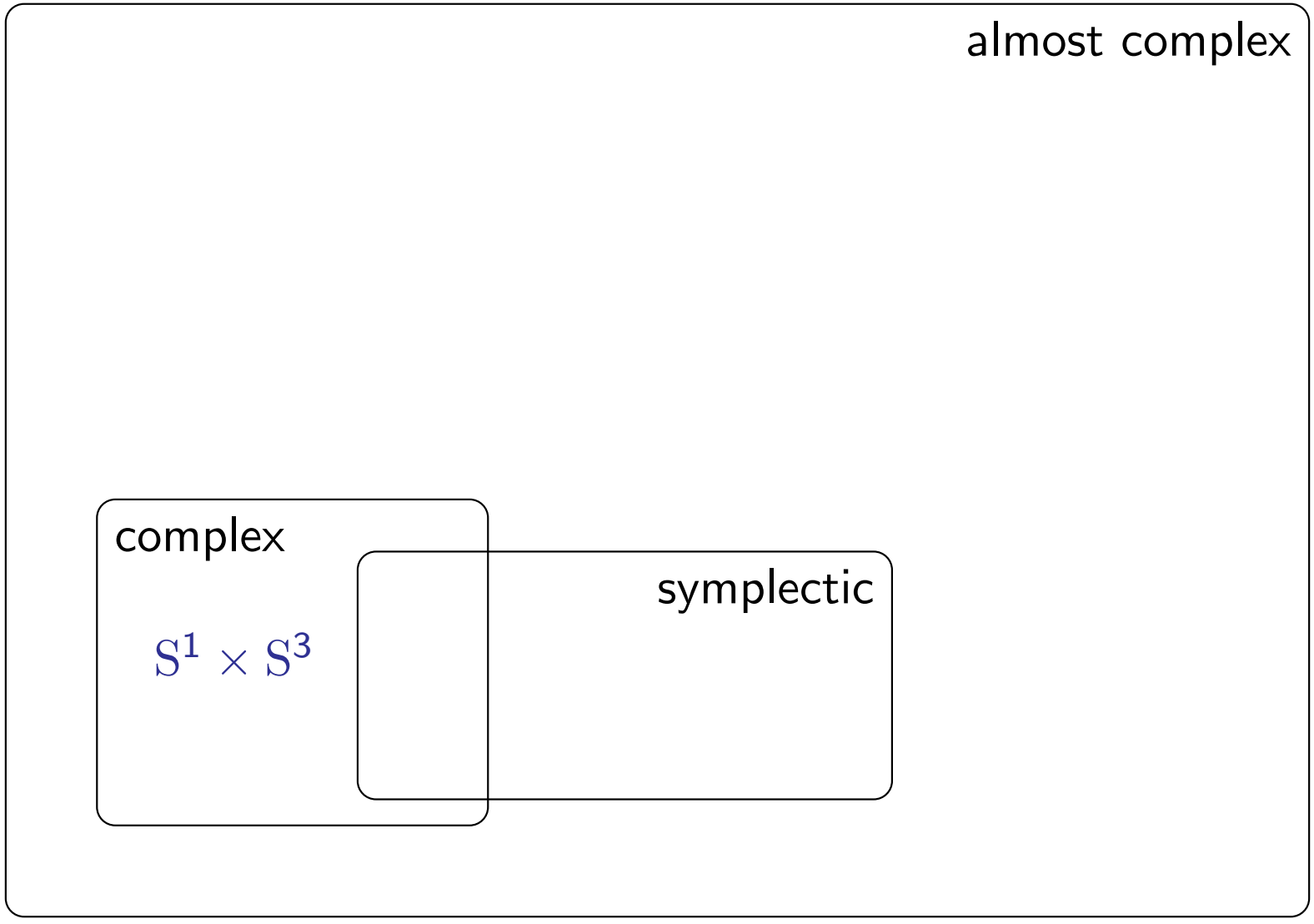
• Geometric structures now have a 'linear part' (in terms of TM) plus a 'geometric' or integrability condition.

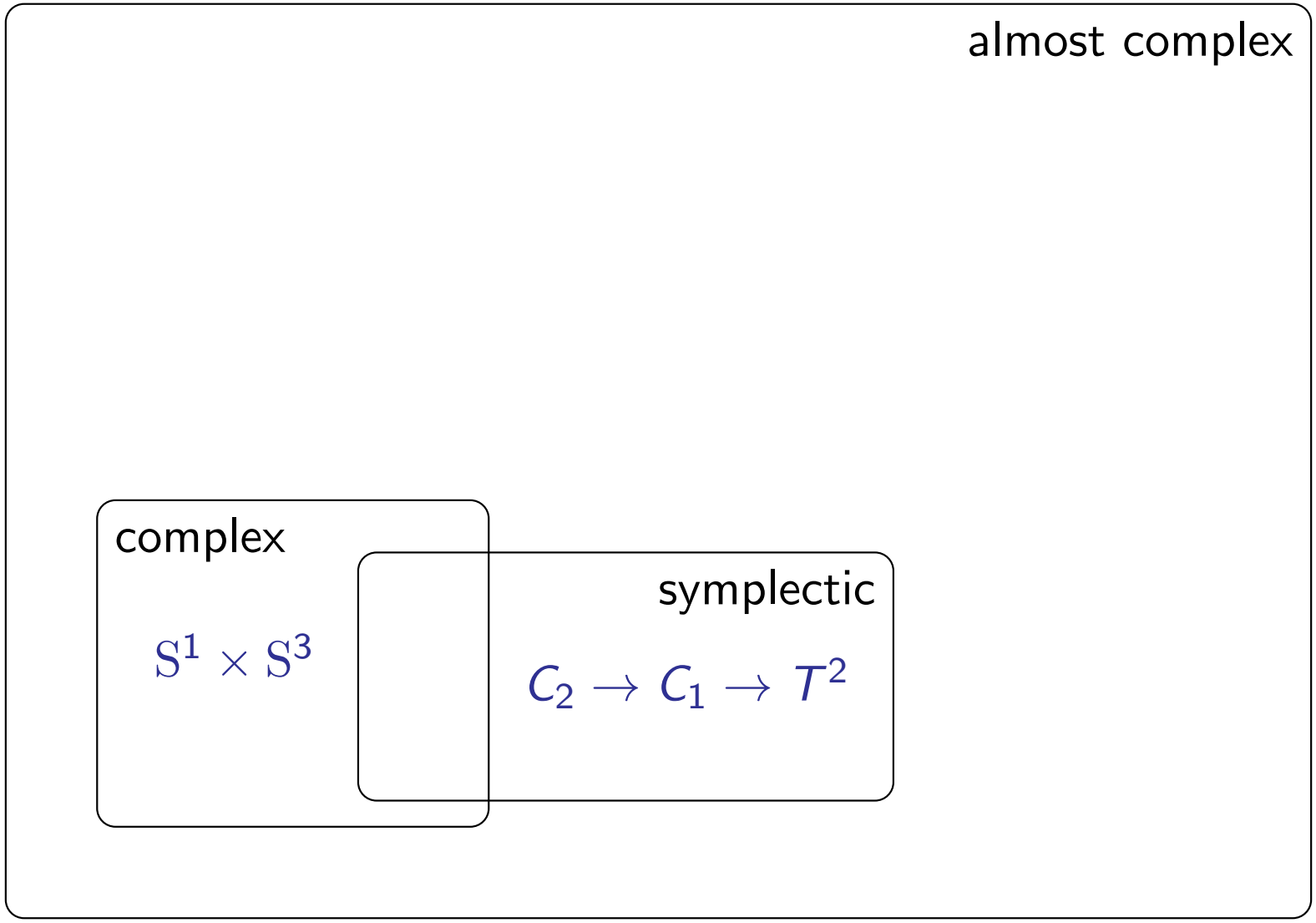
• By removing the integrability condition: almost complex structures and almost symplectic structures (non-degenerate 2-forms).

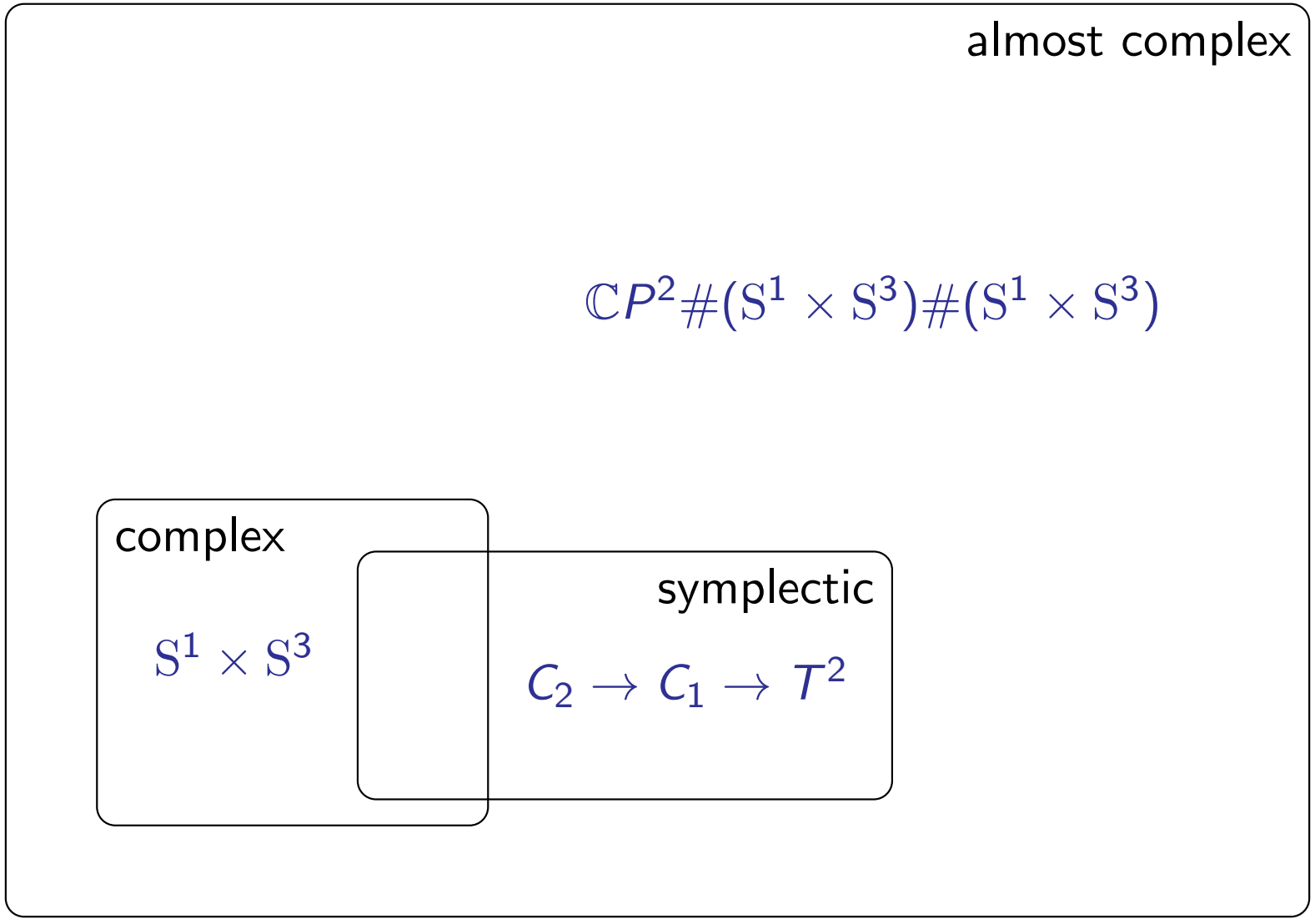
• A manifold admits an almost complex structure if and only if it admits an almost symplectic structure.

How strong are the integrability conditions?


almost complex
complex
symplectic


almost complex
complex
$S^1 \times S^3$
symplectic


almost complex
complex
$\mathrm{S}^1 \times \mathrm{S}^3$
symplectic
$C_2 \to C_1 \to T^2$


almost complex
$\mathbb{C}P^2\#(\mathrm{S}^1\times\mathrm{S}^3)\#(\mathrm{S}^1\times\mathrm{S}^3)$
complex
$\mathrm{S}^1\times\mathrm{S}^3$
symplectic
$C_2\to C_1\to T^2$

Main ideas:

going beyond the tangent bundle


TM

TM

$J \in \mathrm{End}(TM)$
$J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$
skew-symmetric

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

$J \in \mathrm{End}(TM)$
$J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$
skew-symmetric

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

Pairing $\langle X + \alpha, X + \alpha \rangle = \alpha(X)$

$J \in \mathrm{End}(TM)$
$J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$
skew-symmetric

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

Pairing $\langle X+\alpha, X+\alpha\rangle = \alpha(X)$

$J \in \mathrm{End}(TM)$
$J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$
skew-symmetric

$\mathcal{J} \in \mathrm{End}(TM+T^*M)$
$\mathcal{J}^2 = -\mathrm{Id}$ and $\mathcal{J}$ skew

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

Pairing $\langle X+\alpha, X+\alpha\rangle = \alpha(X)$

$$\left.\begin{array}{c} J \in \mathrm{End}(TM) \\ J^2 = -\mathrm{Id} \\ \\ \omega : TM \xrightarrow{\sim} T^*M \\ \text{skew-symmetric} \end{array}\right\}$$

$$\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$$
$$\mathcal{J}^2 = -\mathrm{Id} \text{ and } \mathcal{J} \text{ skew}$$

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

Pairing $\langle X+\alpha, X+\alpha\rangle = \alpha(X)$

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$ skew-symmetric

$\mathcal{J} \in \mathrm{End}(TM+T^*M)$
$\mathcal{J}^2 = -\mathrm{Id}$ and $\mathcal{J}$ skew

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

Generalized complex geometry (Hitchin-Gualtieri'03)

$TM + T^*M$

Pairing $\langle X+\alpha, X+\alpha\rangle = \alpha(X)$

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$

$\omega : TM \xrightarrow{\sim} T^*M$, skew-symmetric

$\mathcal{J} \in \mathrm{End}(TM+T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$ and $\mathcal{J}$ skew

Q: What about J integrable or $d\omega = 0$?

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left.\begin{array}{c} J \in \mathrm{End}(TM) \\ J^2 = -\mathrm{Id} \\ \\ \omega : TM \xrightarrow{\sim} T^*M \\ \text{skew-symmetric} \end{array}\right\}$$

Q: What about J integrable (or $d\omega = 0$)?

$$\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$$
$$\mathcal{J}^2 = -\mathrm{Id} \text{ and } \mathcal{J} \text{ skew}$$

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left.\begin{array}{c} J \in \mathrm{End}(TM) \\ J^2 = -\mathrm{Id} \\ \\ \omega : TM \xrightarrow{\sim} T^*M \\ \text{skew-symmetric} \end{array}\right\}$$

Q: What about
J integrable
(or $d\omega = 0$)?

A: $+i$-eigenbundle $T_{1,0}$
involutive, that is,
$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subseteq \Gamma(T_{1,0})$,
for the Lie bracket

$$\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$$
$$\mathcal{J}^2 = -\mathrm{Id} \text{ and } \mathcal{J} \text{ skew}$$

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left.\begin{array}{c} J \in \mathrm{End}(TM) \\ J^2 = -\mathrm{Id} \\ \\ \omega : TM \xrightarrow{\sim} T^*M \\ \text{skew-symmetric} \end{array}\right\}$$

Q: What about
J integrable
(or $d\omega = 0$)?

A: $+i$-eigenbundle $T_{1,0}$
involutive, that is,
$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subseteq \Gamma(T_{1,0})$,
for the Lie bracket

$$\mathcal{J} \in \mathrm{End}(TM + T^*M)$$
$$\mathcal{J}^2 = -\mathrm{Id} \text{ and } \mathcal{J} \text{ skew}$$

Examples:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

Here, $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$,
involutive...
but **do we even have**
a bracket on $TM + T^*M$?

The Dorfman bracket

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha$$

The Dorfman bracket

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X, Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

- for $\mathcal{J}_J$ if and only if J is a complex structure.

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

- for $\mathcal{J}_J$ if and only if J is a complex structure.
- for $\mathcal{J}_\omega$ if and only if ω is a symplectic structure.

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

- for $\mathcal{J}_J$ if and only if J is a complex structure.
- for $\mathcal{J}_\omega$ if and only if ω is a symplectic structure.

$J \rightsquigarrow +i$-eigenbundle

$T_{1,0} \subset T_{\mathbb{C}}M$

J complex $\leftrightarrow$ $T_{1,0}$ involutive

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

- for $\mathcal{J}_J$ if and only if J is a complex structure.
- for $\mathcal{J}_\omega$ if and only if ω is a symplectic structure.

$J \rightsquigarrow +i$-eigenbundle

$T_{1,0} \subset T_{\mathbb{C}}M$

J complex $\leftrightarrow$ $T_{1,0}$ involutive

For $\mathcal{J} \rightsquigarrow L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$,

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

The $+i$-eigenbundle $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ is involutive, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$,

- for $\mathcal{J}_J$ if and only if J is a complex structure.
- for $\mathcal{J}_\omega$ if and only if ω is a symplectic structure.

$J \rightsquigarrow +i$-eigenbundle

$T_{1,0} \subset T_{\mathbb{C}}M$

J complex $\leftrightarrow$ $T_{1,0}$ involutive

For $\mathcal{J} \rightsquigarrow L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$,

Definition

A **generalized complex structure** is a bundle map $\mathcal{J} \in \mathrm{End}(TM+T^*M)$ such that $\mathcal{J}^2 = -\mathrm{Id}$, $\mathcal{J}$ is skew and its $+i$-eigenbundle L is involutive.

The Dorfman **bracket**??

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha$$

The Dorfman **bracket**??

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

$$\begin{aligned}[X+\alpha, X+\alpha] &= [X,X] + \mathcal{L}_X\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha + \imath_X d\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha = d\langle X+\alpha, X+\alpha\rangle\end{aligned}$$

It is not skew-symmetric,

The Dorfman **bracket**??

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

$$\begin{aligned}[X+\alpha, X+\alpha] &= [X,X] + \mathcal{L}_X\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha + \imath_X d\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha = d\langle X+\alpha, X+\alpha\rangle\end{aligned}$$

It is not skew-symmetric, but satisfies, for $e, u, v \in \Gamma(TM+T^*M)$,

$$[e,[u,v]] = [[e,u],v] + [u,[e,v]]$$

The Dorfman **bracket**??

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

$$\begin{aligned}[X+\alpha, X+\alpha] &= [X,X] + \mathcal{L}_X\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha + \imath_X d\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha = d\langle X+\alpha, X+\alpha\rangle\end{aligned}$$

It is not skew-symmetric, but satisfies, for $e, u, v \in \Gamma(TM+T^*M)$,

$$[e,[u,v]] = [[e,u],v] + [u,[e,v]]$$

$$\pi_{TM}(e)\langle u,v\rangle = \langle [e,u],v\rangle + \langle u,[e,v]\rangle$$

The Dorfman **bracket**??

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \imath_Y d\alpha$$

$$\begin{aligned}[X+\alpha, X+\alpha] &= [X,X] + \mathcal{L}_X\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha + \imath_X d\alpha - \imath_X d\alpha \\ &= d\imath_X\alpha = d\langle X+\alpha, X+\alpha\rangle\end{aligned}$$

It is not skew-symmetric, but satisfies, for $e, u, v \in \Gamma(TM+T^*M)$,

$$[e,[u,v]] = [[e,u],v] + [u,[e,v]]$$

$$\pi_{TM}(e)\langle u,v\rangle = \langle [e,u],v\rangle + \langle u,[e,v]\rangle$$

Actually, this structure has a name...

The Courant algebroid $(TM+T^*M, \langle\,,\rangle, [\,,], \pi_{TM})$

The Courant algebroid $(TM+T^*M, \langle\,,\rangle, [\,,], \pi_{TM})$

Definition (Liu-Weinstein-Xu'97)

A Courant algebroid over M is a tuple $(E, \langle\,,\rangle, [\,,], \pi)$ consisting of

- a vector bundle $E \to M$,
- a nondegenerate symmetric pairing $\langle\,,\rangle$,
- a bilinear bracket $[\,,]$ on $\Gamma(E)$,
- a bundle map $\pi : E \to TM$ covering id_M,

such that, for any $e \in \Gamma(E)$,

- the map $[e, \cdot]$ is a derivation of both the bracket and the pairing,
- we have $[e, e] = \pi^* d\langle e, e\rangle$.

The Courant algebroid $(TM+T^*M, \langle\,,\rangle, [\,,], \pi_{TM})$

Definition (Liu-Weinstein-Xu'97)

A Courant algebroid over M is a tuple $(E, \langle\,,\rangle, [\,,], \pi)$ consisting of

- a vector bundle $E \to M$,
- a nondegenerate symmetric pairing $\langle\,,\rangle$,
- a bilinear bracket $[\,,]$ on $\Gamma(E)$,
- a bundle map $\pi : E \to TM$ covering id_M,

such that, for any $e \in \Gamma(E)$,

- the map $[e, \cdot]$ is a derivation of both the bracket and the pairing,
- we have $[e, e] = \pi^* d\langle e, e\rangle$.

Example

For $H \in \Omega^3_{cl}$, define the H-twisted bracket

$$[X + \alpha, Y + \beta]_H = [X, Y] + \mathcal{L}_X\beta - \imath_Y d\alpha + \imath_X\imath_Y H$$

The tuple $(TM+T^*M, \langle\,,\rangle, [\,,]_H, \pi_{TM})$ is a Courant algebroid

Automorphisms of Courant algebroids

Automorphisms of Courant algebroids

Definition

The automorphism group $\operatorname{Aut} E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \operatorname{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

$$\langle Fu, Fv \rangle = f_* \langle u, v \rangle,$$

$$[Fu, Fv] = f_* [u, v],$$

$$\pi_{TM} \circ F = f_* \circ \pi_{TM}$$

Automorphisms of Courant algebroids

Definition

The automorphism group $\operatorname{Aut} E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \operatorname{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

$$\langle Fu, Fv \rangle = f_* \langle u, v \rangle,$$

$$[Fu, Fv] = f_* [u, v],$$

$$\pi_{TM} \circ F = f_* \circ \pi_{TM}$$

Example

On $TM + T^*M$, for any $f \in \operatorname{Diff}(M)$

$$f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, \qquad X + \alpha \mapsto f_* X + f_* \alpha$$

Automorphisms of Courant algebroids

Definition

The automorphism group $\operatorname{Aut} E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \operatorname{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

$$\langle Fu, Fv \rangle = f_* \langle u, v \rangle,$$
$$[Fu, Fv] = f_* [u, v],$$
$$\pi_{TM} \circ F = f_* \circ \pi_{TM}$$

Example

On $TM \oplus T^*M$, for any $f \in \operatorname{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$,

$$f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, \qquad X + \alpha \mapsto f_* X + f_* \alpha$$
$$e^B = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}, \quad X + \alpha \mapsto X + \alpha + \imath_X B$$

Automorphisms of Courant algebroids

Definition

The automorphism group $\operatorname{Aut} E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \operatorname{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

$$\langle Fu, Fv \rangle = f_* \langle u, v \rangle,$$
$$[Fu, Fv] = f_* [u, v],$$
$$\pi_{TM} \circ F = f_* \circ \pi_{TM}$$

Example

On $TM+T^*M$, for any $f \in \operatorname{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$,

$$f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, \qquad X + \alpha \mapsto f_* X + f_* \alpha$$
$$e^B = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}, \quad X + \alpha \mapsto X + \alpha + \imath_X B$$
$$\in \operatorname{Aut}(TM+T^*M)$$

Automorphisms of Courant algebroids

Definition

The automorphism group $\operatorname{Aut} E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \operatorname{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

$$\langle Fu, Fv \rangle = f_* \langle u, v \rangle,$$
$$[Fu, Fv] = f_* [u, v],$$
$$\pi_{TM} \circ F = f_* \circ \pi_{TM}$$

Example

On $TM+T^*M$, for any $f \in \operatorname{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$,

$$f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, \qquad X + \alpha \mapsto f_* X + f_* \alpha$$
$$e^B = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}, \quad X + \alpha \mapsto X + \alpha + \imath_X B$$
$$\in \operatorname{Aut}(TM+T^*M)$$

Actually, the so-called **generalized diffeomorphisms** are

$$\operatorname{Aut}(TM+T^*M) = \operatorname{Diff}(M) \ltimes \Omega^2_{cl}(M)$$

Enough structure, back to generalized complex structures

M generalized complex $\Longrightarrow$ M almost complex $\Longrightarrow$ $\dim M = n = 2m$

Enough structure, back to generalized complex structures

M generalized complex $\Longrightarrow$ M almost complex $\Longrightarrow$ $\dim M = n = 2m$

The **triple life** of complex structures is revealing and helpful:

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$	$\mathcal{J} \in \mathrm{End}(TM + T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$
$T_{1,0}$ involutive (Lie)	L involutive (Dorfman)
	$\mathcal{J}$ skew ($\mathcal{J} + \mathcal{J}^* = 0$)

Enough structure, back to generalized complex structures

M generalized complex $\Longrightarrow$ M almost complex $\Longrightarrow$ $\dim M = n = 2m$

The **triple life** of complex structures is revealing and helpful:

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$	$\mathcal{J} \in \mathrm{End}(TM + T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$
$T_{1,0}$ involutive (Lie)	L involutive (Dorfman)
	$\mathcal{J}$ skew ($\mathcal{J} + \mathcal{J}^* = 0$)
$L \subset T_{\mathbb{C}}M$, $L \cap \overline{L} = \{0\}$	$L \subset (TM + T^*M)_{\mathbb{C}}$, $L \cap \overline{L} = \{0\}$
L involutive (Lie)	L involutive (Dorfman)
	L maximally isotropic

Enough structure, back to generalized complex structures

M generalized complex $\Longrightarrow$ M almost complex $\Longrightarrow$ $\dim M = n = 2m$

The **triple life** of complex structures is revealing and helpful:

$J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$	$\mathcal{J} \in \mathrm{End}(TM + T^*M)$, $\mathcal{J}^2 = -\mathrm{Id}$
$T_{1,0}$ involutive (Lie)	L involutive (Dorfman)
	$\mathcal{J}$ skew ($\mathcal{J} + \mathcal{J}^* = 0$)
$L \subset T_{\mathbb{C}}M$, $L \cap \overline{L} = \{0\}$	$L \subset (TM + T^*M)_{\mathbb{C}}$, $L \cap \overline{L} = \{0\}$
L involutive (Lie)	L involutive (Dorfman)
	L maximally isotropic
local $\zeta \in \Omega^m_{\mathbb{C}}$, $\zeta \wedge \overline{\zeta} \neq 0$, $d\zeta = 0$	What is ζ here?
(Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$)	What is Ann here?
($T_{1,0} = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$)	What extra property?

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$?

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X + \alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X + \alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.
So $(X + \alpha) \cdot \varphi = 0$ implies $\langle X + \alpha, X + \alpha \rangle = 0 \Longrightarrow \mathrm{Ann}(\varphi)$ isotropic.

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X + \alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.
So $(X + \alpha) \cdot \varphi = 0$ implies $\langle X + \alpha, X + \alpha \rangle = 0 \Longrightarrow \mathrm{Ann}(\varphi)$ isotropic.

Chevalley pairing ($\wedge^{top} T^*_{\mathbb{C}} M$-valued): $\quad (\varphi, \psi) = [\varphi^T \wedge \psi]_{top}$.

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X + \alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.
So $(X + \alpha) \cdot \varphi = 0$ implies $\langle X + \alpha, X + \alpha \rangle = 0 \Longrightarrow \mathrm{Ann}(\varphi)$ isotropic.

Chevalley pairing ($\wedge^{top} T^*_{\mathbb{C}} M$-valued): $\quad (\varphi, \psi) = [\varphi^T \wedge \psi]_{top}$.

$$L = \mathrm{Ann}(\rho)$$

$$L \cap \overline{L} = \{0\} \;\Big|\; L \text{ involutive (Dorfman)} \;\Big|\; L \text{ maximally isotropic}$$

Idea: $\zeta = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_m$, $L = \mathrm{Ann}(\zeta) = \{X : \imath_X \zeta = 0\}$

What is Ann for $(TM + T^*M)_{\mathbb{C}}$? Define, for $\varphi \in \Omega^\bullet_{\mathbb{C}}(M)$,

$$(X + \alpha) \cdot \varphi = \imath_X \varphi + \alpha \wedge \varphi.$$

Unlike $\imath_X \imath_X \varphi = 0$, it satisfies $(X + \alpha)^2 \cdot \varphi = (\imath_X \alpha)\varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.
So $(X + \alpha) \cdot \varphi = 0$ implies $\langle X + \alpha, X + \alpha \rangle = 0 \Longrightarrow \mathrm{Ann}(\varphi)$ isotropic.

Chevalley pairing ($\wedge^{top} T^*_{\mathbb{C}} M$-valued): $\quad (\varphi, \psi) = [\varphi^T \wedge \psi]_{top}$.

$$L = \mathrm{Ann}(\rho)$$

$L \cap \overline{L} = \{0\}$	L involutive (Dorfman)	L maximally isotropic
$(\rho, \overline{\rho}) \neq 0$	$d\rho = (X + \alpha) \cdot \rho$	$\rho \sim e^{B + i\omega} \theta_1 \wedge \ldots \wedge \theta_r$
(nowhere	for some $X + \alpha$	for $B, \omega \in \wedge^2 T^*M$
vanishing)	in $(TM + T^*M)_{\mathbb{C}}$	$\theta_j \in T^*_{\mathbb{C}} M$, ($\rho$ **pure**)

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \left(\begin{smallmatrix} -J & 0 \\ 0 & J^* \end{smallmatrix}\right)$	$\mathcal{J}_\omega = \left(\begin{smallmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{smallmatrix}\right)$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$	$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$	$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$	$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$
$d\rho = 0$	$d\rho = 0$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$	$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$
$d\rho = 0$	$d\rho = 0$
pure $(B = \omega = 0)$	pure $(B = 0, r = 0)$

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \left(\begin{smallmatrix} -J & 0 \\ 0 & J^* \end{smallmatrix}\right)$	$\mathcal{J}_\omega = \left(\begin{smallmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{smallmatrix}\right)$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$
$d\rho = 0$	$d\rho = 0$
pure $(B = \omega = 0)$	pure $(B = 0, r = 0)$

For pure ρ, ρ': $\mathrm{Ann}(\rho) = \mathrm{Ann}(\rho') \iff \rho = f\,\rho'$ for non-vanishing f.

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \left(\begin{smallmatrix} -J & 0 \\ 0 & J^* \end{smallmatrix}\right)$	$\mathcal{J}_\omega = \left(\begin{smallmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{smallmatrix}\right)$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$
$d\rho = 0$	$d\rho = 0$
pure $(B = \omega = 0)$	pure $(B = 0, r = 0)$

For pure ρ, ρ': $\mathrm{Ann}(\rho) = \mathrm{Ann}(\rho') \iff \rho = f\,\rho'$ for non-vanishing f.

Definition

For $\rho = \rho_0 + \ldots + \rho_n$, define **type**: least index j with $\rho_j \neq 0$ (function).

Examples, recall: pure $\rho \sim e^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_r$

J complex	ω symplectic
$\mathcal{J}_J = \left(\begin{smallmatrix} -J & 0 \\ 0 & J^* \end{smallmatrix}\right)$	$\mathcal{J}_\omega = \left(\begin{smallmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{smallmatrix}\right)$
$\rho = dz_1 \wedge \ldots \wedge dz_m$	$\rho = e^{i\omega}$
$(\rho, \overline{\rho}) \sim \rho \wedge \overline{\rho} \neq 0$	$(\rho, \overline{\rho}) \sim \omega^m \neq 0$
$d\rho = 0$	$d\rho = 0$
pure $(B = \omega = 0)$	pure $(B = 0, r = 0)$

For pure ρ, ρ': $\mathrm{Ann}(\rho) = \mathrm{Ann}(\rho') \iff \rho = f\,\rho'$ for non-vanishing f.

Definition

For $\rho = \rho_0 + \ldots + \rho_n$, define **type**: least index j with $\rho_j \neq 0$ (function).




symp.
cplx.
gen.cplx.

symp. — cplx.

gen.cplx.

Proposition (Gualtieri'04)

The type determines the structure at each point: up to B-equivalence, there are some symplectic directions and some transversal complex directions (as many as the type).

Theoretically, the type may vary within a manifold, preserving the parity and upper continuously. In particular, there is no unique local model.

Is it really possible for the type to change?

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\rho = dz$$

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$$

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$$

Pure: $z \neq 0$, $\rho \sim 1 + \frac{dz \wedge dw}{z} = e^{\frac{dz \wedge dw}{z}}$, pure and type 0

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$$

Pure: $z \neq 0$, $\rho \sim 1 + \frac{dz \wedge dw}{z} = e^{\frac{dz \wedge dw}{z}}$, pure and type 0
$z = 0$, $\rho = dz \wedge dw$, pure and type 2

Type change example, recall $(\rho, \overline{\rho}) \neq 0$, $d\rho = u \cdot \rho$, ρ pure

On $\mathbb{R}^4 \cong \mathbb{C}^2$, with complex coordinates (z, w),

$$\rho = z + dz \wedge dw$$

We check all the conditions:

$$(\rho, \overline{\rho}) = [(z + dz \wedge dw)^T \wedge (\overline{z} + d\overline{z} \wedge d\overline{w})]_{top} = dw \wedge dz \wedge d\overline{z} \wedge d\overline{w}$$

$$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$$

Pure: $z \neq 0$, $\rho \sim 1 + \frac{dz \wedge dw}{z} = e^{\frac{dz \wedge dw}{z}}$, pure and type 0
$z = 0$, $\rho = dz \wedge dw$, pure and type 2

Type change: generically B-transform of symplectic, blows up along $z = 0$

What are generalized complex structures geometrically?

In other words, how are symplectic and complex combined?

What are generalized complex structures geometrically?

In other words, how are symplectic and complex combined?

To start with, for any generalized complex structure

$$\mathcal{J} = \begin{pmatrix} * & P \\ * & * \end{pmatrix},$$

$P : T^*M \to TM$ is Poisson, which gives a symplectic foliation.

What are generalized complex structures geometrically?

In other words, how are symplectic and complex combined?

To start with, for any generalized complex structure

$$\mathcal{J} = \begin{pmatrix} * & P \\ * & * \end{pmatrix},$$

$P : T^*M \to TM$ is Poisson, which gives a symplectic foliation.

The Newlander-Nierenberg/Darboux-like theorem is:

Theorem (Bailey'13)

Locally, a generalized complex structure is B-equivalent to the product of a symplectic structure by holomorphic Poisson structure.

What is interesting about generalized complex structures?

Within generalized complex structures:

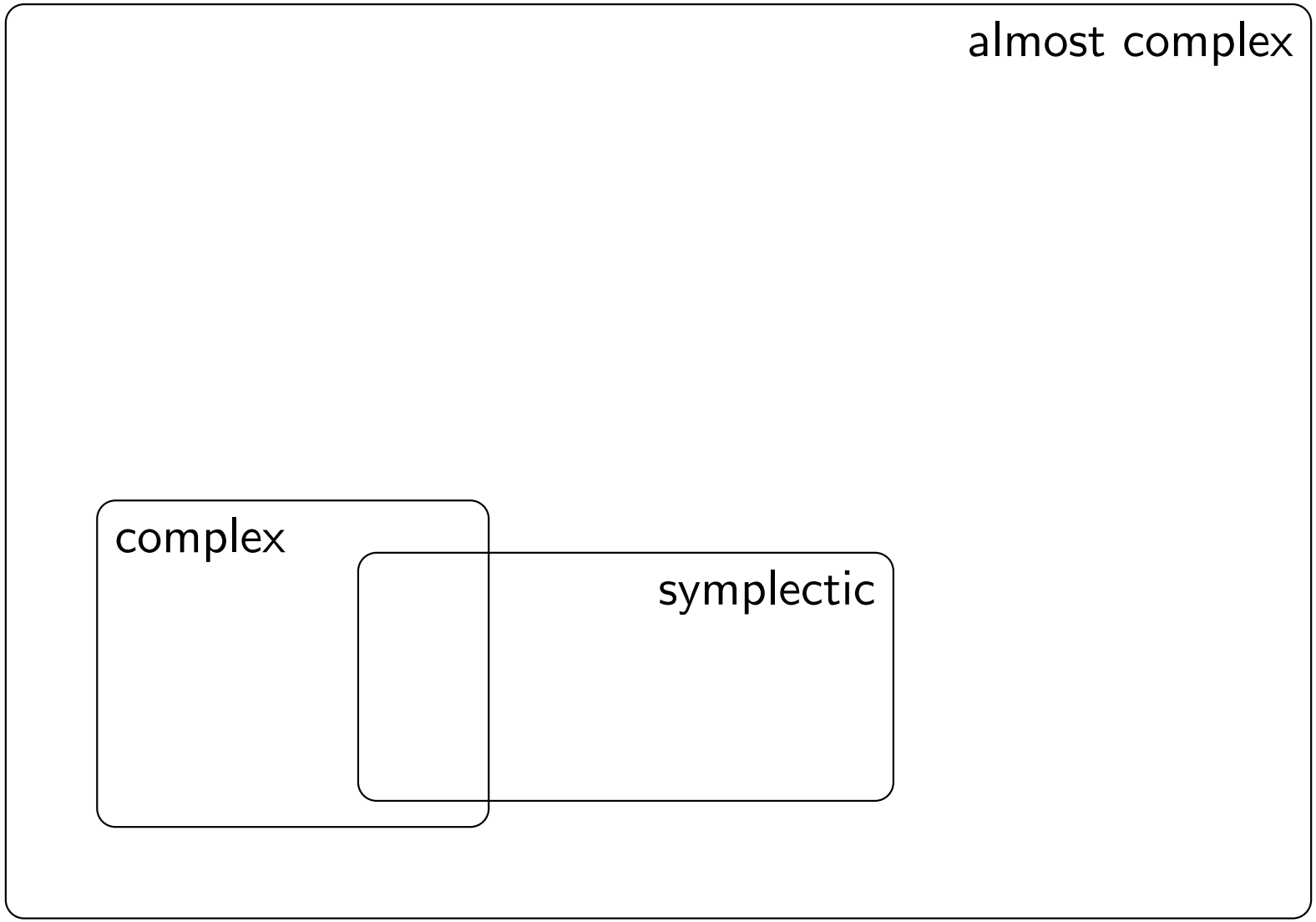


Done by a convex combination of $\mathcal{J}_J$ and $\mathcal{J}_\omega$,

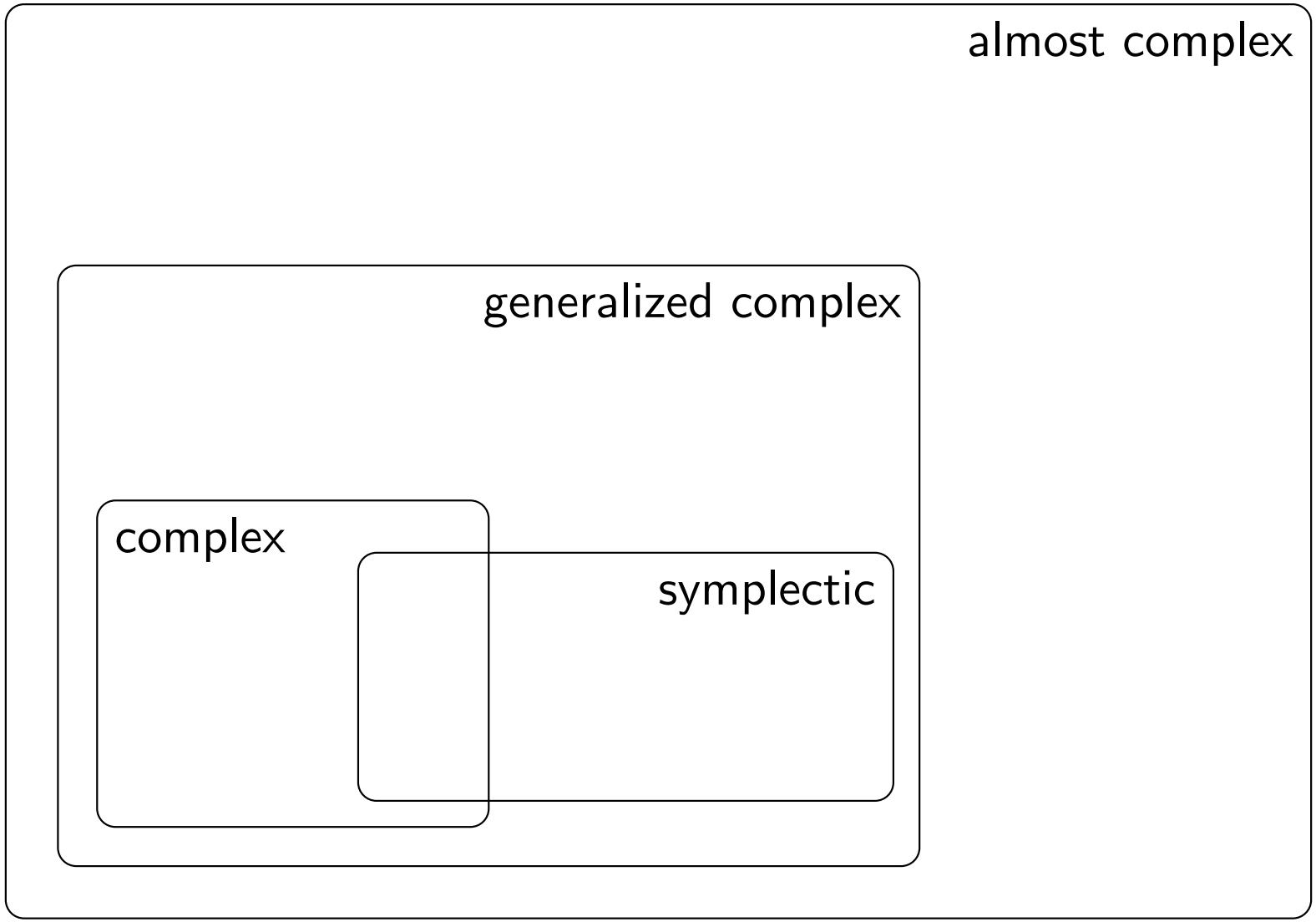
coming from hyperKähler or holomorphic symplectic structures.

The interior of the curve is e^B-equivalent to symplectic structures.

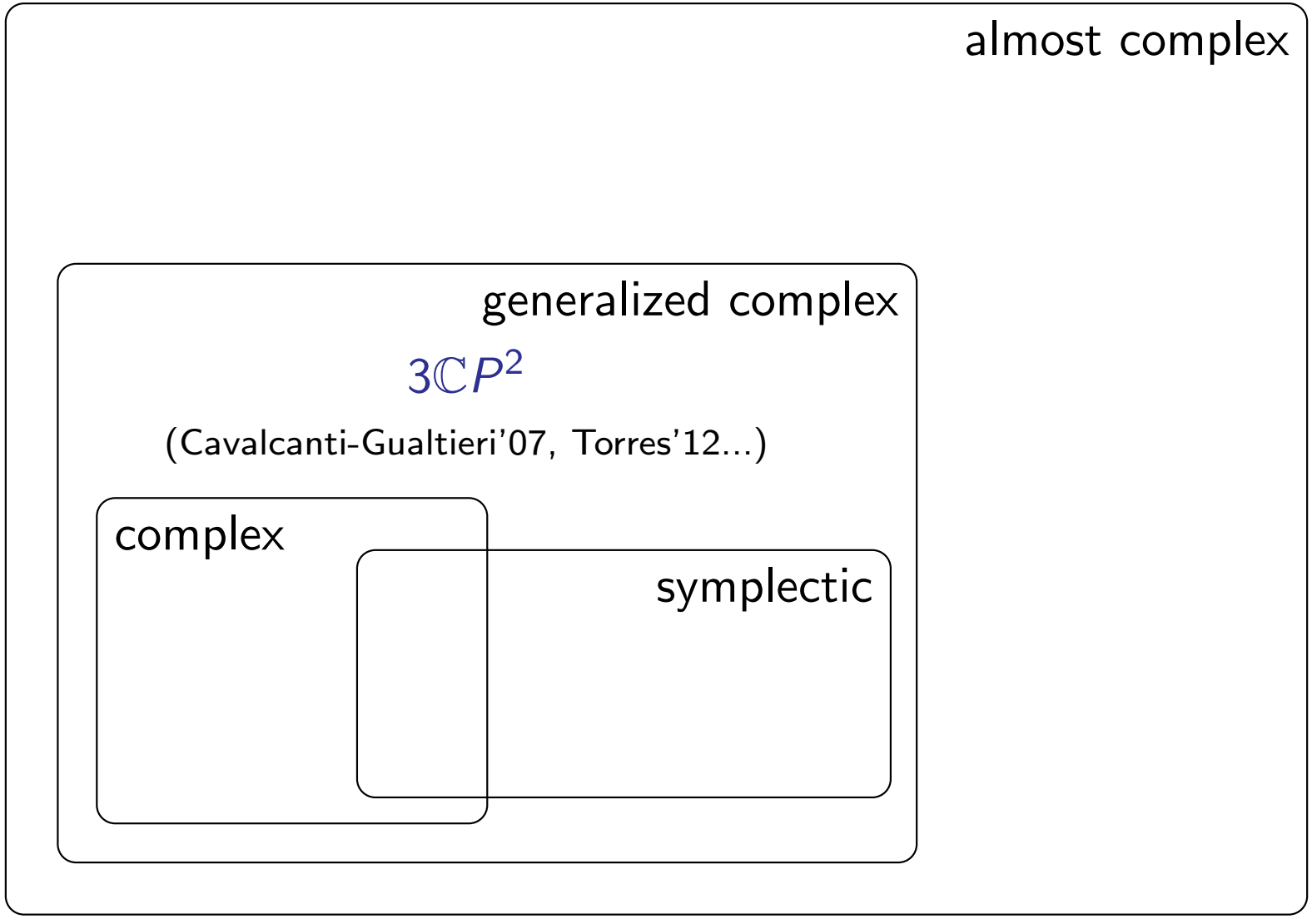
Beyond complex and symplectic



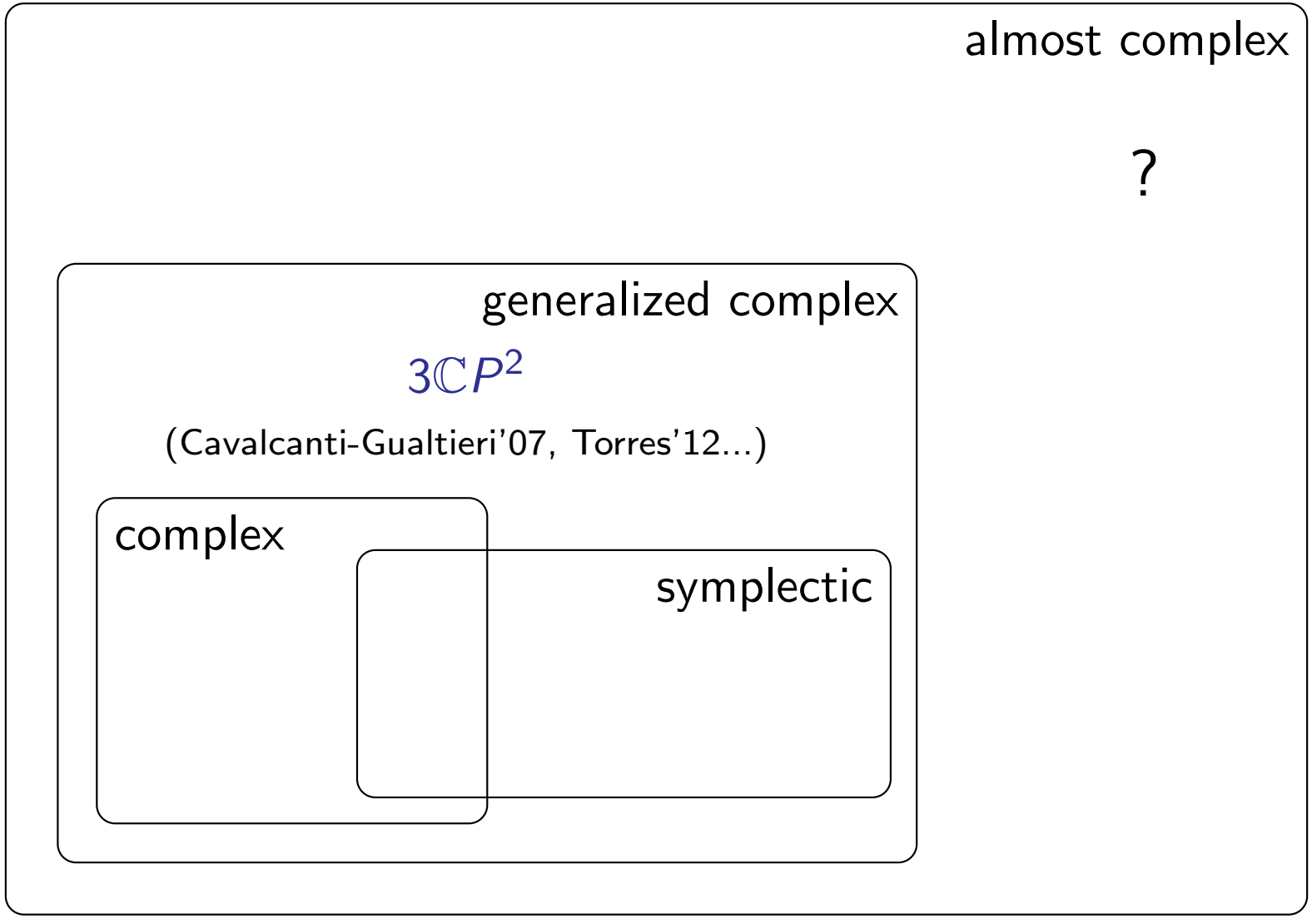
Beyond complex and symplectic



Beyond complex and symplectic



Beyond complex and symplectic



A comment on the pre-history: Dirac structures

A comment on the pre-history: Dirac structures

Dirac structures, introduced by Courant'90 and Weinstein, are lagrangian and Dorfman-involutive subbundles of $TM + T^*M$.

A comment on the pre-history: Dirac structures

Dirac structures, introduced by Courant'90 and Weinstein, are lagrangian and Dorfman-involutive subbundles of $TM + T^*M$.

They encompass presymplectic and Poisson structures, as well as integrable distributions.

A comment on the pre-history: Dirac structures

Dirac structures, introduced by Courant'90 and Weinstein, are lagrangian and Dorfman-involutive subbundles of $TM + T^*M$.

They encompass presymplectic and Poisson structures, as well as integrable distributions.

Geometrically, they correspond to presymplectic foliations.

A comment on the pre-history: Dirac structures

Dirac structures, introduced by Courant'90 and Weinstein, are lagrangian and Dorfman-involutive subbundles of $TM + T^*M$.

They encompass presymplectic and Poisson structures, as well as integrable distributions.

Geometrically, they correspond to presymplectic foliations.

The Dorfman bracket, first appeared in 1987, describes, at **the same time**, the integrability conditions of complex, symplectic and Poisson structures, as well as distributions!

$2 + \varepsilon$ recent directions

1. Another generalized geometry is possible

1. Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

1. Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

1. Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

$$\begin{aligned}
\pi(X + f + \alpha) &= X \\
\langle X + f + \alpha, X + f + \alpha \rangle &= \imath_X \alpha + f^2 \\
[X + f + \alpha, Y + g + \beta\rangle] &= [X, Y] + X(g) - Y(f) \\
&\quad + \mathcal{L}_X \beta - \imath_Y d\alpha + 2g df .
\end{aligned}$$

1. Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

$$\begin{aligned}
\pi(X + f + \alpha) &= X \\
\langle X + f + \alpha, X + f + \alpha \rangle &= \imath_X \alpha + f^2 \\
[X + f + \alpha, Y + g + \beta\rangle] &= [X, Y] + X(g) - Y(f) \\
&\quad + \mathcal{L}_X \beta - \imath_Y d\alpha + 2g df .
\end{aligned}$$

is a Courant algebroid, and moreover an $\mathrm{O}(n+1, n)$-bundle.
As $\mathrm{O}(n+1, n)$ is a real form of $\mathrm{O}(2n+1, \mathbb{C})$, of Lie type B_n:

1. Another generalized geometry is possible

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

$$\begin{aligned}
\pi(X + f + \alpha) &= X \\
\langle X + f + \alpha, X + f + \alpha \rangle &= \imath_X \alpha + f^2 \\
[X + f + \alpha, Y + g + \beta\rangle] &= [X, Y] + X(g) - Y(f) \\
&\quad + \mathcal{L}_X \beta - \imath_Y d\alpha + 2g df .
\end{aligned}$$

is a Courant algebroid, and moreover an $\mathrm{O}(n+1, n)$-bundle.
As $\mathrm{O}(n+1, n)$ is a real form of $\mathrm{O}(2n+1, \mathbb{C})$, of Lie type B_n:

Generalized geometry of type B_n

B_n-generalized complex geometry: examples and properties

• A B_n-generalized complex structure (R.'14) is a lagrangian and involutive subbundle $L \subset (TM + 1 + T^*M)_{\mathbb{C}}$ such that $L \cap \bar{L} = 0$.

B_n-generalized complex geometry: examples and properties

- A B_n-generalized complex structure (R.'14) is a lagrangian and involutive subbundle $L \subset (TM + 1 + T^*M)_{\mathbb{C}}$ such that $L \cap \bar{L} = 0$.

- Any usual generalized complex is B_n-generalized complex.

B_n-generalized complex geometry: examples and properties

• A B_n-generalized complex structure (R.'14) is a lagrangian and involutive subbundle $L \subset (TM + 1 + T^*M)_{\mathbb{C}}$ such that $L \cap \bar{L} = 0$.

• Any usual generalized complex is B_n-generalized complex.

• Cosymplectic and normal almost contact structures are examples, in odd dimensions!

B_n-generalized complex geometry: examples and properties

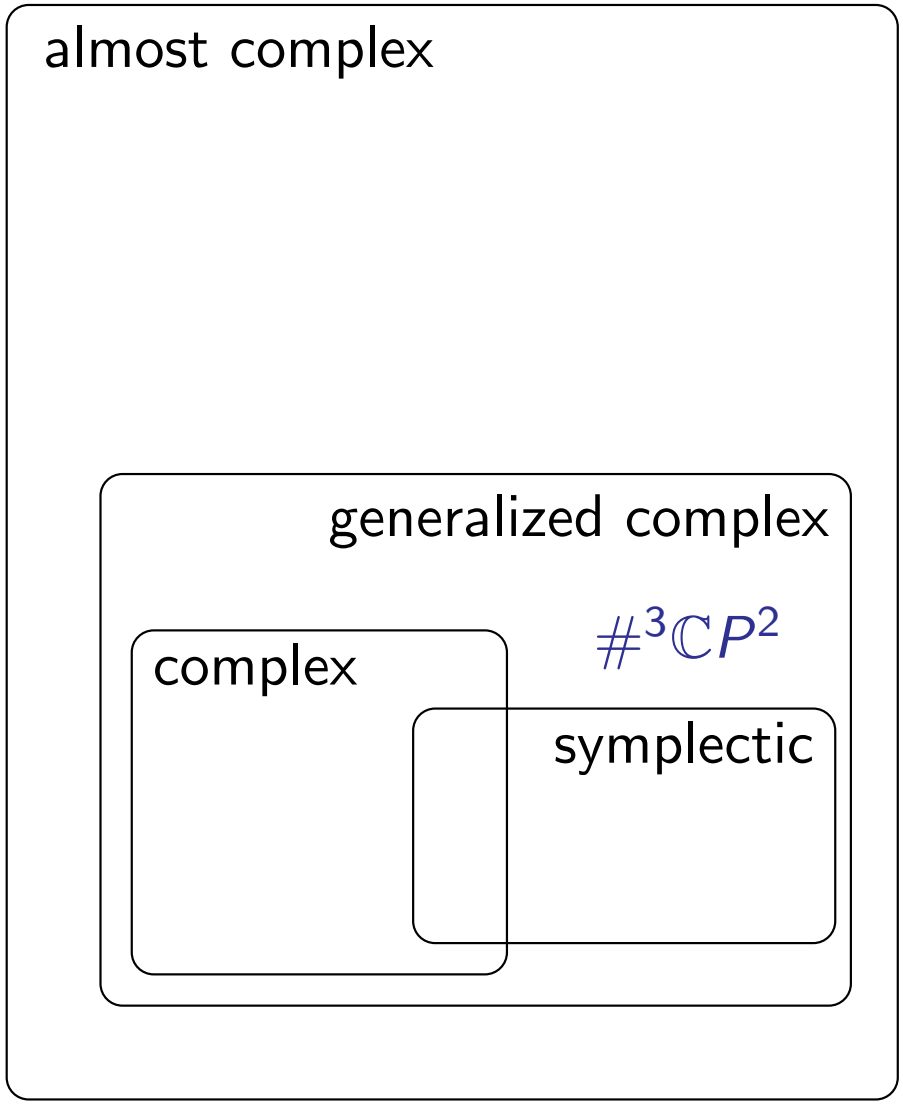
- A B_n-generalized complex structure (R.'14) is a lagrangian and involutive subbundle $L \subset (TM + 1 + T^*M)_{\mathbb{C}}$ such that $L \cap \bar{L} = 0$.

- Any usual generalized complex is B_n-generalized complex.

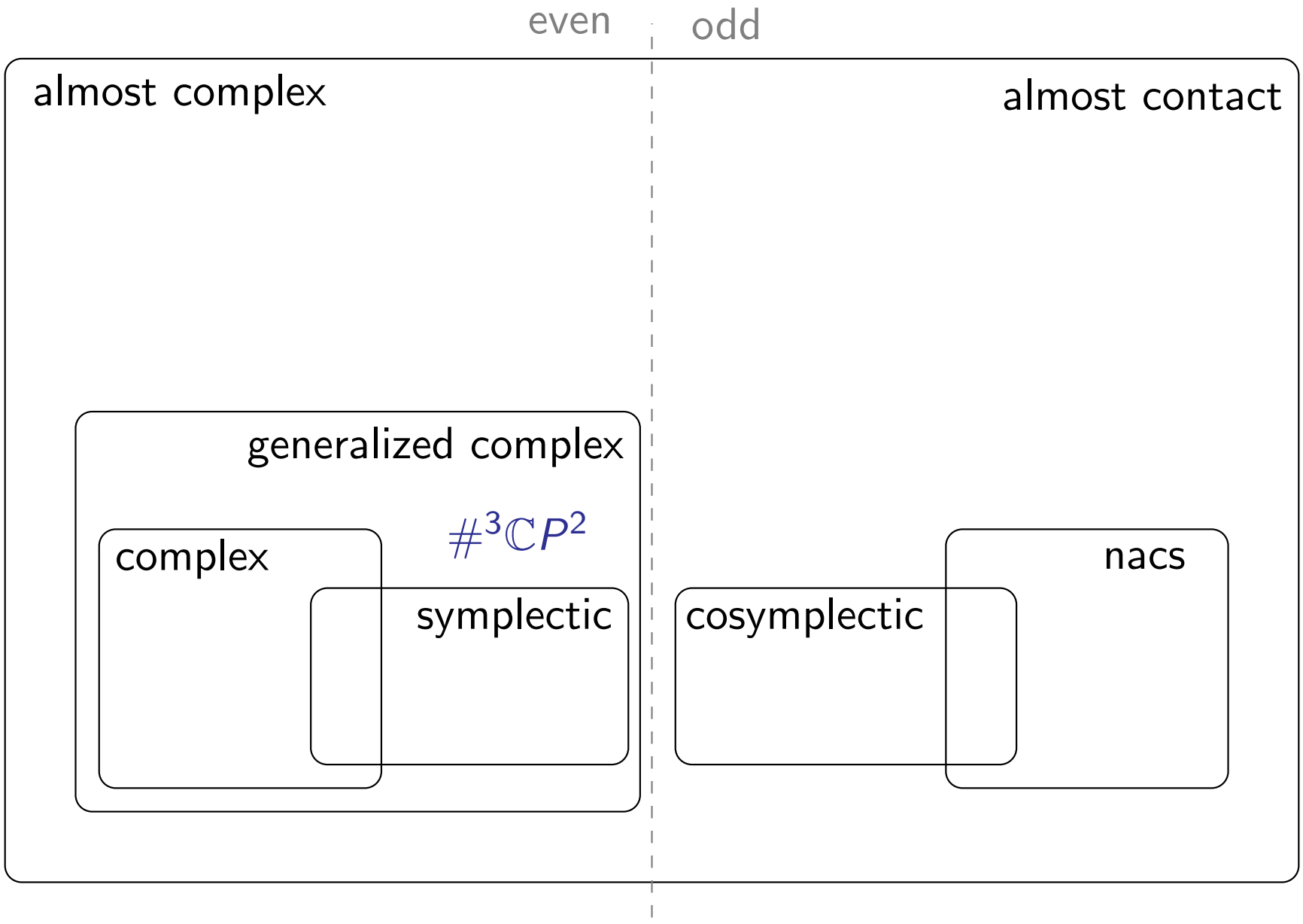
- Cosymplectic and normal almost contact structures are examples, in odd dimensions!

- Type change is now possible from dimension 3.

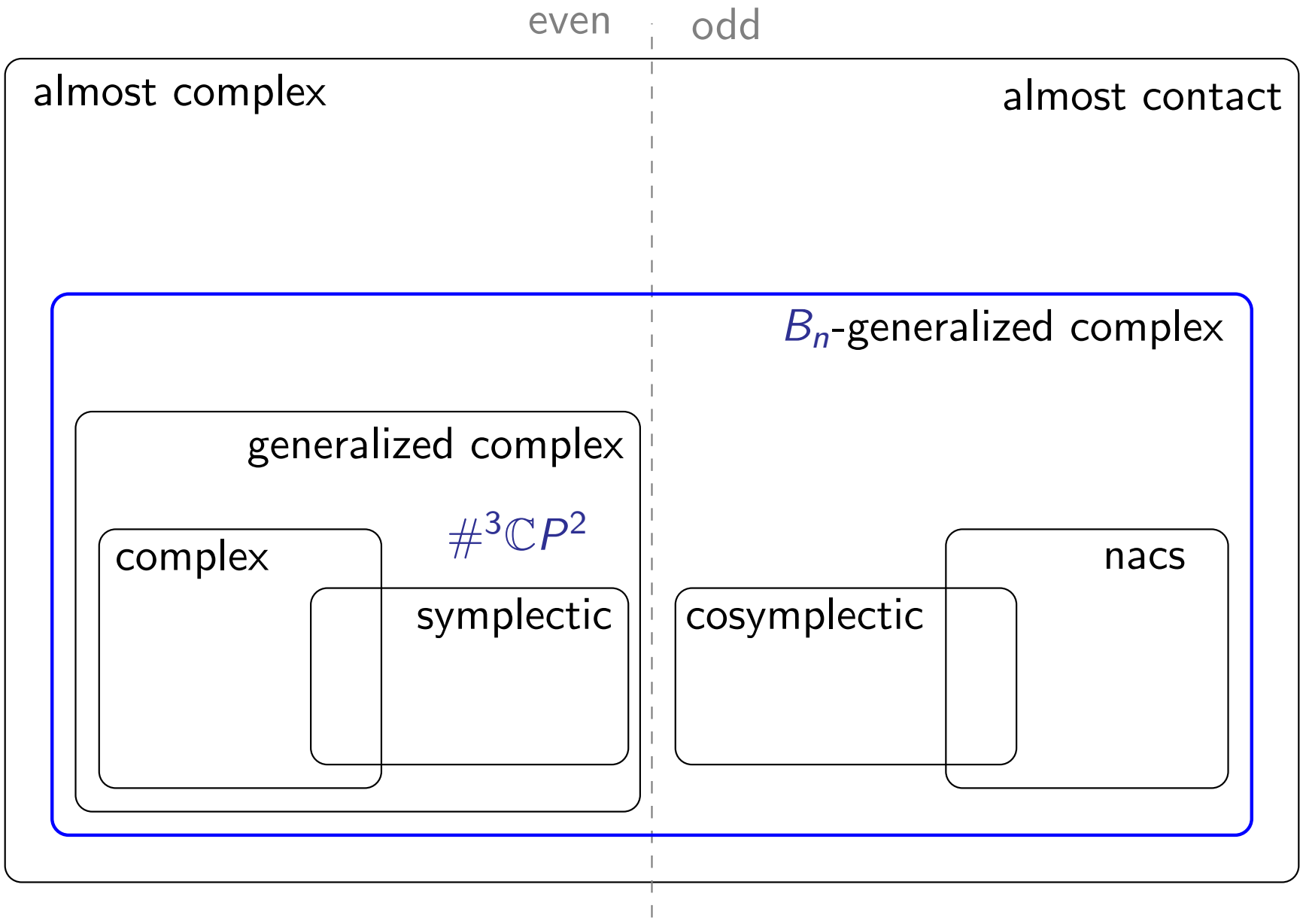
What we had


almost complex
generalized complex
$\#^3\mathbb{C}P^2$
complex
symplectic

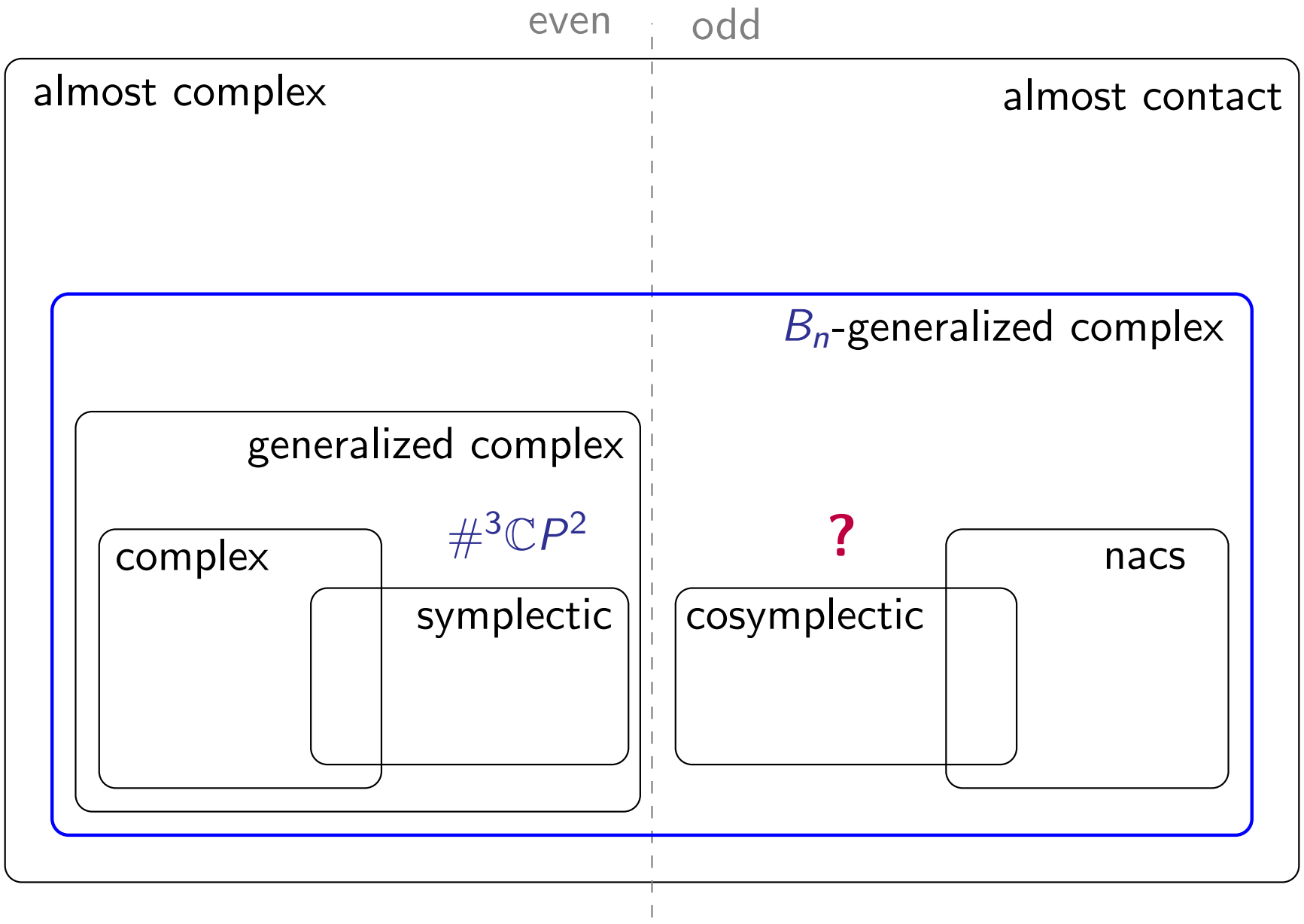
What we are getting



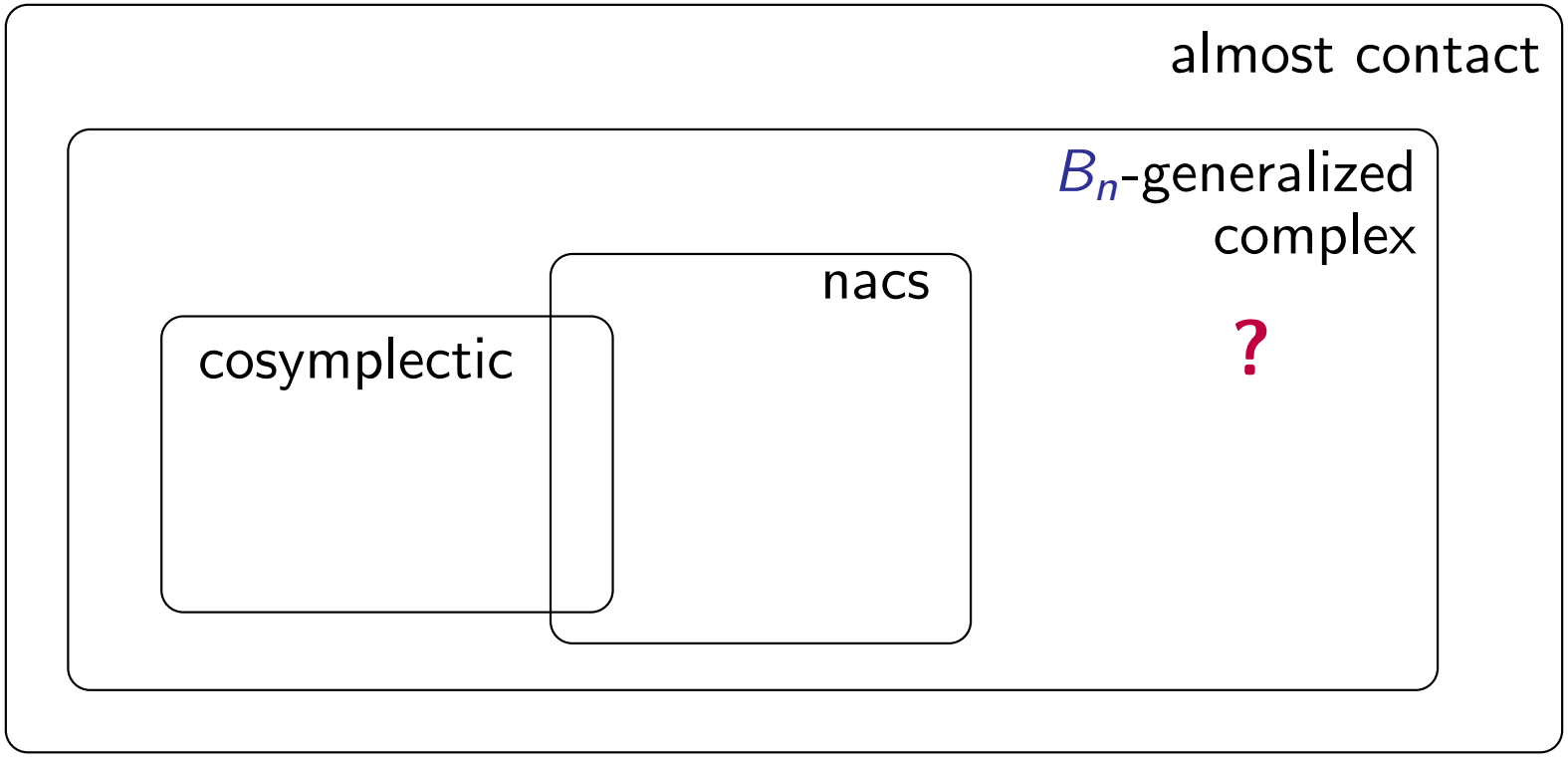
What we are getting



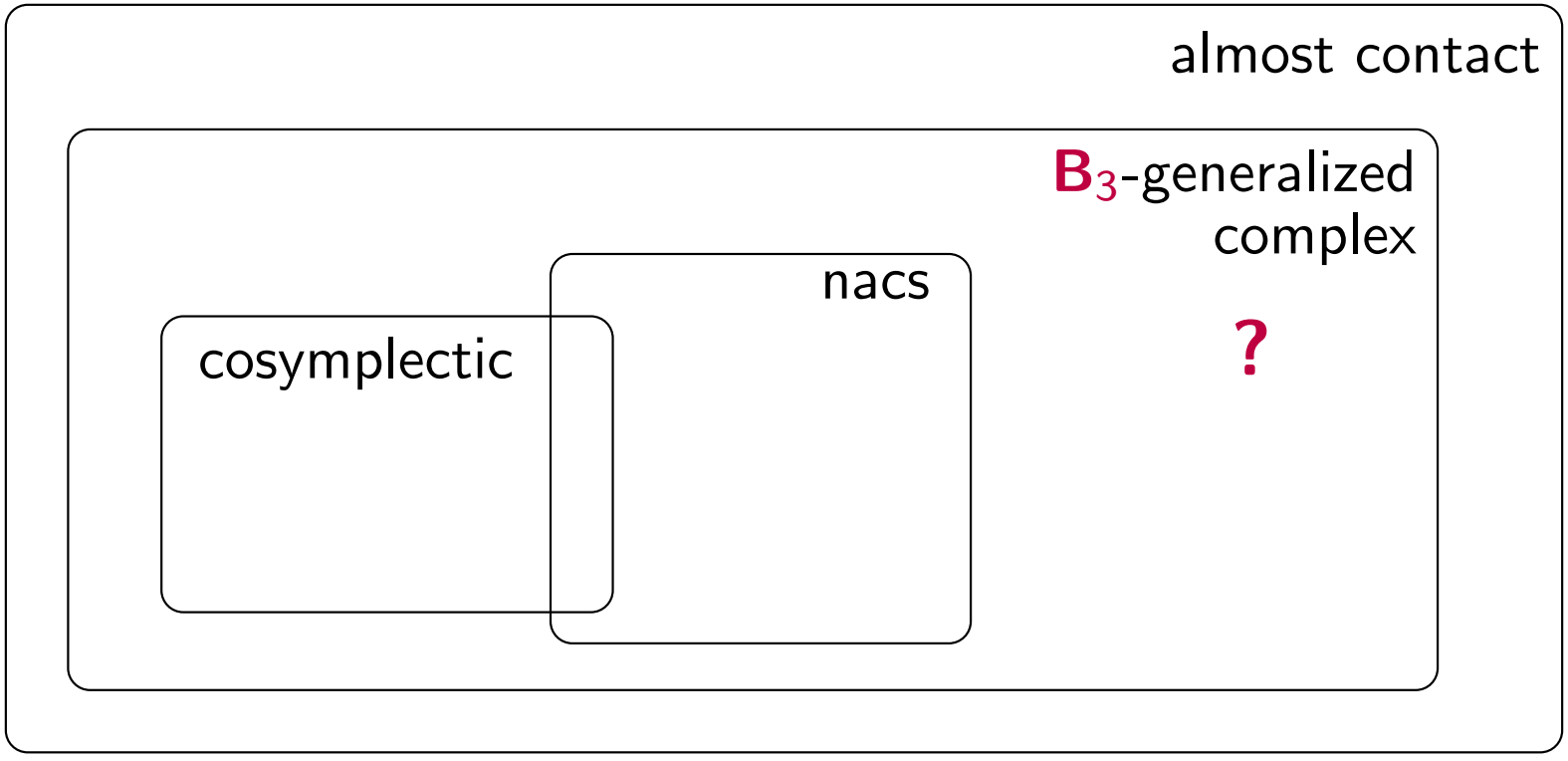
What we are getting



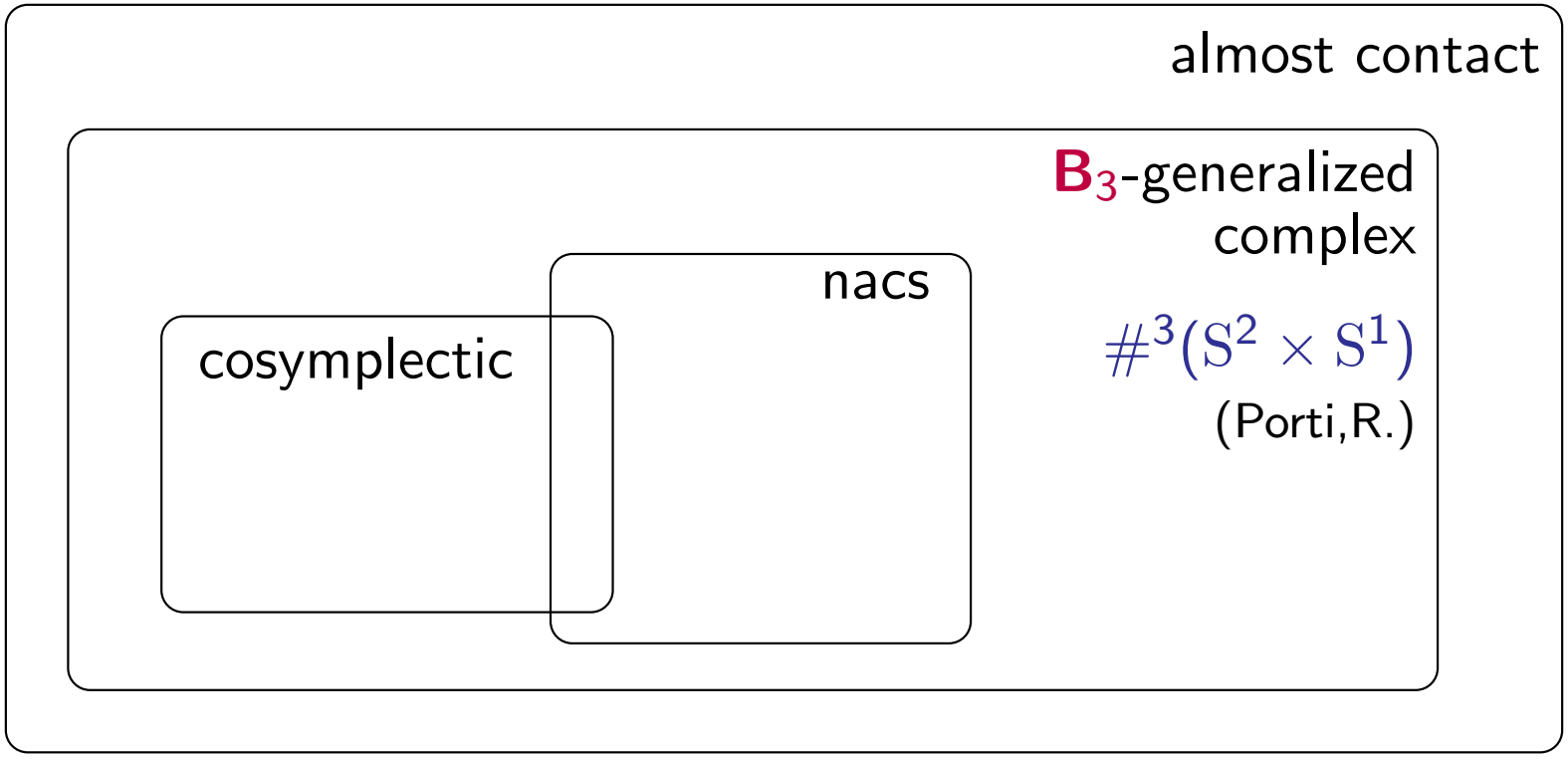
Zooming in



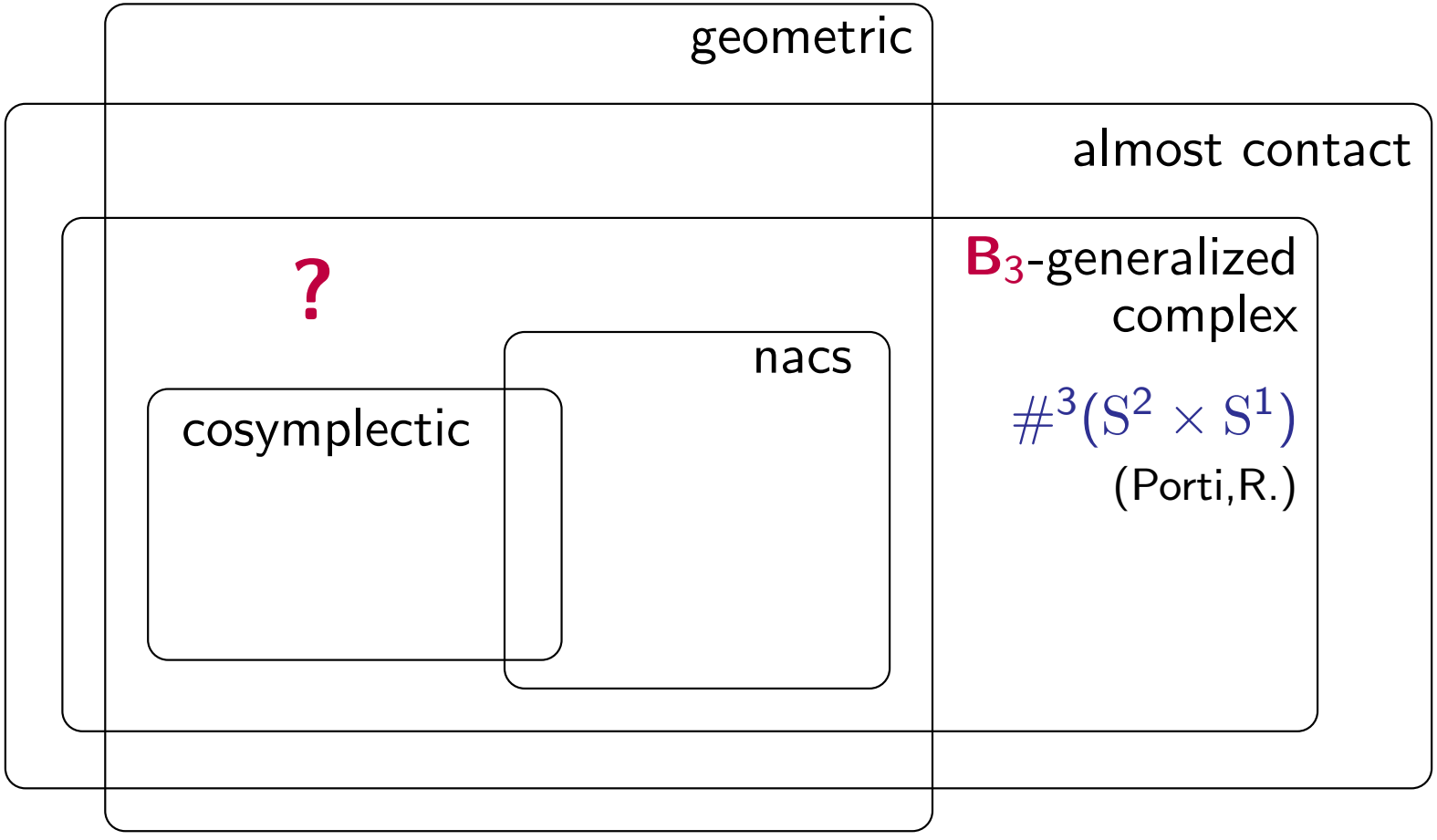
Zooming in


almost contact
$\mathbf{B}_3$-generalized complex
?
nacs
cosymplectic

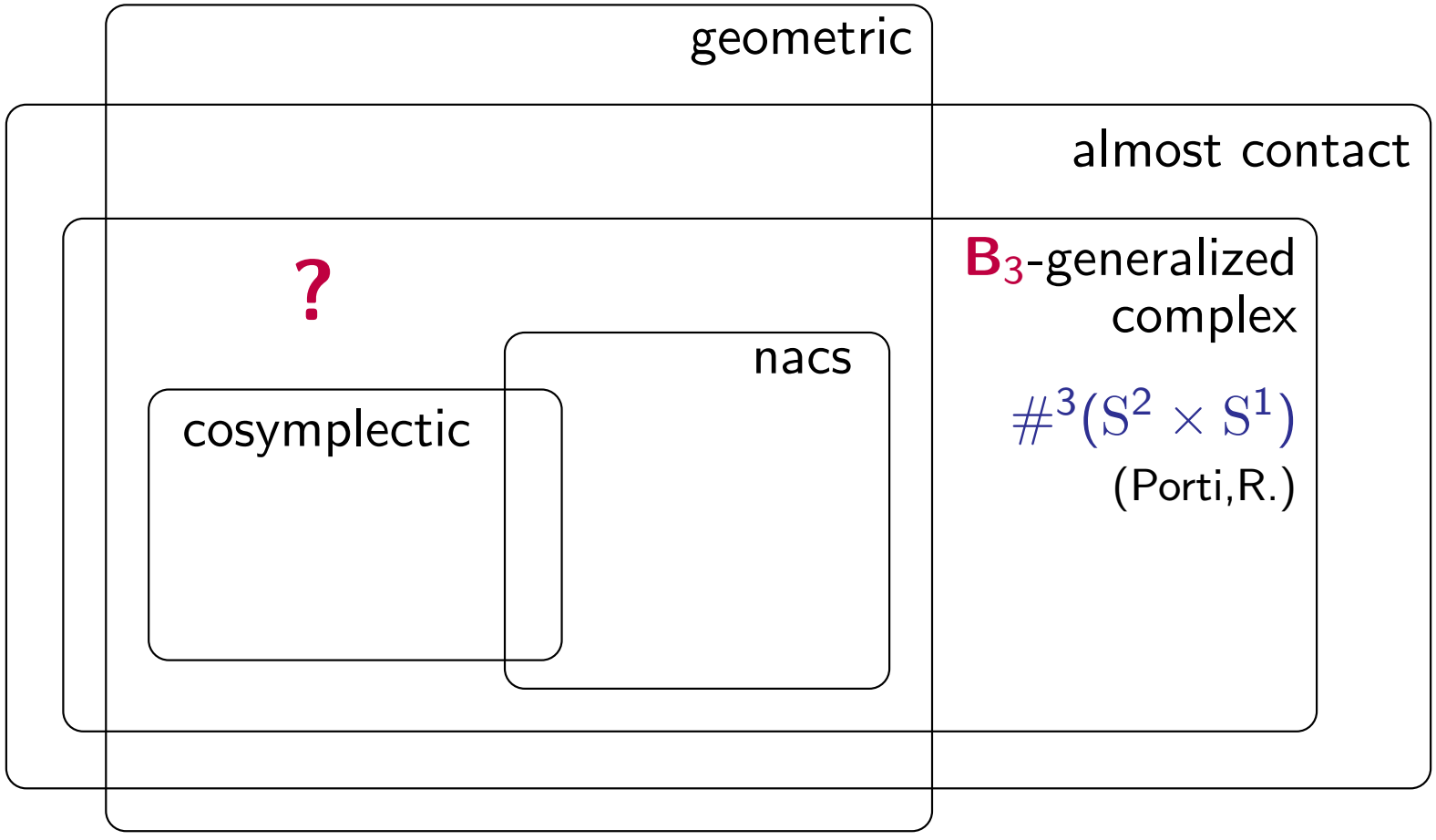
Zooming in


almost contact
$\mathbf{B}_3$-generalized complex
$\#^3(\mathbb{S}^2 \times \mathbb{S}^1)$
(Porti,R.)
nacs
cosymplectic

Zooming in


geometric
almost contact
?
$\mathbf{B}_3$-generalized complex
nacs
cosymplectic
$\#^3(\mathrm{S}^2 \times \mathrm{S}^1)$
(Porti,R.)

Zooming in



Theorem (Porti, R.'26)

Any closed oriented 3-manifold admits a B_3-generalized complex structure, which is moreover stable.

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

Theorem (Agüero, R.'22)

Complex Dirac structures are determined at each point by three invariants:

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

Theorem (Agüero, R.'22)

Complex Dirac structures are determined at each point by three invariants:

- *the (normalized) type,*
- *the real index,*
- *the order.*

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

Theorem (Agüero, R.'22)

Complex Dirac structures are determined at each point by three invariants:

- *the (normalized) type,*
- *the real index,*
- *the order.*

At each point: presymplectic directions + transverse CR directions.

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

Theorem (Agüero, R.'22)

Complex Dirac structures are determined at each point by three invariants:

- *the (normalized) type,*
- *the real index,*
- *the order.*

At each point: presymplectic directions + transverse CR directions.

These invariants may vary (satisfying constraints like parity, upper semi-continuity, order $\leq$ real-index,).

2. Complex Dirac structures

Complex Dirac $\equiv_{def}$ lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$.

Theorem (Agüero, R.'22)

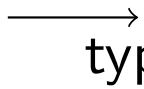
Complex Dirac structures are determined at each point by three invariants:

- *the (normalized) type,*
- *the real index,*
- *the order.*

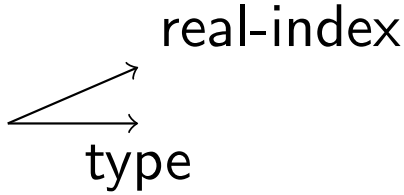
At each point: presymplectic directions + transverse CR directions.

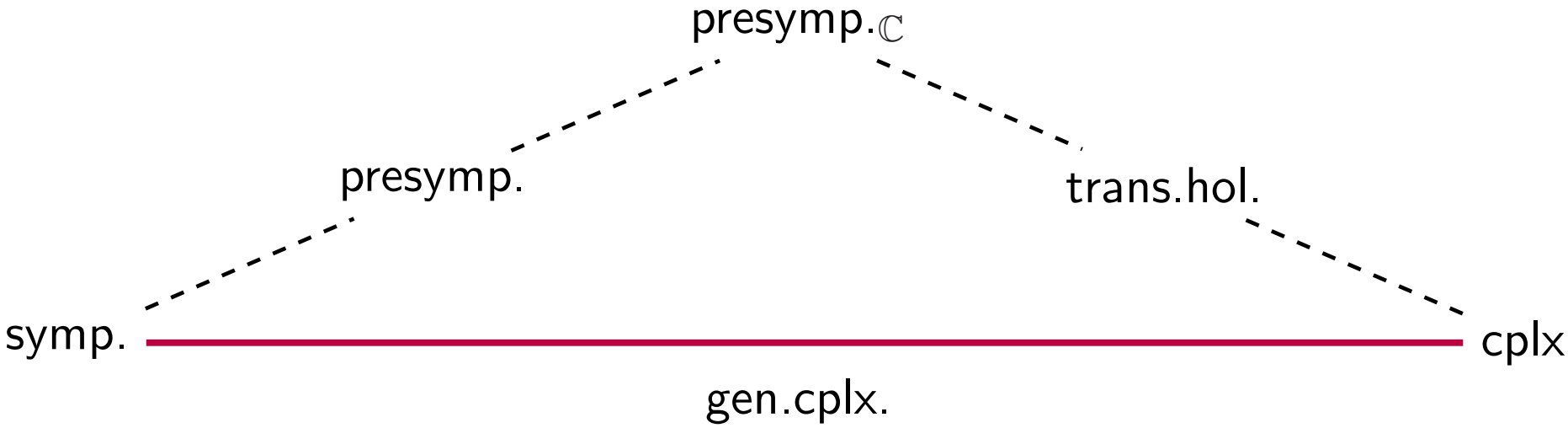
These invariants may vary (satisfying constraints like parity, upper semi-continuity, order $\leq$ real-index,).

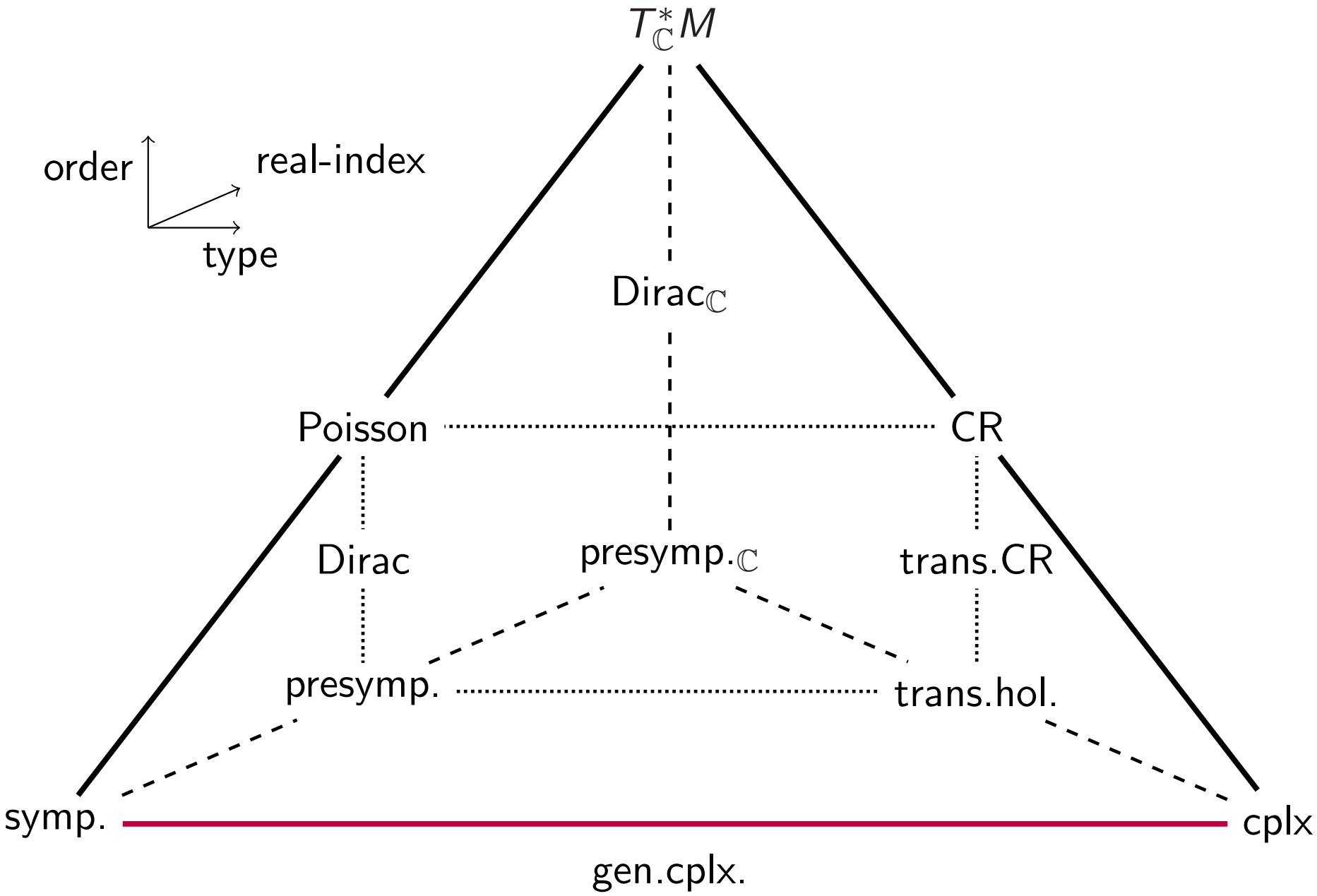
For constant order, a complex Dirac has associated a real Dirac.

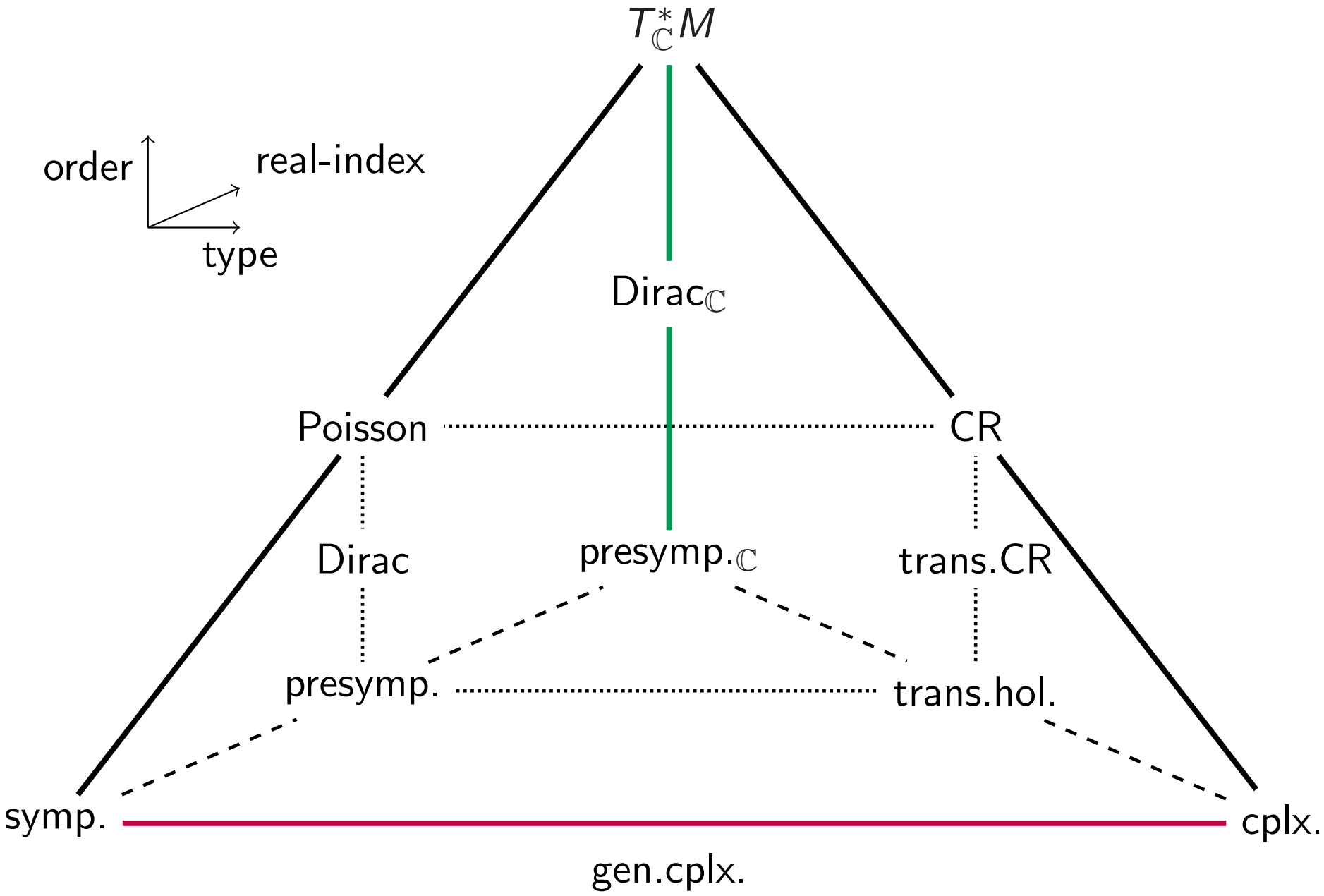

type


symp.
cplx.
gen.cplx.


real-index
type


presymp.$\mathbb{C}$
presymp.
trans.hol.
symp.
cplx.
gen.cplx.


$T^*_{\mathbb{C}}M$
order
real-index
type
Dirac$_{\mathbb{C}}$
Poisson
CR
Dirac
presymp.$_{\mathbb{C}}$
trans.CR
presymp.
trans.hol.
symp.
cplx.
gen.cplx.


$T^*_{\mathbb{C}}M$
order
real-index
type
Dirac$_{\mathbb{C}}$
Poisson
CR
Dirac
presymp.$_{\mathbb{C}}$
trans.CR
presymp.
trans.hol.
symp.
cplx.
gen.cplx.

$2+\varepsilon$: C_n-generalized complex structures

$2+\varepsilon$: C_n-generalized complex structures

Example (Moučka, R.; to appear)

Given an anti-Kähler manifold, that is, (g, J) such that

$$g(JX, JY) = -g(X, Y), \qquad {}^{g}\nabla J = 0,$$

we get a curve going from a complex structure to a metric!



I hope it was an interesting tasting session,
I visited Seoul before in 2000, I will be glad
to come back sooner if you'd like to learn more.

I hope it was an interesting tasting session,
I visited Seoul before in 2000, I will be glad
to come back sooner if you'd like to learn more.

Thanks for your attention



RYC2020-030114-I, PID2022-137667NA-I00, CNS2024-154695, PID2025-173178NB-I00

Slides will be available at mat.uab.cat/gentle