MOSER TRICK IN ALGEBRA AND GEOMETRY

Structure of the talk

(1) Moser's trick in symplectic geometry
(2) Poisson versions of the Moser lemma
(3) Moser trick for foliations (a direct proof)
(4) Moser trick for Lie subalgebras

Part (3) and (4) are based on the preprint [-, 26'], arXiv:2606.25848

(M, ω) symplectic manifold

$I \subseteq \mathbb{R}$ an interval containing the origin

Definition: A smooth map

$$\widetilde{\omega} : I \times M \longrightarrow \Lambda^2 T^*M$$

is called a smooth deformation of $\omega \in \Omega^2_{symp}(M)$, if $\omega_t := \widetilde{\omega}(t, \cdot)$ is a symplectic form $\forall t \in I$ and $\omega_0 = \omega$.

A smooth deformation $(\omega_t)_{t \in I}$ of $\omega \in \Omega^2_{symp}(M)$ is called

trivial if there exists an isotopy $(\varphi_t)_{t\in I} \subset \mathrm{Diff}(M)$ satisfying $\varphi_t^* \omega_t = \omega, \forall t \in I$.

Moser's Trick (Moser 65')

(M, ω) compact symplectic mfd

$(\omega_t)_{t\in I}$ smooth deformation of ω.

If there exists $(\sigma_t)_{t\in I} \subset \Omega^1(M)$ such that

$$\frac{d}{dt}\omega_t = d\sigma_t, \quad \forall t \in I$$

then $(\omega_t)_{t\in I}$ is trivial.

Comment: Compactness of M is essential for Moser's trick to hold, e.g. take $(\mathbb{R}^2, \omega_0)$, and

$\omega_t = (e^{-(x^2+y^2)} + t^2)\, dx \wedge dy$, the

$\frac{d}{dt}\omega_t = d\sigma_t$, where $\sigma_t = t(x\,dy - y\,dx)$

BUT $\int_{\mathbb{R}^2} \omega_0 = \pi$, whereas $\int_{\mathbb{R}^2} \omega_t = +\infty$.

Moser's trick is a key technique in proving:

(I) Darboux's Theorem $\Leftarrow$ local normal form theorem

(II) Moser's Stability Theorem:
(M, ω) compact symplectic mfd
$(\omega_t)_{t\in I} \subset \Omega^2(M)$ smooth deform. of ω
If $[\omega_t] \in H^2_{dR}(M)$ does not depend on t, then $(\omega_t)_{t\in I}$ is trivial.

(M, π) Poisson manifold

$x_0 \in M$ zero of $\pi \in \mathfrak{X}^2(M)$, i.e. $\pi_{x_0} = 0$,

$(T_{x_0}M, \pi_0)$ linear Poisson,

$(\pi_0)_v(d_{x_0}f, d_{x_0}g) := d_{x_0}(\pi(df, dg))(v)$,

where $v \in T_{x_0}M$, $f, g \in C^\infty(M)$.

$\mathfrak{g}_{x_0} := (T^*_{x_0}M, [\cdot,\cdot])$ isotropy Lie algebra at $x_0 \in M$

(M, π) is called linearizable at $x_0 \in M$ if there exists a germ of a Poisson diffeomorphism

$\Phi : T_{x_0}M \longrightarrow M$ s.t. $\Phi(0) = x_0$

and $d_0\Phi = id_{T_{x_0}M}$.

Conn's Linearization Thm [Conn 85']

(M, π) Poisson mfd

$x_0 \in M$ zero of $\pi \in \mathfrak{X}^2(M)$

If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ is semisimple of compact type, then (M, π) is linearizable at $x_0 \in M$.

In [Crainic-Fernandes 11'], the authors made use of a specific instance of the theorem below to give a soft geometric proof of Conn's linearization theorem.

Thm (Poisson version (I) of the Moser lemma)

(M, π) compact Poisson mfd

$(\pi_t)_{t\in I} \subset \mathfrak{X}^2(M)$ smooth deform. of $\pi \in \mathfrak{X}^2_{Poisson}(M)$

If $\exists\ (Y_t)_{t\in I} \subset \mathfrak{X}^2(M)$ s.t.

$$\frac{d}{dt}\pi_t = d_{\pi_t}(Y_t),$$

then the flow ϕ_y^t of Y_t, whenever defined, satisfies $(\phi_y^t)^* \pi_t = \pi$.

$(G, \cdot, \pi)$ Poisson-Lie group

Thm [Alekseev-Meinrenken 16'] Let $\mathfrak{g}$ be a coboundary Lie

bialgebra and G^* the Poisson-Lie group with Lie bialgebra $\mathfrak{g}^*$. Then G^* is linearizable at the unit.

The authors proved the result above, making use of another Poisson version of the Moser trick, which has been proved by themselves in an earlier work [Alekseev-Meinrenken 00's]

Thm (Poisson version (II) of the Moser lemma)

(M, π) compact Poisson mfd

$(B_t)_{t \in I} \subset \Omega^2_{cl}(M)$, with $B_0 = 0$

The smooth deformation $(\pi_t)_{t\in I} \subset \mathfrak{X}^2(M)$ of $\pi \in \mathfrak{X}^2_{Poisson}(M)$ given by

$$\pi_t = e^{B_t}\,\pi$$

is trivial, if there exists $(\alpha_t)_{t\in I} \subset \Omega^2(M)$ s.t. $B_t = d\alpha_t$.

$\mathcal{F} \subseteq M$ foliation $\rightsquigarrow$ $F := T\mathcal{F} \subseteq TM$ involutive distribution

$\nu(\mathcal{F}) := \frac{TM}{F}$ normal bundle of $\mathcal{F}$

$\left(\Omega^\cdot(\mathcal{F}; \nu(\mathcal{F})), d_{\nabla^F}\right)$ deformation complex of $\mathcal{F} \subseteq M$.

∇^F: Bott connection

$(F_t)_{t\in I} \subseteq TM$ is called a smooth deformation of $F \subset TM$ if

it is a smooth family of involutive distributions, with $F_0 = F$.

- Fix a Riemannian metric on M.
- $V(F_t) \simeq F_t^\perp$
- $\begin{cases} \pi_t : TM \to F_t \\ \pi_t^\perp : TM \to F_t^\perp \end{cases}$

Proposition [Heitsch 85']

$(F_t)_{t \in I} \subseteq TM$ smooth deform. of $F \subseteq TM$

The element $\frac{d}{dt} \pi_t \in \Omega^1(F; \nu(F))$

is a 1-cocycle.

Moser trick for foliations

[Crainic-Moerdijk 08']

[del Hoyo - Fernandes 19']

If $(F_t)_{t\in I} \subseteq TM$ is a trivial deformation of $F \subseteq TM$, meaning that there exists an isotopy $(\varphi_t)_{t\in I} \subset Diff(M)$ satisfying $(\varphi_t)_* F = F_t$, then there exists a time-dependent vector field $(Y_t)_{t\in I} \subset \mathfrak{X}(M)$ such that

$$\frac{d}{dt}\eta_t = d_{\nabla F_t}(Y_t).$$

Conversely, if M is compact, and such a family $(Y_t)_{t\in I} \subset \mathfrak{X}(M)$ exists, then $(F_t)_{t\in I} \subseteq TM$ is trivial.

Main steps of a direct proof of the theorem above [–, §6']

"Total foliation": $\widetilde{\mathcal{F}} \subseteq M \times I$ foliation tangent to the slices $M \times \{t\}$, satisfying $\widetilde{\mathcal{F}}|_{M \times \{0\}} = \mathcal{F}$.

Notation: $\widetilde{F} := T\widetilde{\mathcal{F}} \subseteq T(M \times I)$.

Direct implication: Let $(Y_t)_{t \in I}$ be the time-dependent vector field generated by the isotopy $(\varphi_t)_{t \in I} \subset Diff(M)$; $Y_t := \left(\frac{d}{dt}\varphi_t\right) \circ \varphi_t^{-1}$.

$D \subseteq T(M \times I)$, $D := \widetilde{F} \oplus \langle \widetilde{Y} \rangle$,

where $\widetilde{Y} := Y_t + \partial_t \in \mathfrak{X}(M \times I)$.

$\widetilde{\varphi} \in Diff(M \times I)$,

$\widetilde{\varphi}(p,t) = (\varphi_t(p), t)$, $\forall (p,t) \in M \times I$

$$(\tilde{\varphi})_*(F \oplus \langle \partial_t \rangle) = D$$

$\Downarrow$

D is involutive

$\Updownarrow$

$$[\Gamma(\tilde{F}), \tilde{Y}] \subseteq \Gamma(\tilde{F})$$

$\Downarrow$

Cohomological equation in the Moser trick.

(For further details, and for the proof of the inverse implication, see [-, 26'], arXiv: 2606.25848)

Example: $\mathbb{T}^2$ standard flat 2-torus

$\alpha: I \to \mathbb{R}$ smooth function with $\alpha'(t) \neq 0$.

$(X_t)_{t\in I} \subseteq \mathfrak{X}(\mathbb{T}^2)$ defined by

$$X_t = \partial_x + \alpha(t)\partial_y .$$

$$F_t := \langle X_t \rangle$$

$$F_t^{\perp} := \langle N_t \rangle, \text{ where}$$

$$N_t := \partial_y - \alpha(t)\partial_x .$$

A short calculation gives that:

$$\frac{d}{dt}\pi_t(X_t) = \frac{\alpha'(t)}{1+\alpha(t)^2}\cdot N_t .$$

If $(F_t)_{t\in I}$ were trivial, then there would exist $S \in \Gamma(F_t^{\perp})$ s.t.

$$\frac{d}{dt}\pi_t = d_{pF_t}(S)$$

Writing $s = f \cdot N_t$, for some $f \in C^\infty(M)$, we get that:

$$\frac{d}{dt} \Pi_t(X_t) = \nabla^{F_t}_{X_t}(f \cdot N_t) = X_t(f) \cdot N_t$$

We conclude that:

$$X_t(f) = \frac{\alpha'(t)}{1+\alpha(t)^2} \neq 0.$$

BUT

$$0 = \int_{\mathbb{T}^2} X_t(f)\, dx\, dy = X_t(f) \cdot \mathrm{Vol}(\mathbb{T}^2)$$

Contradiction!

$(\mathfrak{g}, [\cdot,\cdot])$ Lie algebra

$\mathfrak{h} \subseteq \mathfrak{g}$ Lie subalgebra, $\dim(\mathfrak{h}) = k$

$$Gr_k(\mathfrak{g}) = \{ W \subseteq \mathfrak{g} \mid \dim W = k \}$$

↑ k-Grassmannian of $\mathfrak{g}$.

Definition: A smooth deformation $(\mathfrak{h}_t)_{t \in I} \subset Gr_k(\mathfrak{g})$ of $\mathfrak{h} \subseteq \mathfrak{g}$ is a smooth curve

$$\tilde{\mathfrak{h}} : I \to Gr_k(\mathfrak{g})$$

such that $\mathfrak{h}_t := \tilde{\mathfrak{h}}(t)$ is a Lie subalgebra $\forall t \in I$ and $\mathfrak{h}_0 = \mathfrak{h}$.

$(\mathfrak{h}_t)_{t \in I} \subset Gr_k(\mathfrak{g})$ is called trivial if $\exists (\varphi_t)_{t \in I} \subset Aut(\mathfrak{g})$ s.t. $\varphi_t(\mathfrak{h}) = \mathfrak{h}_t$ and $\varphi_0 = id_{\mathfrak{g}}$.

Thm [-, 26'] Let $\mathfrak{h}$ be a

Lie algebra together with an inclusion $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Furthermore, consider $(\iota_t)_{t \in I} \subset \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ a smooth family of injective Lie algebra morphisms, with $\iota_0 = \iota$.

The deformation

$$(\mathfrak{h}_t)_{t \in I} := (\mathrm{Im}(\iota_t))_{t \in I}$$

is trivial if and only if

$\exists\, (X_t)_{t \in I} \subset \mathrm{Der}(\mathfrak{g})$

satisfying for each $t \in I$:

$$\pi_t^{can} \circ \frac{d}{dt} \iota_t = \pi_t^{can} \circ X_t \circ \iota_t ,$$

where $\pi_t^{can} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}_t$ the canonical projection.

Example: $\mathfrak{g} = \underbrace{\mathfrak{sl}(2)}_{\{H_1, E_1, F_1\}} \oplus \underbrace{\mathfrak{sl}(2)}_{\{H_2, E_2, F_2\}}$

$\mathfrak{sl}(2) = \langle H, E, F \rangle$ satisfying: $[H,E]=2E$, $[H,F]=-2F$, and $[E,F]=H$.

Let $\mathfrak{h} := \langle H \rangle$, together with the inclusion $\iota(H) = H_2$.

Consider the smooth family of embeddings

$$\iota_t : \mathfrak{h} \hookrightarrow \mathfrak{D}_g, \quad H \mapsto \iota_t(H) = H_2 + t \cdot H_1.$$

Therefore:

$$\mathfrak{h}_t := \langle H_2 + t H_1 \rangle \subseteq Gr_2(\mathfrak{D}_g) \simeq \mathbb{RP}^6,$$

and $$\frac{d}{dt} \iota_t = H_1.$$

If $(\mathfrak{h}_t)_{t \in I}$ were trivial, then there would exist $(X_t)_{t \in I} \subset Der(\mathfrak{D}_g)$ s.t.

(*) $\pi_t^{can}(H_2) = \pi_t^{can}(X_t(H_2 + tH_1))$.

But, D_f is semisimple and so $X_t = ad_{Z_t}$, for some $Z_t = (A_t, B_t) \in D_f$.

Consequently, by equation (*), we obtain that

(**) $H_1 - [Z_t, H_2 + t \cdot H_1) = \lambda_t \cdot (H_2 + t \cdot H_1)$

for some smooth function

$\widetilde{\lambda} : I \to \mathbb{R}$, $t \longmapsto \lambda_t := \widetilde{\lambda}(t)$.

BUT, $[Z_t, H_2 + t \cdot H_1) \in \langle E_2, F_1, E_2, F_2 \rangle$

and so by comparing the components of both sides in equation (**), we obtain: $t \cdot \lambda_t = 1$, and $\lambda_t = 0$, which is impossible!

MUCHAS GRACIAS!