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Image zooming based on sampling theorems

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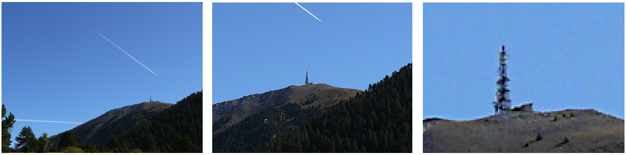
In this paper we introduce two digital zoom methods based on sampling theory and we study their mathematical foundation. The first one (usually known by the names of 'sinc interpolation', 'zero-padding' and 'Fourier zoom') is commonly used by the image processing community.



1 Introduction

Image zooming is a direct application of image interpolation procedures. In fact, a zoom can be easily seen as a homogeneous scaling of the image. Many image interpolation methods have been proposed in the literature and the mathematical foundation for them goes from Fourier analysis (where sampling theory is the key tool) [1], [7], [16], [15], [22], [24], [26], [32], [36], to Wavelet analysis [30], partial differential equations (in combination with the maximum principle and/or some techniques from calculus of variations) [4], [9], [23], [28], [29], [39], fractal geometry [31], [38], statistical filtering [20], mathematical morphology [21], machine learning [12], [17], [18], etc., or to a combination of several of these methods.

With respect to applications of zooming, there are many reasons to be interested in resizing an image retaining as most as possible of the information it contains. Just to mention a few cases where this is a main objective, let us cite what other people say. In [39], the author comments

The applications of image zooming range from the commonplace viewing of online images to the more sophisticated magnification of satellite images. With the rise of consumer-based digital photography, users expect to have a greater control over their digital images. Digital zooming *has a role in picking up clues and details in surveillance images and video. As high-definition television (HDTV) technology enters the marketplace, engineers are interested in fast interpolation algorithms for viewing traditional low-definition programs on HDTV. Astronomical images from rovers and probes are received at an extremely low transmission rate (about 40 bytes per second), making the transmission of high-resolution data infeasible. In medical imaging, neurologists would like to have the ability to zoom in on specific parts of brain tomography images. This is just a short list of applications, but the wide variety cautions us that our desired interpolation result could vary depending on the application and user.*

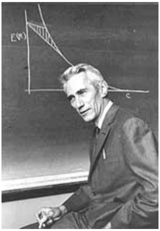
In [12] the authors comment other motivations for zooming algorithms:

A common application occurs when we want to increase the resolution of an image while enlarging it using a digital imaging software (such as Adobe Photoshop®). Another application is found in web pages with images. To shorten the response time of browsing such web pages, images are often shown in low-resolution forms (as the so-called "thumbnail images"). An enlarged, higher resolution *image is only shown if the user clicks on the corresponding thumbnail. However, this approach still requires the high-resolution image to be stored on the web server and downloaded to the user's client machine on demand. To save storage space and communication bandwidth (hence download time), it would be desirable if the low-resolution image is downloaded and then enlarged on the user's machine. Yet another application arises in the restoration of old, historic photographs, sometimes known as image inpainting. Besides reverting deteriorations in the photographs, it is sometimes beneficial to also enlarge them with increased resolution for display purposes.*

Thus, it is evident that there are many interesting applications of zooming. The main aim of this paper is to show how the construction of zooms of digital gray-level images can be approached as a consequence of the well-known digital and analog uniform sampling theorems in dimension two. These theorems are used widely in signal processing and in interpolation (for some applications of these theorems in dimension one we recommend to see [5], [15], [2]), and are a central tool for digital and analog signal processing.

Although 'Fourier zoom' is by no way the best choice for zooming images, since it produces Gibbs oscillations near the boundaries of the image, it is an interesting application of Fourier analysis that can be explained both to mathematicians and engineers at the undergraduate level. It is because of this that we decided to write this paper on vulgarization of mathematics. But we should not loss the opportunity of stand up here that partial differential equations have found a huge field of applications in the mathematical foundation of image processing and, nowadays, they can be considered the truly key stone for the mathematical treatment of images. A very nice paper where this topic is described with the exact detail for the beginner (and a paper we strongly recommend) is [9]. Just to motivate the interest of the reader on this subject, let us comment here that the main idea behind the use of PDE in image processing is a natural one. You can consider an image as a topographic map, with many level curves (i.e., curves where the intensity [or the gray-scale level] is the same). A set of level curves of this kind can be easily zoomed to another one. Then, in order to fill the space of the image between two “adjacent” level curves, you solve a Laplacian (or another PDE satisfying the maximum principle) with boundary values the values associated to the level curves. This method has shown to be very useful not only for zooming images but also for many other purposes. Meanwhile, we should also comment that commercial software, like Photoshop®, does not uses ‘Fourier zoom’ nor PDE based methods but bicuadratic and bicubic interpolation, with the extra aid of some filters after zooming the original image [14].

In section 2 the basic background on digital and analog images is given, together with a formal definition of image interpolation. The next three sections (3, 4 and 5) contain the main results of this work: in section 3, a zooming procedure (sinc interpolation) is derived from the digital uniform sampling theorem, obtaining some formal properties. Section 4 is devoted to the description of the space of d-zoomed digital images in the frequency domain. In section 5, the classical analog sampling theorem of Shannon-Whittaker-Kotelnikov is used, in dimension two, to build another image zooming procedure. Last section contains some examples of the algorithm performance, empirically comparing both approaches.



C. E. Shanon
(1916-2001)

E. T. Whittaker
(1873-1956)

V. Kotelnikov
(1908-2005)

2 Preliminaries

2.1 Digital images

A digital (gray-scale) image is an array of gray-level values. These values (sometimes called 'samples') are a discrete representation of a continuous function (an analog image), after being applied the process of sampling and quantization. In particular, it is usually assumed that there are only 256 gray levels, so that, given a real number $h \in [0, 1]$ the gray level assigned to h is the $k-$th gray level if $h \in \left[\frac{k}{256}, \frac{k+1}{256}\right)$ for $k = 0, 1, \ldots, 254$ and the 256-th gray level corresponds with $h \in \left[\frac{255}{256}, 1\right]$. Moreover, the gray level scale is such that 1 corresponds to the white color and 0 corresponds to the black one.

From a formal point of view, we can think of a digital image of size $N \times M$ as a matrix $I = (I(n,m))_{n=0,\ldots,N-1}^{m=0,\ldots,M-1}$ of real or complex numbers, being the unique digital images we can visualize those with all entries belonging to the real interval $[0,1]$. This model allow us to identify the set of digital images with the complex vector space

$$\begin{aligned}\ell^2(\mathbb{Z}_N \times \mathbb{Z}_M) &= \Big\{ I : \{0, \ldots, N-1\} \times \{0, \ldots, M-1\} \to \mathbb{C} \\ &\qquad\qquad : I \text{ is a map} \Big\} \\ &= \Big\{ I : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} : I \text{ is a map and} \\ &\qquad \forall (k,l) \in \mathbb{Z} \times \mathbb{Z}, \\ &\qquad I(k+N, l) = I(k,l) = I(k, l+M) \Big\},\end{aligned}$$

which is naturally equipped with the following scalar product:

$$\langle I, J\rangle = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} I(n,m)\,\overline{J(n,m)},$$

where the bar denotes complex conjugation. From now on, and for simplicity, we will only consider square images, so that $N = M$.

The standard basis of the space $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ is given by

$$\mathbf{T} = \{\mathbf{T}_{i,j}\}_{0\leq i,\, j<N},$$

where

$$\mathbf{T}_{i,j}(n,m) = 0 \text{ if } (n,m) \neq (i,j) \text{ and } \mathbf{T}_{i,j}(i,j) = 1,$$

so that any digital image, when viewed as a two-dimensional signal in the so called "time domain", is given by the expression

$$I = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\,\mathbf{T}_{i,j}.$$

Now, as it occurs for digital one-dimensional signals, there exists another special orthogonal basis which is naturally interpreted in terms of "frequencies". This basis is given by

$$\mathbf{F} = \{\mathbf{Exp}_{k,l}\}_{0 \leq k,l < N},$$

where

$$\mathbf{Exp}_{k,l}(n,m) = e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

The usual notation for a digital image I when viewed in the "frequency domain", is

$$I = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \widehat{I}(n,m)\, \mathbf{Exp}_{n,m}.$$

and the map $\mathcal{F} : \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ given by

$$\mathcal{F}\Big((I(n,m))_{0 \leq n,m < N} \Big) = \Big(\widehat{I}(n,m) \Big)_{0 \leq n,m < N}$$

is the so called discrete Fourier transform (DFT).

This map is realized by the formula

$$\widehat{I}(k,l) = \frac{\langle I, \mathbf{Exp}_{k,l}\rangle}{\|\mathbf{Exp}_{k,l}\|^2}$$
$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} I(n,m)\, e^{\frac{-2\pi \mathbf{i}(kn+lm)}{N}}.$$

Moreover, the following inversion formula holds

$$I(k,l) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

Now, using the periodicity of the elements of $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, one can also introduce negative frequencies and rewrite the inversion formula as follows

$$I(k,l) = \sum_{n=-N/2}^{N/2}\sum_{m=-N/2}^{N/2} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

Finally, we say that the image I is band-limited with band-size $M < N/2$, and we write this as

$I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$, if

$$I(k,l) = \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}}.$$

2.2 Analog images

Analog images are the elements of $L^2(\mathbb{R}^2)$. Moreover, we will say that $f \in L^2(\mathbb{R}^2)$ is band limited with band size $\leq M$ if

$$\forall |\xi|, |\nu| > M, \hat{f}(\xi,\nu) = 0,$$

where

$$\hat{f}(\xi,\nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp(-2\pi \mathbf{i}(x\xi + y\nu))dxdy$$

denotes the Fourier transform of f. Obviously, these images are precisely those satisfying the formula

$$f(x,y) = \int_{-M}^{M} \int_{-M}^{M} \widehat{f}(\xi,\nu) \exp(2\pi \mathbf{i}(x\xi + y\nu))d\xi d\nu$$

J. Fourier (1768-1830)

2.3 Image interpolation based on filters

Image interpolation [26] is the process of determining the unknown values of an image at positions lying between some known values, called samples. This task is often achieved by fitting a continuous function through the discrete input samples.

Interpolation methods are required in various tasks in image processing and computer vision such as image generation, compression, and zooming. In fact, the last one can be considered as a special case of interpolation, where the zoomed image results from interpolation at certain uniformly distributed samples which are taken to coincide with the original image. This will be our approach in this paper.

The most usual methods to obtain an analog image $f(x, y)$ by using interpolation are expressed as the convolution of the image samples $f_s(k, l)$ with a continuous 2D filter H_{2D}, which is called interpolation kernel:

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} u(k, l) H_{2D}(x - k, y - l)$$

Usually the interpolation kernel is selected to have the following properties [26]:

(a) Separability: $H_{2D}(x, y) = H_1(x) H_2(y)$.

(b) Symmetry for the separated kernels: $H_i(-x) = H_i(x)$, for $i \in \{1, 2\}$.

(c) Image invariance: $H_i(0) = 1$, and $H_i(x) = 0, \forall |x| = 1, 2, \ldots$ and $i \in \{1, 2\}$.

(d) Partition of the unity condition:

$$\sum_{k=-\infty}^{\infty} H_i(d+k) = 1, \forall d : 0 \leq d < 1,$$

$$\text{and } i \in \{1, 2\}.$$

Conditions (a) and (b) are needed to avoid computational complexity. With property (c), we do not modify original image samples. Separated kernels that fulfill (c) are called interpolators, and those which do not verify that, are named approximators. The condition (d) implies that the brightness of the image is not altered when the image is interpolated, i.e. the energy (the standard ℓ^2 norm) of the image remains unchanged after the interpolation.

3 Digital sampling and zoom

Let $I(n,m)$, $0 \leq n,m \leq N-1$, be a digital image of size $N \times N$. It is quite natural to ask for a simple algorithm to zoom this image into another image J of size $dN \times dN$ for $d = 2, 3, \ldots$ (we would say that J is a $(d \times 100)\%$ zoom of I). Clearly, there are several ways to zoom a digital image and all of them imply a certain amount of arbitrariness, since the original image I only gives information about the zoomed image at the points (kd, ld), $k, l \in \{0, \ldots, N-1\}$, where the identities $J(kd, ld) = I(k,l)$ are assumed. Thus, we will have a $(d \times 100)\%$ zoom of I as soon as we define a process to reconstruct J at the other points of the array $\{0, \ldots, dN-1\}^2$. More precisely, given $E \subseteq \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ and $F \subseteq \ell^2(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ two spaces of digital images of size $N \times N$ and $dN \times dN$

respectively, we say that a map

$$Z : E \to F$$

defines a d-zoom process if for all $I \in E$ and all $(k, l) \in \{0, \ldots, N-1\}^2$, we have that

$$Z(I)(kd, ld) = I(k, l).$$

All the standard algorithms used to zoom digital images lie into this notion. Of course, this definition is too general because it allows too many zoom processes. For example, defining $Z(I)(t, s) = 1$ whenever $(t, s) \notin \{1, d, 2d, \ldots d(N-1)\}^2$ would be considered "a very poor zoom" of I. Thus, it is usual to define zooms via some average process which takes into account the topological or morphological properties of the image I, a process that usually receives the name of "spacial zoom". The easiest methods of spacial zooming are *nearest neighbor interpolation* and *pixel replication* (see

[20] for details). These are fast methods and very easy to implement. On the other hand, they produce an undesirable checkerboard effect when applied to get a high factor of magnification. Because of this, several other zooming methods based on the use of bilinear interpolation or low degree spline interpolation (such as bicubic and bicuadratic interpolation) have been proposed (see [20], [26] for instance).

We recall now the digital uniform sampling theorem (see [22], [2] for the proof in dimensions two and one, respectively).

Theorem 1 (Digital uniform sampling theorem). *Let $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ be a digital image of size $N \times N$ and limited band size M. Let d be a divisor of N and let us assume that $d(2M+1) \leq N$. Then I is completely determined by its samples $I(kd, ld)$, $0 \leq k, l \leq r := N/d - 1$. In particular, the follow-*

ing synthesis formula holds

$$I(n,m) = d^2 \sum_{i=0}^{r} \sum_{j=0}^{r} I(di,dj)\,\mathrm{sinc}_M(n-di)\,\mathrm{sinc}_M(m-dj) \qquad (1)$$

with $n,m = 0,\ldots,N-1$, *and where* $\mathrm{sinc}_M(n) = \frac{\sin(\pi(2M+1)n/N)}{N\sin(\pi n/N)}$ *for* $n \neq 0$ *and* $\mathrm{sinc}_M(0) = \frac{2M+1}{N}$.

Taking into account that the only imposed restriction by our definition of a d-zoom process Z is given by the knowledge of some sampling values of the zoomed image $J = Z(I)$, it follows that a natural question is to study wether Theorem 1 is applicable in order to recover all the entries of J from the known samples. More precisely, we would like to know if J can be chosen as a band-limited digital image, and for which band-size we can guarantee a unique $J = Z(I)$ verifying $J(kd,ld) = I(k,l)$. This is solved by the following result:

Theorem 2. *Let $M < \frac{N}{2}$, $d \in \{2, 3, \dots\}$ and let us assume that $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$. Then there exists a unique band-limited digital image $J \in \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ satisfying $J(kd, ld) = I(k, l)$, $k, l \in \{0, \dots, N-1\}$. In particular, there exists a unique d-zoom process (which will be called a sampling d-zoom) $Z_S : \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N) \to \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$.*

Proof. Let $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ be a band-limited digital image of size N and band-size $M < \frac{N}{2}$. We have

$$
\begin{aligned}
I(k,l) &= \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(kn+lm)}{N}} \\
&= \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}((kd)n+(ld)m)}{dN}} \\
&= Z_S(I)(kd, ld),
\end{aligned}
$$

where

$$Z_S(I)(t,s) := \sum_{n=-M}^{M} \sum_{m=-M}^{M} \widehat{I}(n,m)\, e^{\frac{2\pi \mathbf{i}(tn+sm)}{dN}}. \tag{2}$$

Obviously, Z_S is well defined. Moreover, we can use Theorem 1 to guarantee both the uniqueness of Z_S and the fact that

$$Z_S(I)(n,m) = d^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, \mathrm{sinc}_M(n-di)\, \mathrm{sinc}_M \tag{3}$$

where $n, m = 0, \ldots, dN-1$, since $d(2M+1) \leq dN$. $\square$

For an arbitrary digital image $I \in \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ we will have $M = N/2$, so that $d(2M+1) > dN$ violates one of the assumptions of Theorem 1. This is the reason because we have restricted our attention to the space $\mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$ in Theorem 2. Now, in practice this is just a formalism because images from $\ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ are very well approximable by images from $\mathcal{B}_{[N/2]-1}(\mathbb{Z}_N \times \mathbb{Z}_N)$, $[N/2]$ being the integer part of $N/2$. Indeed, they are visually identical. Moreover, we can also use formula (2) instead of (3) to construct Z_S for arbitrary digital images I. Finally, as we will see below, there is an argument (see the comments after Theorem 4 below) which shows that Z_S is the unique digital d-zoom $Z : \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N) \to \mathcal{B}_{N/2}(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ that exists.

Remark 3. It is interesting to note that the map $Z_S : \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N) \to \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$ is an isometry. Thus, we have proved that there exists just one way to introduce a d-zoom between these spaces, and this zoom is also an isometry. On the other hand, if one wants to improve the quality of this zoom, it is natural to look for signals with a bigger band size than the original one. This may be difficult since it is not clear how to introduce the new frequencies from the information given by the old ones.

4 Description, in the frequency domain, of d-zoomed digital images

It is natural to ask about the implications of the zooming condition $I(n,m) = Z(I)(dn, dm)$ on the spectrum of $Z(I)$. This question is fully solved by the following characterization:

Theorem 4. *The following are equivalent claims:*

(a) *Z is a d-zoom process.*

(b) *For all I and all k, l satisfying $0 \leq k, l \leq N-1$, we have that*

$$\widehat{I}(k,l) = \sum_{p=0}^{d-1} \sum_{q=0}^{d-1} \widehat{Z(I)}(k+pN, l+qN). \quad (4)$$

Proof. In order to prove $(a) \Rightarrow (b)$, let us note that

$$\begin{aligned}\widehat{I}(k,l) &= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} I(n,m)e^{\frac{-2\pi\mathbf{i}(kn+lm)}{N}} \\ &= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} Z(I)(dn,dm)e^{\frac{-2\pi\mathbf{i}(kn+lm)}{N}}.\end{aligned}$$

Now, for any $(p,q) \in \{0,1,\cdots,d-1\}^2$ we have that:

$$\begin{aligned}&\widehat{Z(I)}(k+pN, l+qN) \\ &\qquad = \frac{1}{(dN)^2}\sum_{n=0}^{dN-1}\sum_{m=0}^{dN-1} Z(I)(n,m)e^{\frac{-2\pi\mathbf{i}((k+pN)n+(l+qN)m)}{dN}} \\ &\qquad = \frac{1}{(dN)^2}\sum_{n=0}^{dN-1}\sum_{m=0}^{dN-1} Z(I)(n,m)e^{\frac{-2\pi\mathbf{i}(kn+lm)}{dN}}e^{\frac{-2\pi\mathbf{i}(pn+qm)}{d}}\end{aligned}$$

Moreover, it is easy to check that:

$$\sum_{p=0}^{d-1}\sum_{q=0}^{d-1} e^{\frac{-2\pi\mathbf{i}(pn+qm)}{d}} = \begin{cases} d^2 & \text{if } (n,m)\in d\mathbb{Z}\times d\mathbb{Z} \\ 0 & \text{otherwise.}\end{cases}$$

Indeed, the equation above is just a restatement of the fact that the inverse discrete Fourier transform of the $d \times d$ matrix $A = (A_{ij}) = \sum_{0 \leq i,j < d} \mathbf{T}_{i,j}$ (given by $A_{ij} = 1$ for all i, j), is the matrix $\mathbf{T}_{0,0}$. Hence

$$
\begin{aligned}
&\sum_{p=0}^{d-1}\sum_{q=0}^{d-1} \widehat{Z(I)}(k+pN, l+qN) \\
&= \frac{1}{(dN)^2} \sum_{n=0}^{dN-1} \sum_{m=0}^{dN-1} Z(I)(n,m) \left[\sum_{p=0}^{d-1}\sum_{q=0}^{d-1} e^{\frac{-2\pi \mathbf{i}(pn+qm)}{d}} \right] \\
&\qquad e^{\frac{-2\pi \mathbf{i}(kn+lm)}{dN}} \\
&\quad = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} Z(I)(dn, dm) e^{\frac{-2\pi \mathbf{i}(kn+lm)}{N}} = \widehat{I}(k,l),
\end{aligned}
$$

which is what we wanted to prove.

To prove $(b) \Rightarrow (a)$ it is enough to take into account that the computations above show that

$$
\begin{aligned}
&\sum_{p=0}^{d-1}\sum_{q=0}^{d-1} \widehat{Z(I)}(k+pN, l+qN) \\
&\qquad = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} Z(I)(dn, dm) e^{\frac{-2\pi \mathbf{i}(kn+lm)}{N}}
\end{aligned}
$$

holds for all images $Z(I)$. Thus, (4) can be rewritten as

$$\widehat{I}(k,l) = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} Z(I)(dn,dm)e^{\frac{-2\pi\mathbf{i}(kn+lm)}{N}}$$

The proof follows just taking the inverse Fourier transform to prove that $I(k,l) = Z(I)(dk,dl)$. $\square$

Thus, to make a zoom $Z(I)$ of I means to distribute the frequency content of $\widehat{I}(k,l)$ between the frequencies $\widehat{Z(I)}(k+pN, l+qN)$, $p,q \in \{0,1,\cdots,d-1\}$ in such a way that relation (4) holds. As a particular case, 'Fourier zoom' appears when, for $I \in \mathcal{B}_M(\mathbb{Z}_N \times \mathbb{Z}_N)$, we impose $Z(I) \in \mathcal{B}_M(\mathbb{Z}_{dN} \times \mathbb{Z}_{dN})$. This, in conjunction with (4), forces a unique choice for the values $\widehat{Z(I)}(n,m)$, which proves (with a new argument) Theorem 2, this time including also the case $M = N/2$. For example, for $d = 2$ the only distribution of frequencies valid for $Z(I)$ is given by:

- Case $0 \leq k, l \leq M$:

$$\widehat{Z(I)}(k,l) = \widehat{I}(k,l)$$

and

$$\begin{aligned}\widehat{Z(I)}(k+N,l) &= \widehat{Z(I)}(k,l+N)\\ &= \widehat{Z(I)}(k+N,l+N) = 0.\end{aligned}$$

- Case $N - M \leq k < N, 0 \leq l \leq M$:

$$\widehat{Z(I)}(k+N,l) = \widehat{I}(k,l)$$

and

$$\begin{aligned}\widehat{Z(I)}(k,l) &= \widehat{Z(I)}(k,l+N)\\ &= \widehat{Z(I)}(k+N,l+N) = 0.\end{aligned}$$

- Case $N - M \leq l < N, 0 \leq k \leq M$:

$$\widehat{Z(I)}(k,l+N) = \widehat{I}(k,l)$$

and

$$\widehat{Z(I)}(k,l) = \widehat{Z(I)}(k+N,l)$$
$$= \widehat{Z(I)}(k+N,l+N) = 0.$$

- Case $N-M \leq k,l < N$:

$$\widehat{Z(I)}(k+N,l+N) = \widehat{I}(k,l)$$

and

$$\widehat{Z(I)}(k,l) = \widehat{Z(I)}(k+N,l)$$
$$= \widehat{Z(I)}(k,l+N) = 0.$$

- $\widehat{Z(I)}(n,m) = 0$ for all frequencies (n,m) not appearing in the above mentioned cases.

What is more, an analogous result appears for zooming bandlimited images to images containing only high frequencies (i.e. the frequencies appearing in a box of the form

$$\Gamma_M = \{(n,m) : \max\{|n-N|,|m-N|\} \leq M/2\})$$

The only difference is the forced choice for the new distribution of frequencies. Unfortunately, this new zoom is, in general, a very bad one, since the presence of high frequencies everywhere in a picture produces a highly non-smooth aspect. This proves that, for working with digital images, a mixture of mathematical motivation and real experience -guided by experiments- are simultaneously necessary.

Thus, in order to look for a good d-zoom process, one should add certain impositions to equation (4). A very reasonable way would be to minimize certain energy functional defined on the space of solutions of (4) (which is an affine space). Several authors have studied this situation and usually their algorithms take a gradient descend like method with initial value given by the 'Fourier zoom'. Moreover, we can impose that, for a certain fixed linear filter $S : \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, the zoomed image $Z(I)$ belongs to the image of S . This would serve to guarantee a certain regularity on $Z(I)$. In this case, if we write $Z(I) = S(W(I)) = s * W(I)$, then $\widehat{Z(I)} = \widehat{s}\widehat{W(I)}$ and equation (4) is transformed into the new equation

$$\widehat{I}(k,l) = \sum_{p=0}^{d-1}\sum_{q=0}^{d-1} \widehat{s}(k+pN, l+qN) \widehat{W(I)}(k+pN, l+qN). \quad (5)$$

In fact, Guichard and Malgouyres (see [23], [27], [28], [29]) have studied a method of this kind for the construction of zooms. To be more precise, we should mention the filter they use is a little bit different. They take $S : L^2(\mathbb{T}_N \times \mathbb{T}_N) \to L^2(\mathbb{T}_N \times \mathbb{T}_N)$, where $\mathbb{T}_N$ is the unidimensional torus of size N. Thus the relation between I and $Z(I)$ is that I is just a sampling of the analog image $s * W(I)$, where both s and $W(I)$ belong to $L^2(\mathbb{T}_N \times \mathbb{T}_N)$. In any case, they use Poisson summation formula to get an equation for $W(I)$ analogous to the equation (5) above. If we denote by $\mathcal{W}_{I,s}$ the space of solutions $W(I)$ of this new equation, the method proposed by Guichard and Malgouyres consists of looking for an element $W \in \mathcal{W}_{I,s}$ that minimizes the total variation of the image W. They have proved this method is a very reasonable one and, in particular, their results greatly improve 'Fourier zoom'.

At this point, it is interesting to note that Theorem 4 alerts us that to sample an image and to reconstruct it from the samples via formula (1) is not an ideal low-pass filter. This process will produce a band-limited image, but the low frequencies of the new image will not coincide with the original ones. To be more precise, assume that $I \in \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ and d is a divisor of N, $M = N/d$. We define $R(I) \in \ell^2(\mathbb{Z}_M \times \mathbb{Z}_M)$ by $R(I)(k,l) = I(kd, ld)$ for all $0 < k, l \leq M - 1$ and $J(I) = Z_S(R(I))$.

Then Theorem 4 implies that, for example, for $0 \leq k, l \leq M/2$,

$$\widehat{R(I)}(k,l) = \widehat{J}(k,l) = \sum_{p=0}^{d-1}\sum_{q=0}^{d-1} \widehat{I}(k+pM, l+qM).$$

Thus, $J(I)$ is not the result of the application of an ideal low-pass filter to I (which would left unmodified the low frequencies of I) but it is just the unique band-limited image of band size $\leq M/2$ that interpolates the original image I at the points $\{(kd, ld)\}_{0 \leq k,l < M}$. Of course, the same arguments apply to one-dimensional signals.

5 Analog sampling and zoom

The classical analog uniform sampling theorem, in dimension two, reads as follows (see [22], [32]):

Theorem 5 (2-Dimensional analog sampling theorem). *Let $f(x, y)$ be an analog image of finite band* *size $M < \infty$. Then*

$$f(x,y) = 4M^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f\left(\frac{k}{2M}, \frac{l}{2M}\right) \operatorname{sinc}(2Mx - k)\operatorname{sinc}(2My - l)$$

In practice, this theorem implies that, for $f(x,y)$ an analog image of finite band size $M < \infty$, the partial sums

$$P_N(x,y) = 4M^2 \sum_{k=-N}^{N} \sum_{l=-N}^{N} f\left(\frac{k}{2M}, \frac{l}{2M}\right) \operatorname{sinc}(2Mx - k)\operatorname{sinc}(2My - l) \quad (6)$$

are good approximations of $f(x,y)$ inside the square $[\frac{-N}{2M}, \frac{N}{2M}] \times [\frac{-N}{2M}, \frac{N}{2M}]$. In fact, these approximation should be visually good except near the border of the square, where some waves will distort the original image.

Now, the analog d-zoom of $f(x,y)$ is obviously given by the scaling $g(x,y) = f(\frac{x}{d}, \frac{y}{d})$, so that

$$g(x,y) = \int_{-M}^{M} \int_{-M}^{M} \widehat{f}(\xi,\tau) e^{2\pi \mathbf{i}(\frac{x}{d}\xi + \frac{y}{d}\tau)} d\xi d\tau$$

$$= d^2 \int_{-M/d}^{M/d} \int_{-M/d}^{M/d} \widehat{f}(d \cdot u, d \cdot v) e^{2\pi \mathbf{i}(xu+yv)}\, dudv.$$

It follows that $g(x,y)$ is band limited with band size M/d, so that

$$g(x,y) = \frac{4M^2}{d^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g\Big(\frac{kd}{2M}, \frac{ld}{2M}\Big) \\ \operatorname{sinc}\Big(\frac{2M}{d}x - k\Big) \operatorname{sinc}\Big(\frac{2M}{d}y - l\Big) \quad (7)$$

and the partial sums

$$Q_N(x,y) = P_N\Big(\frac{x}{d}, \frac{y}{d}\Big) \quad (8)$$

are good approximations of $g(x,y)$ inside the square $[\frac{-Nd}{2M}, \frac{Nd}{2M}] \times [\frac{-Nd}{2M}, \frac{Nd}{2M}]$.

Let us now assume that $I \in \ell^2(\mathbb{Z}_{2N+1} \times \mathbb{Z}_{2N+1})$ is a digital image which has been constructed by sampling the analog image $f(x, y)$ of finite band size $M < \infty$ on the square $[\frac{-N}{2M}, \frac{N}{2M}] \times [\frac{-N}{2M}, \frac{N}{2M}]$ exactly at the Nyquist rate, so that

$$I(k, l) = f\Big(\frac{k-N}{2M}, \frac{l-N}{2M}\Big),\ k, l = 0, \ldots, 2N. \quad (9)$$

If we denote by J a digital d-zoom of I we have that

$$J(kd, ld) = I(k, l) = g\Big(\frac{d(k-N)}{2M}, \frac{d(l-N)}{2M}\Big), \quad (10)$$

so that these samples are enough to recover $g(x, y)$ approximately inside the square $[\frac{-Nd}{2M}, \frac{Nd}{2M}]$ $\times[\frac{-Nd}{2M}, \frac{Nd}{2M}]$. In particular, using (9) and (8), the formula

$$\begin{aligned} J(n,m) &= Q_N\left(\frac{n-dN}{2M}, \frac{m-dN}{2M}\right) \\ &= \frac{4M^2}{d^2} \sum_{k=-N}^{N} \sum_{l=-N}^{N} I(k+N, l+N) \\ &\qquad \times \operatorname{sinc}\left(\frac{n}{d} - N - k\right) \\ &\qquad \times \operatorname{sinc}\left(\frac{m}{d} - N - l\right) \end{aligned} \tag{11}$$

defines a reasonable digital d-zoom of I.

The previous discussion is summarized on the following result:

Theorem 6. *Let $f(x,y)$ be an analog image of finite band size $M < \infty$ and let us set*

$$I(k,l) = f\left(\frac{k-N}{2M}, \frac{l-N}{2M}\right), \; k,l = 0, \dots, 2N.$$

Then,

$$J(n,m) = \frac{4M^2}{d^2} \sum_{k=-N}^{N} \sum_{l=-N}^{N} I(k+N, l+N)$$
$$\operatorname{sinc}\Big(\frac{n}{d} - N - k\Big) \operatorname{sinc}\Big(\frac{m}{d} - N - l\Big)$$

defines a digital d-zoom of I.

It seems natural to normalize the zoomed image given by (11) to another whose entries belong to the interval $[0,1]$. Surprisingly, this defines an image $Z_A(I)$ which is independent of the value of M and is given by the formula

$$Z_A(I) = \frac{J - \min(J)U}{\max(J)} = \frac{E - \min(E)U}{\max(E)}, \quad (12)$$

where E is defined by $E = \frac{1}{M^2}J$, and $U \in \ell^2(\mathbb{Z}_{d(2N+1)} \times \mathbb{Z}_{d(2N+1)})$ is given by $U(i,j) = 1$ for all i,j. It is important to note that E satisfies $E = E(I)$ (i.e., M has no role for the computation of the entries of E).

Particularly this is the "digital zoom" we wanted to introduce in this section.

Remark 7. Properly speaking, $Z_A(I)$ does not define a zoom of I, since the imposed normalization may have the negative effect that the new image does not satisfy the interpolation condition $I(n, m) = Z_A(I)(dn, dm)$. On the other hand, the experiments show that this is indeed a very reasonable "zoom of I".

Remark 8. It should be noticed that, in practice, the images are not finite band sized, in general, so that a high band size M is needed to get a good approximation of them, according our procedure. This fact implies that our assumption that we have a digital image which has been constructed by sampling an analog image at the Nyquist rate is not a reasonable one, since each pixel covers a square of size bigger than $1/2M$. Moreover, for analog finite band sized images, it is a main problem to know their exact band size M. These objections have motivated the normalized version of the zoom, previously introduced.

Remark 9. In general, the zoomed image defined by (12) is not a band limited digital signal. This could be used to improve the high frequency content of the sampling zoom given by (3). Moreover, the pictures that one visualizes when drawing the frequency content of the zoomed images are highly nonlinear and unpredictable. This should put some light on the difficulty of the problem of improving the sampling d-zoom mentioned at the very end of the section above.

6 A few examples

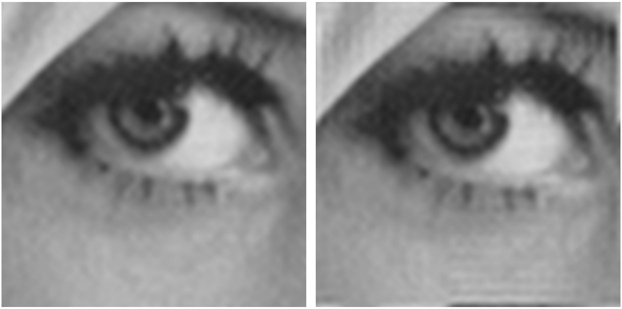
We have implemented in Matlab 7.0 the algorithm for sampling d-zoom of arbitrary images by using formula (3). The algorithm, for $I \in \ell^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, takes a piece T of I and uses the sampling d-zoom Z_S to produce the image $J = Z_S(T)$. Moreover, we have implemented another algorithm which takes the entire image and, after zooming the whole picture with the d-zoom Z_S, extracts the desired fragment. Finally, we have also implemented the "analog" zoom Z_A given by formula (12). In this case we have tested too the zoom on the fragment of the image, and on the entire image.

We show the two algorithms working over a fragment of two well known test images in computer vision: "Lena" and "Living room" (size $= 512 \times 512$). We have compared our approach with the simpler one, pixel replication, using $d = 8$ for both cases, showing original images too.

(a) Original image with fragment marked

(b) Fragment zoomed with pixel replication

Figure 1: “Lena” image



(a) Using all the information from the original image

(b) Using only information from the fragment

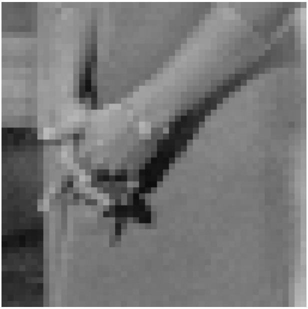
Figure 2: “Lena” image, fragment zoomed with Z_S for $d = 8$

(a) Using all the information from the original image

(b) Using only information from the fragment

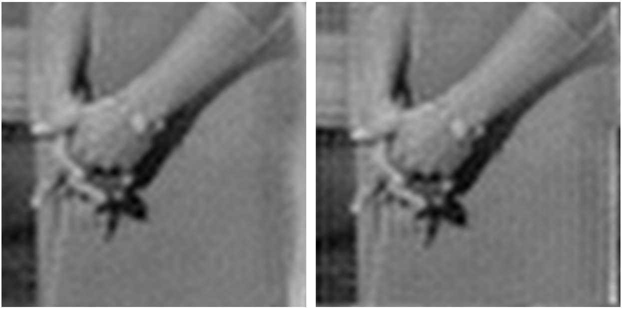
Figure 3: "Lena" image, fragment zoomed with Z_A for $d = 8$

(a) Original image with fragment marked



(b) Fragment zoomed with pixel replication

Figure 4: “Living room” image



(a) Using all the information from the original image

(b) Using only information from the fragment

Figure 5: "Living room" image, fragment zoomed with Z_S for $d = 8$

(a) Using all the information from the original image

(b) Using only information from the fragment

Figure 6: “Living room” image, fragment zoomed with Z_A for $d = 8$

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