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# Polygons, conics and billiards

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In this expository article we will describe some elementary properties of billiards and Poncelet maps and special attention is dedicated to $N$-periodic orbits. In general, problems involving billiards are easy to state and understanding, and difficult or laborious to solve.

## Contents

# Introduction

Billiards are objects that lie at the crossroads of several areas in mathematics and physics, such that: dynamical systems (rotations of the circle, interval exchange transformations, twist maps, K.A.M theory, etc), algebraic geometry, projective and symplectic geometry, ergodic theory, singularity theory, number theory, probability, abelian varieties, classical mechanics, statistical physics, optics, acoustics, Aubry–Mather variational theory and others.

Historically, a systematic study of billiards begun with Birkhoff [13]. But the antecedents are the works of William Chapple, Euler, Poncelet, Jacobi, Chasles, Joachimstall, Cayley, Poincaré, Darboux and many others.

Following L. Bunimovich [18], billiard dynamics can be seen as the propagation of rays of light with an uniform speed within some closed region which boundary consists of dispersing (convex outwards), focusing (convex inwards) and neutral (planar) mirrors. Upon reaching the boundary the rays experience specular (elastic) reflections. i.e., the angle of incidence is equal to the angle of reflection as in mirrors.

In Fig. 1 we show the evolution of a wave that is initially parallel to the elliptic billiard. See also [5] and [75].

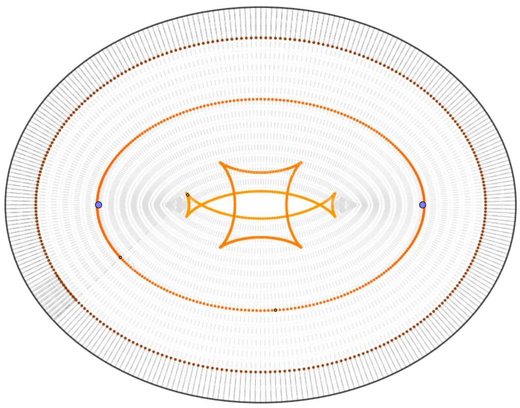

Figure 1: Family of parallel waves to an elliptic billiard.

A billiard trajectory in a smooth convex body $\Omega \in \mathbb{R}^d$ is a polygon $P \subset \Omega$, with all its vertices on the boundary of $\Omega$, and at each vertex the direction of line changes according to the elastic reflection rule. Motion is in a straight line, with unit velocity, except when the ball hits $\partial\Omega$, where it rebounds with incidence angle equal to reflection angle. In a more general setting the billiards can be considered as a geodesic flow on a manifold with a boundary.

Billiards are conservative dynamical systems, and so periodic behavior is expected and, typical questions are:

i) Classification of periodic orbits (elliptic, hyperbolic and others).

ii) How many $N$-periodic orbits do they have?

iii) Is the set of periodic points dense in configuration space?

iv) Can a recurrent orbit be approximated by a periodic one?

v) What is the structure of the envelope of the billiard orbits?

vi) What is the minimal length of a billiard N-periodic orbit in a convex billiard?

vii) How is the billiard theory in other geometries (spherical, hyperbolic, affine, projective, etc)?

ix) When two billiard flows are $C^k-$topologically equivalent? Equivalently, when the associated billiard maps are $C^k$-conjugated?

The main goal of this paper is to present some elementary facts about billiards and Poncelet porisms that is still a source of problems. We will concentrate essentially in the case of billiards in a convex region with smooth boundary. Some proofs are sketched and the reader is invited to look at the original works. Some books introducing billiards and/or Poncelet porisms include [10], [14], [19], [23], [34], [37], [39], [50], [59], [69], [76], [74], [86], [90], [95], [104], [107].

For historical aspects, problems and surveys see [24], [25], [33], [55], [65], [69], [105], [109].

# 1 Circular and polygonal billiards

In this section we will discuss briefly two examples of billiards and the outer billiard.

## 1.1 Circular billiard

Consider a unit circle $\mathcal{C}_1$ given by $x^2+y^2=1$. Let $N=-[x,y]$ and $T=[-y,x]$ the positive frame of $\mathcal{C}_1$. The straight line $\ell(\varphi,\theta)$ passing through $P_1=[\cos\varphi,\sin\varphi]$ and direction $d_1=\cos\theta T(\varphi)+\sin\theta N(\varphi)$ intersects the circle again at the point $P_2=[\cos(\varphi+2\theta),\sin(\varphi+2\theta)]$. The reflection of $d_1=\lambda(P_1-P_2)$ about $N(\varphi+2\theta)$ is the vector $d_2=\cos\theta T(\varphi+2\theta)+\sin\theta N(\varphi+2\theta)$.

Therefore, the billiard map in coordinates $(\varphi,\theta)$ with $\varphi\in[0,2\pi) \bmod 2\pi$ and $\theta\in[0,\pi]$ is the map $T:=\mathbb{S}^1\times[0,\pi]\to\mathbb{S}^1\times[0,\pi]$ with

$$T(\varphi,\theta)=(\varphi+2\theta,\theta),\;\; T(\varphi,0)=(\varphi,0) \;\text{ and }\; T(\varphi,\pi)=(\varphi,\pi) \qquad (1)$$

Therefore a billiard orbit is the polygon with vertices $P_n = T^n(P_1) = [\cos(\varphi + 2n\theta), \sin(\varphi + 2n\theta)]$.

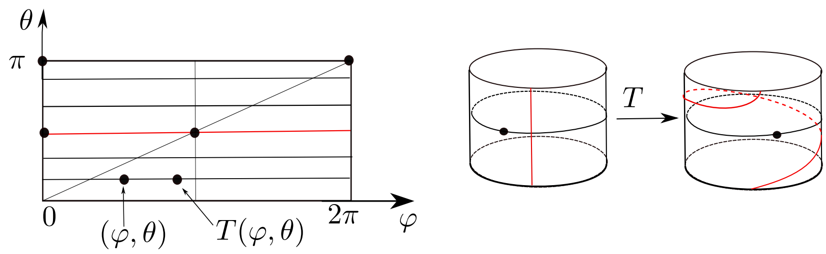


Figure 2: Billiard map defined in the cylinder $\mathbb{S}^1 \times [0,\pi]$ preserving the horizontal fibers.

**Proposition 1.** *The billiard map preserves the fibers $\theta$, i.e., for fixed $\theta$ the map $T_1 : \mathbb{S}^1 \to \mathbb{S}^1$ $T_1(\varphi) = \varphi + 2\theta$ is a rotation of angle $2\theta$. All sides of the billiard orbit have length $2\sin\theta$ and it is tangent to the circle centered at $(0,0)$ and radius $|\cos\theta|$.*

*Proof.* Direct calculations or geometrically observing the symmetry of the circle. □

**Proposition 2.** *When $\frac{\theta}{\pi} = \frac{m}{n}$, $mdc(m,n) = 1$ all billiard orbits are closed of period $n$ making $m$ turns around the origin. When $\frac{\theta}{\pi}$ is irrational all billiard orbits are dense.*

*Proof.* This is a direct consequence of the properties of rotations of the circle. □

**Proposition 3.** *The number of polygonal orbits of period $n$ ($n$-sides) in the circular billiard is $\phi(n)$, where $\phi$ is the Euler function. Moreover, the turning number $m < n$ of a $n$-gonal periodic orbit is $n/mdc(m,n)$. See Fig. 17.*

**Remark 1.** *The study of billiards which are small deformations of the circle, including the elliptic billiard, has a long tradition and the literature about this subject is vast and rich. See for example [6], [9, Chapter XI], [13], [95], [107] and references cited in Math. Review and Zentralblatt.*

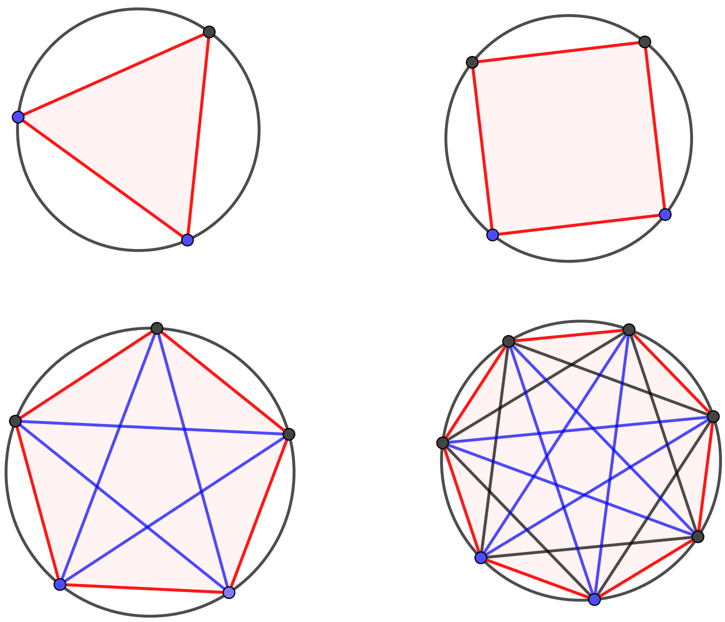

Figure 3: Regular polygons (billiard orbits of period 3, 4, 5 (two types, red and blue, bottom left) and 7 (three types, red, blue and black, bottom right).

## 1.2 Polygonal billiards

Consider a convex polygon $\mathcal{Q}$ counterclockwise oriented with vertices $V_n = \{q_1, \ldots, q_n\}$. A billiard orbit is defined by a map $T : \mathcal{Q} \setminus V_n \times [0, \pi] \to \mathcal{Q} \times [0, \pi]$. As usual, the angle $\theta \in [0, \pi]$ is identified with the direction $d = (\cos\theta, \sin\theta)$ making an angle $\theta$ with a side of $\mathcal{Q}$. A billiard orbit is a polygon $P_n$ with vertices $p_i \in \mathcal{Q}$ and angles $\theta_i$ with the sides of $\mathcal{Q}$. In this case, when a orbit reach a vertex $q_i$ it ends there. See Figs. 4 and 5.

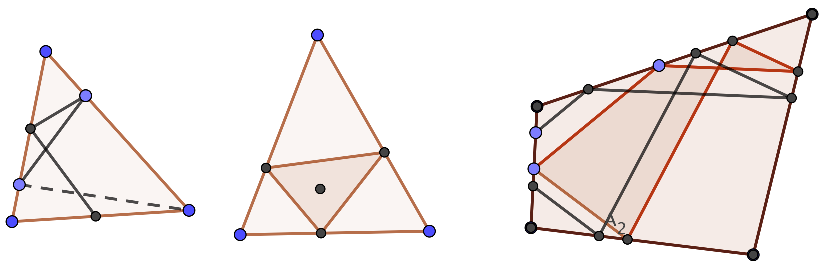

Figure 4: Billiard orbits in triangles and a quadrilateral.

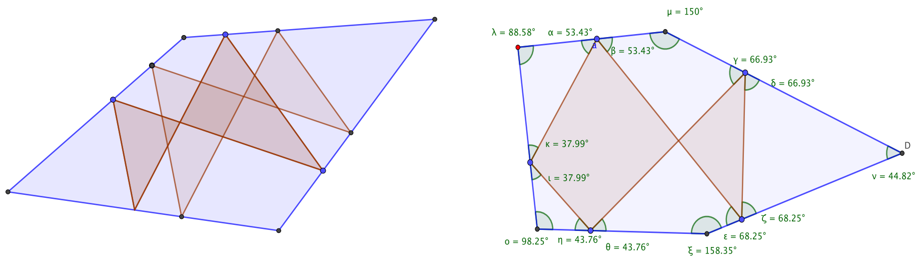


Figure 5: Self-intersecting billiard 4-periodics orbits in a quadrilateral and a 5-periodic in a pentagon.

**Proposition 4.** *In any acute triangle the pedal triangle of the orthocenter is the unique 3-periodic billiard orbit. See Fig. 6.*

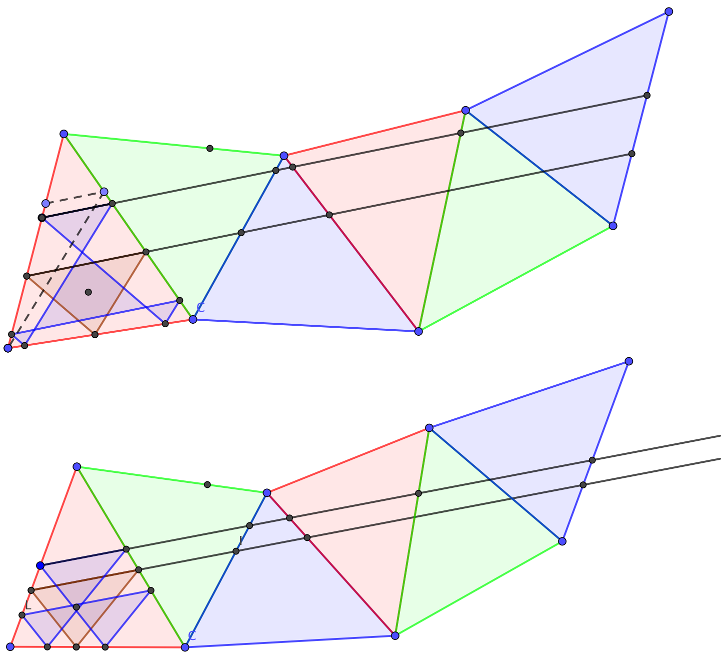

Figure 6: In a acute triangle, the pedal triangle is the unique 3-periodic orbit. All 6-period orbits have the same perimeter.

*Proof.* The geometric proof follows from the Fagnano construction as shown in Fig. 6. It is constructed a chain of triangles such that adjacent triangles are obtained by reflection of a vertex with respect to the opposite side. See also [56] and [58]. We will proceed as follows. Any triangle is similar to a triangle with vertices $A = [-1, 0]$, $B = [1, 0]$, $C = [a, b]$. Let $P_1 = [x_1, 0]$ and $d_1 = [1, m]$. Computing the billiard map $T$ it follows that the third return map is given by:

$$\begin{aligned} T^3(x_1,m) =&[x_4,m_4] \\ x_4 =&\frac{m a_1 a_2 x_1}{(a_3-2b)(a_3+2b)m-4\,b a_3} - \frac{4b\left(b\left(2\,a+a_3\right)m+a a_3-2\,b^2\right)}{(a_3-2b)(a_3+2b)m-4\,b a_3} \\ m_4 =&\frac{(2b-a_3)\left(2\,b+a_3\right)m+4\,a_3\,b}{4\,b a_3\,m+(a_3-2\,b)} \\ a_1 =&a^2+b^2-2\,a+1, \quad a_2=a^2+b^2+2\,a+1, \quad a_3=a^2+b^2-1 \end{aligned} \tag{2}$$

The equation $T^3(x_1,m)=[x_1,m]$ is given by

$$\begin{aligned} (a_3+2bm)(a_3 m-2b)=0 \\ x_1 =\frac{b\left(a_3+2\,a\right)m+a a_3-2\,b^2}{2\,bm+a_3} \end{aligned}$$

Therefore the unique solution is given by: $(x_1,m) = (a,\frac{2b}{a_3})$. Now it is straightforward to check that the solution obtained is the pedal triangle with respect to the orthocenter of the triangle $\{A,B,C\}$. □

**Corollary 1.** *Near a 3-periodic orbit (pedal triangle) there exists a one parameter family of self intersecting 6-periodic orbits as shown in Fig. 6.*

*Proof.* The proof follows directly taking a family of parallel lines of the Fagnano construction. Also it can be checked directly, using symbolic computations, that $T^6(x_1,m)=(x_1,m)$ with $m=\frac{2b}{a_3}$. □

**Proposition 5.** *Consider an obtuse triangle $A=[-1,0]$, $B=[1,0]$, $C=[a,b]$ with vertex $C$ contained in hyperbola $h(x,y)=3x^2-y^2+2x-1=0$. Then there exists a family of 6-periodic orbits which are perpendicular to the sides $AC$ e $BC$. See Fig. 7.*

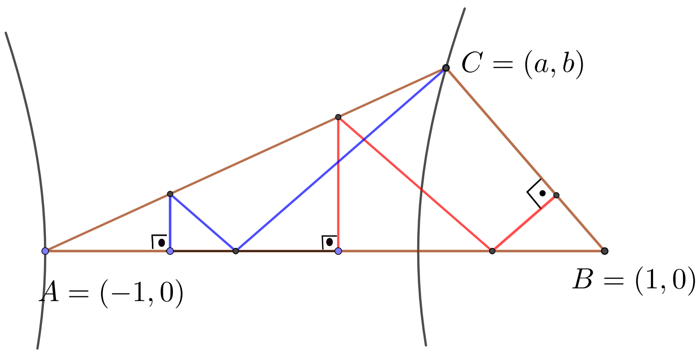


Figure 7: A obtuse triangle with a family of 6-periodic orbits perpendicular to the sides of the triangle.

*Proof.* Let $P_1 = [x_0, 0]$ and $d_1 = [0, 1]$. Computing the first three iterates of the billiard $T$ and imposing the conditions that $P_2 \in AB$, $P_3 \in BC$ and that $P_3 - P_2$ is orthogonal to the side BC it follows that the $T : (\omega_-, \omega_+) \subset AB \to BC$ restricted to the vertical direction $d_1 = [0, 1]$ is defined in the open interval $(\omega_-, \omega_+)$ given by:

$$(\omega_-, \omega_+) = \left(\frac{2a^2 + 2a - 1}{a^2 + a + 1}, \frac{3a}{a^2 + a + 1}\right), \quad a \in (-\frac{1}{2}, 1).$$

The condition that the vertex $C$ is contained in hyperbola $h(x, y) = 0$ is essential to obtain the right calculations. □

**Remark 2.** *In a triangular billiard there are no 2-periodic orbits. In an obtuse triangle there are no 3-periodic periodic orbits. Consider an obtuse triangle defined by $A = [-1, 0]$, $B = [1, 0]$, $C = [a, b]$ and let $H = [a, (1 - a^2)/b]$ be the orthocenter. Then the pedal triangle of $C$ (orthocenter) is a billiard 3-periodic orbit of the acute triangle $\{A, B, H\}$. It is not known if a general obtuse triangle has $n$-periodic orbits. For results in this area see [41] and [99].*

**Remark 3.** *There are $(n-1)!/2$ "essentially" different types of polygons with $n$ vertices. We invite the reader to construct closed billiard orbits in the case $n = 6$ and the Pascal lines associated to the configuration of points. See [113, Pascal lines]*

## 1.3 External and Outer billiard

Let $\mathcal{P}_n$ be a convex polygon. Consider the polygon $\mathcal{P}'_n$ having vertices defined as the intersection of the external bisectors of $\mathcal{P}_n$ as shown in Fig. 8.

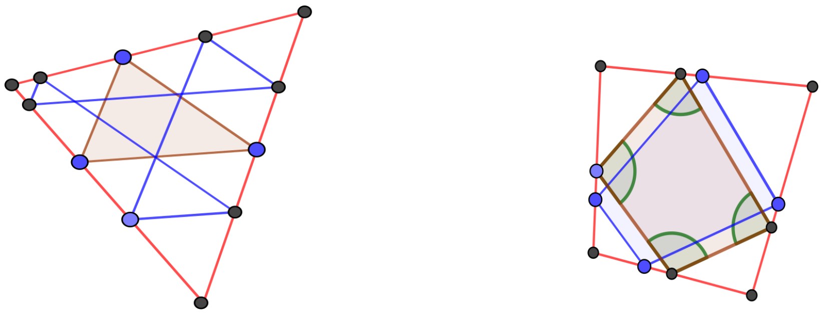

Figure 8: External billiards defined by the external bisectors of a convex polygon.

**Proposition 6.** *In the above conditions we have that $\mathcal{P}_n$ is a $n$-periodic billiard orbit of $\mathcal{P}'_n$. A neighborhood of $\mathcal{P}_n$ is foliated by $n-$periodic orbits when $n$ is even and by $2n$-periodic orbits when $n$ is odd. Moreover these families of orbits are isoperimetric.*

*Proof.* Direct from the construction of the billiard. □

Another possibility to construct "external billiard" is the following.

Consider a convex polygon $\mathcal{P}_n$ with vertices $\{p_1, p_2, \dots, p_n\}$. For a point $P_1$ in the external region delimited by $\mathcal{P}_n$ consider a straight line $\ell_1$ passing though $P_1$ and a vertex of $\mathcal{P}_n$ such that the polygon $\mathcal{P}_n$ is contained in a half plane defined by $\ell_1$.

The reflection of $P_1$ about this vertex defines a point $P_2$ external to $\mathcal{P}_n$. Repeat this process taking into account the orientation (the polygon is always on the same side of the half line) and obtain a sequence $(P_n)$ that is called the *outer billiard orbit*. When a support line $\ell_j$ contains a side of $\mathcal{P}_n$ the reflection is taken about the midpoint of this side. The correspondence $T(P_1) = P_2$ is called outer billiard map. See Fig. 9.

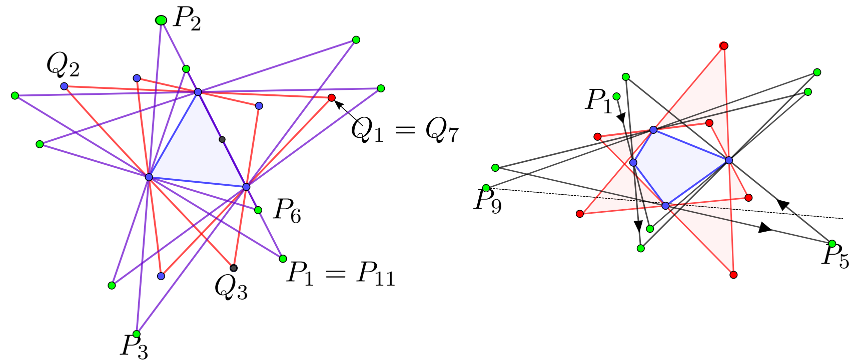


Figure 9: Outer billiards on a convex polygon. Left is shown a 11-periodic with a support line containing a side of the triangle.

**Lemma 1.** *Consider a segment defined by the vertices $A = [0, 0]$ and $B = [a, 0]$. Then the half plane $y > 0$ (resp. $y < 0$) is invariant by $T^2 = T \circ T$ and $T^2$ is a right translation (resp. left translation). See Fig. 10.*

*Proof.* The outer billiard $T$ is given by:

$$T(P) = \begin{cases} -P, & P = (x, y),\ y > 0 \\ 2B - P, & P = (x, y),\ y < 0 \\ B - P, & P = (x, 0) \end{cases}$$

Observe that the orientation is essential to define correctly the transformation $T$ that is discontinuous along the $x$ axis. A particle moving from $P$ to $T(P)$ see the segment $AB$ in the left half plane. Therefore $T^2(x, y) = (x + 2a, y)$ when $y > 0$, and $T^2(x, y) = (x - 2a, y)$ when $y < 0$. □

**Corollary 2.** *The map $T^2 : \mathbb{R}^2 \to \mathbb{R}^2$ induces a twist map $\bar{T} : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$. i.e., a counterclockwise rotation (resp. clockwise) for $y > 0$ (resp. $y < 0$).*

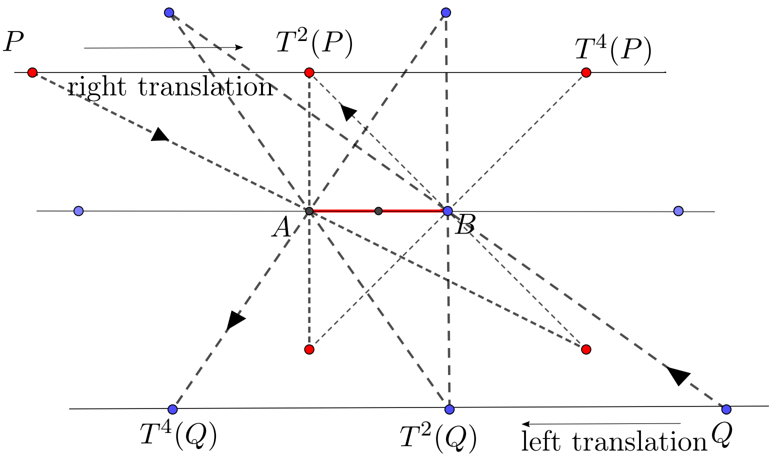


Figure 10: Outer billiard on a segment. The half planes $y > 0$ and $y < 0$ are invariant by $T^2$.

For the study of outer billiard on regular polygons see [108]. For related results see also [4], [16], [40], [48], [57], [60], [84].

The following result is due to R. Schwartz. See [97] and [98].

Consider a convex kite $K_a$ defined by the vertices $A = [0, 1]$, $B = [-1, 0]$, $C = [0, -1]$ and $D = [a, 0]$. An outer billiard on $K_a$ is called special if it is contained in the union of horizontal odd straight lines $\ell_k = \mathbb{R} \times \{2k+1\}$, $k \in \mathbb{Z}$.

**Theorem 1.** *When $a > 0$ is irrational, the outer billiard on $K_a$ has unbounded special orbits.*

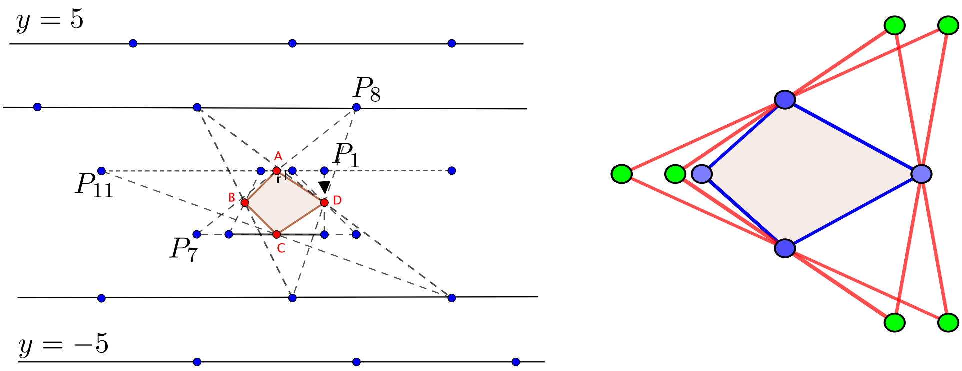


Figure 11: Outer billiards on a kite. Left is shown a possible unbounded orbit. Right is shown a 6-periodic orbit.

This beautiful result, which can be considered a masterpiece of mathematics, solves a problem known as Moser-Neumann problem that asks whether there exists unbounded outer billiard orbits in outer billiard systems. A theory, called "plaid model", was developed in [97]. It is introduced compactification, polyhedral exchange transformations and renormalization.

The dynamic of outer billiards is governed by discontinuous piece-wise linear maps in the plan. In the billiard on the kite above we have the following transformation.

$$T(p)=\begin{cases}2A-p, & p\in R_1\\ 2B-p, & p\in R_2\\ 2C-p, & p\in R_3\\ 2D-p, & p\in R_4\end{cases} \quad \text{and} \quad T(p)=\begin{cases}A+B-p, & p\in R_1\cap R_2\\ B+C-p, & p\in R_2\cap R_3\\ C+D-p, & p\in R_3\cap R_4\\ A+D-p, & p\in R_1\cap R_4\end{cases}$$

The regions $R_i$ are sketched in Fig. 12. For example, the region $R_1$ is defined by the semialgebraic conditions $x \geq 0$ and $(x + ay - a)(x - y + 1) \geq 0$.

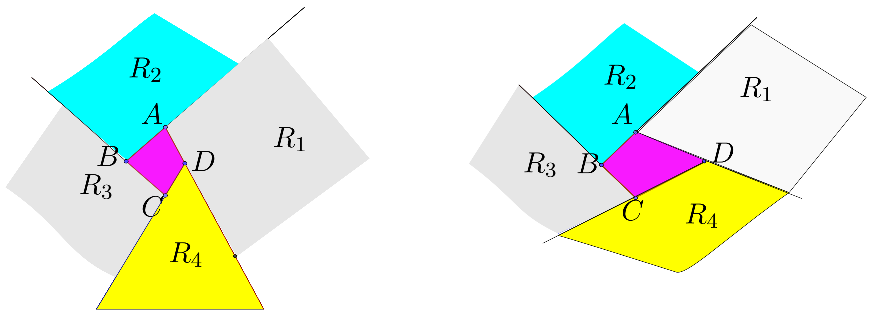


Figure 12: Kite $\{A, B, C, D\}$ and the regions $R_i$. Left when $0 < a < 1$ and right when $a > 1$.

An interactive experiment reveals that the set $\cup_k \ell_k$ is invariant by $T$. Starting at a point $P_1 = (x_1, 1)$, we have $P_2 = T(P_1) = (-x_1, 1)$, $P_3 = T(P_2) = (x_1 - 2, -1)$. Now $T(P_3) = (2 - x_1, -1)$ if $x_1 > 2$ or $T(P_3) = (x_1 - 2a - 2, -1)$ when $x_1 < 2$. In any case, when $T^i(P_1) \in R_1$ it follows that $T^{i+1}(P_1) = (*, 3)$. In general, when $P_n = (*, \pm k) \in R_1 \cup R_3$ we have $T(P_n) = (*, 2 \mp k)$. Also any iterated will be of the form $(f(n)x_1 + g(n, a), h(n))$, where $f, g, h$ are polynomial. As the parameter $a$ is irrational and near the infinity the map is almost the antipodal it is natural to expect unbounded orbits. Also, when $a$ is very large the kite is "almost "a segment and Lemma 1 help us in the intuition of presence of unbounded orbits or $N$-periodic orbits. But periodic orbits are rare in outer billiards. See [110].

## 2 Properties of the chords and variation of length

In this section we obtain some properties of chords of convex curves and applications in billiard orbits.

Consider two regular convex curves $\gamma$ and $\Gamma$ parametrized by arc lengths $s$ and $t$. Let $l(s,t) = |\gamma(s) - \Gamma(t)|$, $\theta(s,t)$ the angle between $\gamma'(s)$ and $V(s,t) = \Gamma(t) - \gamma(s)$ and $\eta(s,t)$ the angle between $\Gamma'(t)$ and $V(s,t)$. See Fig. 13

Consider the Frenet frames $\{\gamma'(s), N_\gamma\}$ and $\{\Gamma'(t), N_\Gamma\}$ along $\gamma$ and $\Gamma$. Denote the curvatures by $k_\gamma$ and $k_\Gamma$.

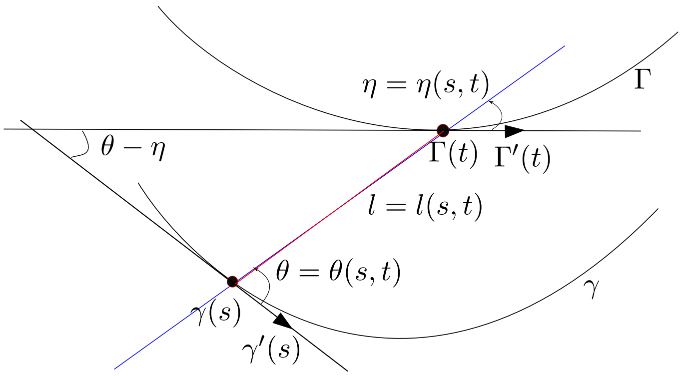


Figure 13: Pair of curves and variations of length and angles.

**Proposition 7.** *In the above conditions it follows that:*

$$\begin{aligned} dl &= -\cos\theta\, ds + \cos\eta\, dt \\ d\theta &= \left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{\sin\eta}{l} dt \\ d\eta &= \frac{\sin\theta}{l} ds - \left(\frac{\sin\eta}{l} + k_\Gamma(t)\right) dt \end{aligned} \tag{3}$$

*Proof.* We have that

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$$

From the equation

$$l^2 = \langle \Gamma(t) - \gamma(s), \Gamma(t) - \gamma(s) \rangle$$

it follows that

$$\begin{aligned} 2l\frac{\partial l}{\partial s} &= -2\langle \gamma'(s), \Gamma(t) - \gamma(s)\rangle = -2l\cos\theta \quad \Rightarrow \quad l_s = -\cos\theta \\ 2l\frac{\partial l}{\partial t} &= 2\langle \Gamma'(t), \Gamma(t) - \gamma(s)\rangle = 2l\cos\eta \quad \Rightarrow \quad l_t = \cos\eta \end{aligned}$$

From the equations

$$l(s,t)\cos\theta = \langle \gamma'(s), \Gamma(t) - \gamma(s)\rangle, \quad l(s,t)\cos\eta = \langle \Gamma'(t), \Gamma(t) - \gamma(s)\rangle$$

it follows that

$$\begin{aligned}
l_s \cos\theta - l\theta_s \sin\theta &= \langle \gamma''(s), \Gamma(t) - \gamma(s)\rangle - \langle \gamma'(s), \gamma'(s)\rangle \\
&= \langle \gamma''(s), l\cos\theta\gamma' + l\sin\theta N_\gamma(s)\rangle - 1 \\
&= l\sin\theta k_\gamma(s) - 1 \\
l_t \cos\theta - l\theta_t \sin\theta &= \langle \gamma'(s), \Gamma'(t)\rangle = \cos(\theta - \eta) = \cos(\eta - \theta) \\
l_s \cos\eta - l\eta_s \sin\eta &= \langle \gamma'(s), \Gamma'(t)\rangle = \cos(\theta - \eta) \\
l_t \cos\eta - l\eta_t \sin\eta &= \langle \Gamma''(t), \Gamma(t) - \gamma(s)\rangle + \langle \Gamma'(t), \Gamma'(t)\rangle \\
&= \langle \Gamma''(t), l\cos\eta\Gamma' + l\sin\eta N_\Gamma(t)\rangle + 1 \\
&= \; k_\Gamma l \sin\eta + 1
\end{aligned}$$

Performing the calculations leads to the result. □

**Proposition 8.** *In the same conditions above but with arc length parameters $s$ and $t$ it follows that*

$$\begin{aligned}
l_{ss} &= \sin\theta\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) \\
l_{st} &= \frac{\sin\theta\sin\eta}{l} \\
l_{tt} &= \sin\eta\left(\frac{\sin\eta}{l} - k_\Gamma(s)\right)
\end{aligned} \tag{4}$$

*Proof.* Follows directly from differentiation of equation (3). □

**Proposition 9.** *In the same conditions above but with arbitrary parameters $s$ and $t$ it follows that*

$$\begin{aligned}
dl &= -|\gamma'(s)|\cos\theta\, ds + |\Gamma'(t)|\cos\eta\, dt \\
d\theta &= |\gamma'(s)|\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{|\Gamma'(t)|\sin\eta}{l}\, dt \\
d\eta &= \frac{|\gamma'(s)|\sin\theta}{l}\, ds - |\Gamma'(t)|\left(\frac{\sin\eta}{l} + k_\Gamma(t)\right)\, dt
\end{aligned} \tag{5}$$

Consider a simple closed convex curve $\gamma$ parametrized by arc length $s$ and of length $L$. A particle moving along straight lines in the region $\Omega = int(\gamma)$ with uniform velocity and reflecting along $\gamma$ elastically and respecting the laws of reflection of geometric optics is called a "billiard orbit". Recall that the reflection about the tangent direction is given by:

$$R(u) = u - 2\langle u, N\rangle N,$$

Let $\ell(s_0, \theta_0)$ be a a straight line (chord) passing through $\gamma(s_0)$ making an angle $\theta_0 \in (0, \pi)$ counterclockwise oriented with the tangent vector $\gamma'(s_0)$. Denote by $s_1$ the coordinate of the intersection of $\ell(s_0, \theta_0)$ with $\gamma$ and $\theta_1$ is the reflected angle of intersection of the chord $\ell(s_0, \theta_0)$ with $\Gamma$ at the point $\Gamma(s_1)$. So it follows that is defined a map

$$T : [0, L] \times (0, \pi) \to [0, L] \times (0, \pi), \quad T(s_0, \theta_0) = (s_1, \theta_1).$$

Identifying the points $s_0$ and $s_0 + L$ the map $T$ can be suspended as map

$$T : \mathbb{R} \times [0, \pi] \to \mathbb{R} \times [0, \pi], \quad T(s_0, \theta_0) = (s_1, \theta_1).$$

The map $T$ is called *billiard map.*

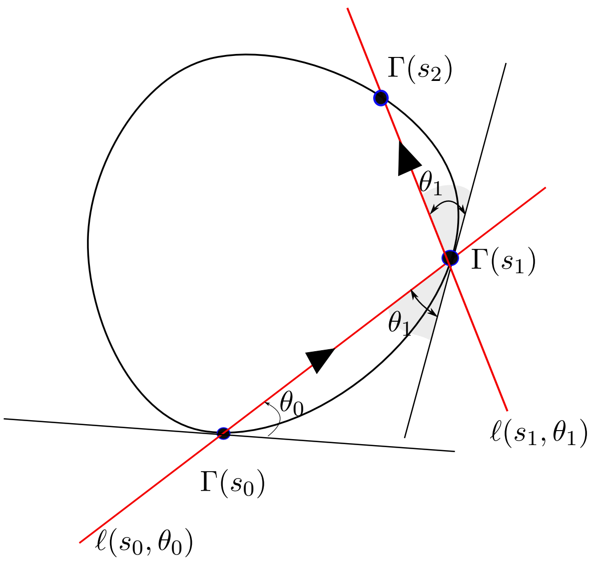


Figure 14: Billiard map $T$ associated to a convex curve.

**Proposition 10.** *Let $r_i = -\cos\theta_i$. In the coordinates $(s, r_0)$ the billiard map $T(s, r_0) = (s_1, r_1)$ is area preserving and $detD(T) = 1$.*

*Proof.* Consider the notation of Proposition 7 and the function $l(s_0, s_1) = |\gamma(s_0) - \gamma(s_1)|$. Then $l_{s_0} = -\cos\theta_0 = r_0$ and $l_{s_1} = \cos\theta_1 = -r_1$. Define $L(s_0, r_0) = l(s_0, s_1(s_0, r_0))$. Then

$$L_{s_0} = r_0 - r_1 \frac{\partial s_1}{\partial s_0}, \quad L_{r_0} = -r_1 \frac{\partial s_1}{\partial r_0}.$$

Differentiating the above relations and using that $L_{s_0 r_0} = L_{r_0 s_0}$ it follows that

$$1 - \frac{\partial r_1}{\partial r_0}\frac{\partial s_1}{\partial s_0} - r_1 \frac{\partial^2 s_1}{\partial r_0 s_0} = -\frac{\partial r_1}{\partial r_0}\frac{\partial s_1}{\partial s_0} - r_1 \frac{\partial^2 s_1}{\partial s_0 r_0}.$$

Therefore,

$$\frac{\partial s_1}{\partial s_0}\frac{\partial r_1}{\partial r_0} - \frac{\partial r_1}{\partial r_0}\frac{\partial s_1}{\partial s_0} = 1.$$

□

In other terms we have.

**Proposition 11.** *The map $T$ preserves the area form $\omega = \sin\theta\, ds \wedge d\theta$.*

*Proof.* See [107]. □

**Remark 4.** *Expressing the map $T$ in the coordinates $(s, p)$, where $p = \sin\varphi$ and $\varphi = \frac{\pi}{2} - \theta$ is the angle between the segment of orbit and the normal vector we write $T(s, p) = (S(s, p), P(s, p))$. Therefore, the derivative of $T$ is given implicitly by:*

$$DT(s,p) = \begin{pmatrix} \frac{\partial S}{\partial s} & \frac{\partial S}{\partial p} \\ \frac{\partial P}{\partial s} & \frac{\partial P}{\partial p} \end{pmatrix} = -\frac{1}{\cos\varphi_1}\begin{pmatrix} \frac{l+r\cos\varphi}{r} & l \\ \frac{l+r\cos\varphi+r_1\cos\varphi_1}{rr_1} & \frac{l+r\cos\varphi_1}{r_1} \end{pmatrix}$$

*In the equation above $r$ (resp. $r_1$) is the radius of curvature of $\Gamma$ at $s$ (resp. $s_1$) and $l = |\Gamma(s) - \Gamma(s_1)|$ is the length of the orbit segment. The same for $\varphi$ and $\varphi_1$. Therefore, $detDT(s, p) = \cos\varphi/\cos\varphi_1$. See [19, Chapter IV] and [68].*

**Proposition 12.** *Let $f : U \to \mathbb{R}$ be a smooth function that the regular levels of $f$ are $T$-invariant. Then $T$ restricted to $f^{-1}(c)$ is conjugated to a rotation.*

*Proof.* Let $T_c = T|_{f^{-1}(c)}$. Consider the 1-form $\psi$ defined by $df \wedge \psi = \omega = \sin\theta\, ds \wedge d\theta$.

Explicitly,

$$\psi = \frac{\sin\theta}{|\nabla f|^2}(-f_\theta ds + f_s d\theta) = -\frac{\sin\theta}{f_\theta} ds.$$

To obtain the last equality we used that $f_s ds + f_\theta d\theta = 0$. By construction $T_c$ preserves $\psi$. □

**Proposition 13.** *Let $\Gamma$ be a planar smooth convex simple curve. The billiard periodic orbits of $T$ are polygonal curves which are critical points of the perimeter function.*

*Proof.* The proof follows from Proposition 7. Consider a 3-periodic billiard orbit with vertices $\Gamma(s_i)$ (i=1,2,3) and lenfth sides $l_i = |\Gamma(s_i) - \Gamma(s_{i+1})|$ for example. Then,

$$\begin{aligned} dl_1 &= -|\Gamma'(s_1)|\cos\theta_1\, ds_1 + |\Gamma'(s_2)|\cos\theta_2\, ds_2 \\ dl_2 &= -|\Gamma'(s_2)|\cos\theta_2\, ds_2 + |\Gamma'(s_3)|\cos\theta_3\, ds_2 \\ dl_3 &= -|\Gamma'(s_3)|\cos\theta_3\, ds_3 + |\Gamma'(s_1)|\cos\theta_1\, ds_1. \end{aligned}$$

Therefore $d(l_1 + l_2 + l_3) = 0$. The general case follows the same ideas. □

**Proposition 14.** *Let $\Gamma$ be a planar smooth convex simple curve. The billiard transformation $T$ associated to $\Gamma$ always has at least two distinct periodic orbits of period 3.*

*Sketch of Proof.* See [22]. Consider the space $\mathcal{T}$ of oriented triangles inscribed in $\gamma$ with the appropriate topology induced from $\mathbb{R}^6$. Considering also degenerated triangles (coincidence of two or three vertices) the space $\mathcal{T}$ is compact. It is a stratified three dimensional manifold with singular boundary. As the perimeter of a triangle (3-periodic orbit) is continuous and $\mathcal{T}$ is compact, there is a non degenerate triangle $T_0 = \{P_1, P_2, P_3\}$ with maximum perimeter. Fixed two vertices, say $P_1$ and $P_2$, the position of the third vertex $P_3$ is such that $|P_1 - P_3| + |P_2 - P_3|$ is extremal. Therefore, the normal at $P_3$ bisects the vectors $P_1 - P_3$ and $P_2 - P_3$. Renumbering the vertices cyclically we obtain three different 3-periodic orbits for which the perimeter is maximum and with the same value. Passing continually from two maximum we need to pass through a saddle point which is also a critical point. The result follows from Moutain Pass Theorem in a compact domain. See Fig. 15.

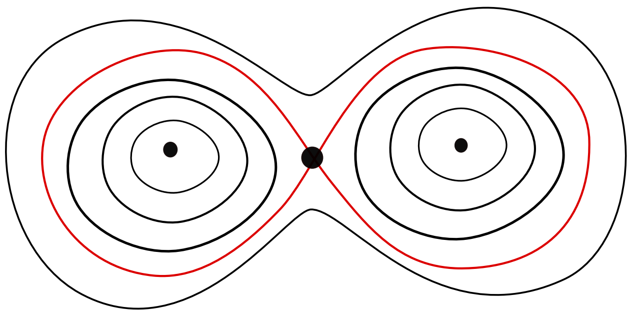

Figure 15: Level sets near two points of maximum in dimension two. In dimension three the saddle point has index 1 (locally a cone).

□

**Remark 5.** *The same result is valid for any $N$-periodic billiard orbit. For more deeper results see [35].*

## 3 About billiard orbits of period 2

A smooth convex billiard has at least two periodic orbits of period 2 (the diameter and its breadth). The diameter is always unstable and the breadth can be stable or unstable. In the ellipse the major contains the foci and is unstable. The breadth is the smaller axis which is stable.

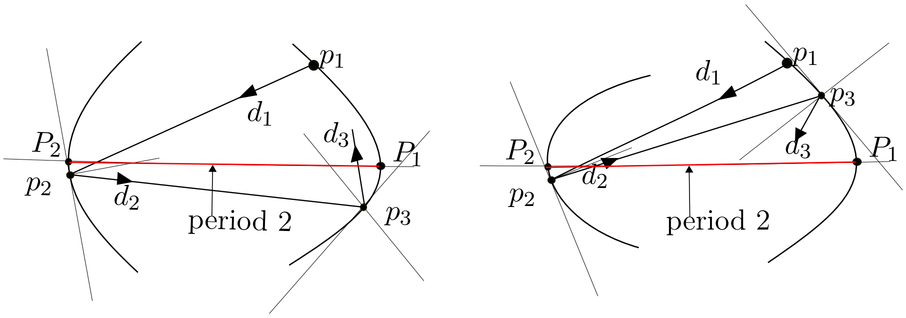


Figure 16: Local billiard map (second return).

**Proposition 15.** *Consider a periodic orbit $P_1P_2$ and suppose that $P_1 = [\frac{l}{2}, 0]$ and $P_2 = [-\frac{l}{2}, 0]$. Suppose that the parametrizations of the boundary near the 2-periodic orbit are given by*

$$\gamma_1(y) = \left[\frac{l}{2} - \frac{a_2y^2}{2} - \frac{a_3y^3}{6} - \frac{a_4y^4}{24} + O(5), y\right] = [h_1(y), y]$$
$$\gamma_2(y) = \left[-\frac{l}{2} + \frac{b_2y^2}{2} + \frac{b_3y^3}{6} + \frac{b_4y^4}{24} + O(5), y\right] = [h_2(y), y]$$

*Let $r_1 = 1/a_2$ and $r_2 = 1/b_2$ the inverses of the curvatures at $P_1$ and $P_2$. Then the derivative of the $T^2 = T \circ T$ is given by*

$$DT^2(0,0) = \begin{bmatrix} \frac{r_2-2\,l}{r_2} & \frac{4(l-r_2)}{r_2} \\ \frac{(r_1+r_2-2l)l}{r_1\,r_2} & \frac{(r_2-2l)r_1+4\,l(l-r_2)}{r_1\,r_2} \end{bmatrix} \tag{6}$$

*Proof.* The proof is straightforward. Using the Implicit Function Theorem we can calculate explicitly the billiard orbit with higher precision for $y$ small. Departing from a point $p_1 = \gamma_1(y)$ and direction $d_1 = (1, m)$ we obtain the point $p_2 = \gamma_2(s(y))$, where $s(y)$ is defined implicitly by the condition that $p_1 + s(y)d_1$ is a point of the curve $\gamma_2$. Therefore,

$$T(p_1, d_1) = (p_2, d_2),\ d_2 = d_1 - 2\langle N(p_2), d_1\rangle N(p_2).$$

Repeat the process and express $T^2(p_1, d_1) = (p_3, d_3)$ in angular coordinates $(\theta, m)$. So, $T(\theta_1, m) = (\theta_3, m_3)$. Here $\theta_1$ is defined by $h_1(y) - \tan\theta_1\, y = 0$. See Fig. 16. □

**Proposition 16.** *The eigenvalues of $DT^2(0,0)$ are given by:*

$$\lambda_{1,2} = \frac{2\,l^2 - 2\,(r_1 + r_2)\,l + r_1\,r_2 \mp 2\,\sqrt{l\,(l - r_1)\,(l - r_2)\,(l - r_1 - r_2)}}{r_1\,r_2}$$

*Therefore, for $min\{r_1, r_2\} < l < max\{r_1, r_2\}$ or $r_1 + r_2 < l$ the 2-periodic orbit is unstable.*

*Proof.* The eigenvalues are defined by $\det(DT^2(0,0)-\lambda)=0$. Direct calculations leads to the result. Under the hypothesis it follows that $0<\lambda_1<1<\lambda_2$ and so the fixed point $(0,0)$ is a hyperbolic saddle and the result follows from Hartman theorem. □

**Proposition 17.** *On the elliptic billiard $\mathcal{E}_{a,b}:\ x^2/a^2+y^2/b^2=1$ it follows that*

$$trace(DT^2(0,0)) = 2\,\frac{8\,a^4-8\,a^2b^2+b^4}{b^4}, \quad \textit{diameter (larger axis)} \tag{7}$$

$$trace(DT^2(0,0)) = 2\,\frac{a^4-8\,a^2b^2+8\,b^4}{a^4}, \quad \textit{width (smaller axis)} \tag{8}$$

*Proof.* Follows directly from equation (6) observing that $l=2a$ and $r_1=r_2=b^2/a$ for the diameter (larger axis) and $l=2b$ and $r_1=r_2=a^2/b$ for the width (smaller axis). □

**Corollary 3.** *Two elliptic billiards $\mathcal{E}_{a,b}$ and $\mathcal{E}_{a_1,b_1}$ are $C^1$-topologically equivalent, if and only if, $a/a_1=b/b_1$, i.e., the ellipses $\mathcal{E}_{a,b}$ and $\mathcal{E}_{a_1,b_1}$ are similar.*

*Proof.* It is well known that $C^1-$equivalence preserves the eigenvalues of derivative of the return map in periodic orbits (see [59], [82]). As $\det DT^2(0,0)=1$ the result follows from equations (7) and (8) applied to the two ellipses. It follows that $(ab_1-a_1b)[(a^2-b^2)b_1^2+a_1^2b^2]=0$ and $(ab_1-a_1b)[(a^2-b^2)a_1^2-a^2b_1^2]=0$. The system above has solution only when $a/a_1=b/b_1$. □

**Remark 6.** *The study of the case of complex eigenvalues is more subtle. It is based on Birkhoff normal forms and Moser's twist theorem [85]. In this case is necessary to consider at least the third jet of $T$ at $0$. In generic case the fixed point $(0,0)$ of $T^2$ is stable, i.e., a neighborhood of the point is foliated by invariant circles (closed curves) which are invariant by $T^2$. See [31] and [68].*

In [96] the following result was obtained.

**Theorem 2.** *The periodic points of a billiard with period 3 form a set of measure zero.*

## 3.1 Billiards in higher dimensions

Consider a compact convex body $K\subset\mathbb{R}^n$.

A billiard orbit is generalized as a sequence of points $(p_i)\in\partial K$ such that

$$n_i=|p_i-p_{i+1}|(p_i-p_{i-1})+|p_i-p_{i-1}|(p_i-p_{i+1})$$

is a support vector of $K$ at $p_i$.

Recall that an outward support vector at $p\in\partial K$ a vector $v$ such that $\langle x-p,v\rangle\le 0$ for all $x\in K$. A point $p\in\partial K$ is called smooth if the outward support vector of $K$ at $p$ is unique. Equivalently, there is a unique hyperplane through $p$ that is disjoint from int$K$.

See [11] for illustration of this kind of generalized billiard orbits in various convex bodies in the plane. See also [9, Chapter XI].

The *width* (or breadth) of $K$ is the infimum of the distances between pairs of parallel supporting hyperplanes of $K$.

The *inradius* of $K$ is the radius of the largest ball (or sphere) contained in $K$. For a triangle $T$ the inradius is the radius of the circle inscribed in $T$.

In a convex body $K$ having boundary the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2 = 1$ the inradius is the small semiaxis $c < b < a$ and the width is $2c$.

The following result due to M. Ghomi [49] establishes bounds for the perimeter of *generalized billiard period orbits*. An algorithm to find the shortest generalised closed billiard orbits on a billiard table was given in [1].

**Theorem 3.** *Let $K \subset \mathbb{R}^n$ be a compact convex body, and $(P_n)$ be a $N$-periodic billiard trajectory in $K$. Then $L(P) \geq 4$ inradius$(K)$. Further, the equality holds for some $(P_n)$, if and only if width$(K) = 2$ inradius$(K)$. In this case, every shortest periodic trajectory of $K$ is a bouncing ball orbit (2-periodic).*

## 4 Poncelet theorem

The classical Poncelet theorem is about a natural generalization of billiards. Consider a pair of nested ellipses $\Gamma$ (outer) and $\gamma$ (inner) oriented by the external normal vector. Consider a point $p \in \gamma$ and $\ell_p$ the tangent line to $\gamma$ at $p$. Let $p_1$ and $p_2$ the intersection of $\ell_p$ with $\Gamma$. The Poncelet map is induced by the correspondence $P_p : \Gamma \to \Gamma$, $P_p(p_1) = p_2$. Through $p_2$ consider the other tangent line to $\gamma$ and let $p_3$ the intersection of this line with $\Gamma$. Therefore, fixed a orientation we have a well defined map $T : \Gamma \to \Gamma$ with $T(p_1) = p_2$, $T(p_2) = p_3$, etc. Changing the orientation defines the inverse $T^{-1}$. See Fig. 17.

The orbit of a point $p_1 \in \Gamma$ is the polygon defined by the vertices $T^k(p_1) = p_k$. A orbit of a point $p_1$ is called $N$-periodic when $T^N(p_1) = p_1$. The $N$-gon is inscribed in $\Gamma$ and circumscribed about $\gamma$. Then Poncelet's Theorem states:

**Theorem 4** (Poncelet's Closure Theorem [86])**.** *Consider a pair of nested ellipses $\gamma \subset \Gamma$. If $T : \Gamma \to \Gamma$ has a periodic orbit for some $p_1$ then all orbits are periodic.*

For a historical and proofs of this theorem see [8], [24], [25], [28], [33] and references therein. See also [9, Chapter IV], [20], [23, Livre III, Chapitres II, III], [27], [50, Chapter 9], [61], [62], [76], [83], [86] and [89].

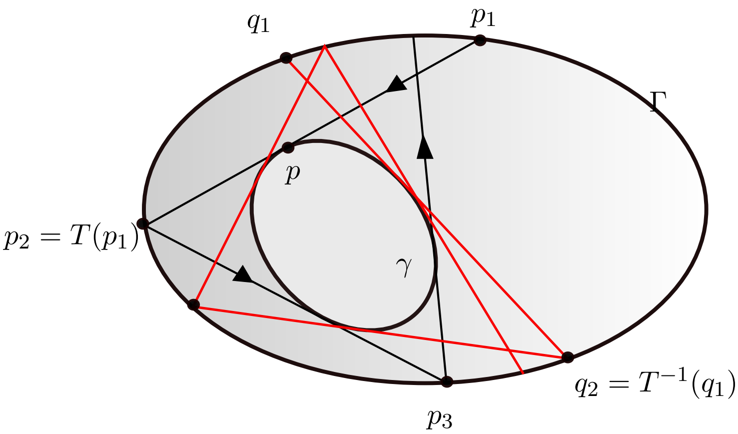


Figure 17: Poncelet map $T : \Gamma \to \Gamma$ with $T(p_1) = p_2$ and $T^{-1} : \Gamma \to \Gamma$ with $T^{-1}(q_1) = q_2$ associated to the pair of nested ellipses $\gamma \subset \Gamma$.

The proof presented below is based in [102]. Consider an ellipse and a circle given by

$$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \mathcal{C}_r : x^2 + y^2 = r^2, \; 0 < r < b < a.$$

For the reduction of a general pair of nested ellipses to the pair above and a proof of Poncelet's Porism theorem see [17].

Let $P = (a\cos\varphi, b\sin\varphi)$ and $P' = (a\cos\varphi, a\sin\varphi)$ as shown in Fig. 18.

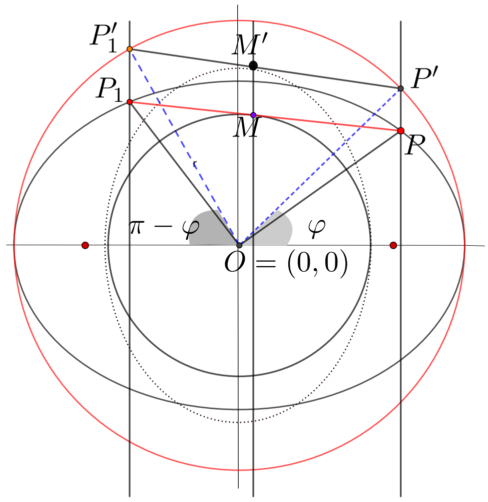


Figure 18: Poncelet maps $T : \mathcal{E}_{a,b} \to \mathcal{E}_{a,b}$ with $T(P) = P_1$ and $T : \mathcal{C}_a \to \mathcal{C}_a$ with $T'(P') = P_1'$ associated to the pairs $\{\mathcal{E}_{a,b}, \mathcal{C}_r\}$ and $\{\mathcal{C}_a, \mathcal{E}_{r,ra/b}\}$.

Consider the tangent line $PP_1$ to $\mathcal{C}_r$ at the point $M$, and let

$$\angle AOP' = \varphi, \quad \angle AOP_1' = \varphi_1.$$

**Proposition 18.** *In the above conditions it follows that:*

$$\left(\frac{d\varphi_1}{d\varphi}\right)^2 = \frac{1 - k^2 \sin^2 \varphi_1}{1 - k^2 \sin^2 \varphi} \tag{9}$$

*Proof.* Let $A(x, y) = (x, \frac{b}{a} y)$ be the affine transformation sending the circle $\mathcal{C}_a$ of radius $a$ to the ellipse $\mathcal{E}_{a,b}$. Define the points $M' = T^{-1}M$ and $P_1' = T^{-1}(P_1)$. The envelope of the family of lines $P'P_1'$ is an ellipse $\mathcal{E}_{r,ar/b}$ of semiaxes $r$ and $\frac{ar}{b} > r$ wich is tangent to $M_1'$. See Fig 18.

The line $P'M'P_1'$ intersects the circle $\mathcal{C}_a$ in equal angles, and it follows that

$$\frac{d\varphi_1}{d\varphi} = \frac{|M'P_1'|}{|M'P'|} = \frac{|MP_1|}{|MP|}.$$

Now observe that

$$\begin{aligned}
|MP|^2 =& |OP|^2 - |OM|^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - r^2 \\
&= a^2 - r^2 - (a^2 - b^2) \sin^2 \varphi \\
&= (a^2 - r^2) \left(1 - \frac{a^2 - b^2}{a^2 - r^2} \sin^2 \varphi\right) \\
&= (a^2 - r^2)(1 - k^2 \sin^2 \varphi), \quad k^2 = \frac{a^2 - b^2}{a^2 - r^2} < 1. \\
|MP_1|^2 =& |OP_1|^2 - |OM_1|^2 = (a^2 - r^2)\left(1 - k^2 \sin^2 \varphi_1\right)
\end{aligned}$$

□

**Proposition 19.** *Let*

$$J(\varphi, \varphi_1) = \int_{\varphi}^{\varphi_1} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}.$$

*Then $J(\varphi, \varphi_1) = cte$ is independent of the initial position $\varphi$.*

*Proof.* Follows directly from equation (9) and Fundamental Theorem of Calculus. □

**Remark 7.** *By the construction above it follows that $A \circ T' = T \circ A$ and therefore the two billiar maps $t$ and $T'$ are conjugated by the affine map $A$.*

**Theorem 5** (Poncelet)**.** *Consider the nested pair of an ellipse $\mathcal{E}_{a,b}$ and a circle $\mathcal{C}_r$. Consider a polygonal orbit $\mathcal{P}_n = P_1P_2 \dots P_n$ inscribed in $\mathcal{E}_{a,b}$ and circumscribed about $\mathcal{C}_r$. If after one revolution along $\mathcal{E}_{a,b}$ the polygon $\mathcal{P}_n$ is closed, then all orbits will be closed.*

*Proof.* Denote $P_k = (a\cos\varphi_k, b\sin\varphi_k)$ with $\varphi_0 = \varphi$. As $J(\varphi,\varphi_1) = \omega =$ cte, it follows that for a n-periodic orbit with turning number $N$ we have $J(\varphi, \varphi + 2\pi N) = n\omega$. □

The following proposition can be found originally in [13].

**Proposition 20.** *Consider a billiard of a convex curve. Then the least number of polygonal orbits of period n (n-sides) is $\phi(n)$, where $\phi$ is the Euler function. Moreover, the turning number $m < n$ of a n-gonal periodic orbits is $n/mdc(m,n)$.*

$$\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right), \quad n = \prod_{i=1}^{r} p_i^{k_i}.$$

**Remark 8.** *For $n \geq 2$ it follows that*

$$\frac{\sqrt{n}}{2} \leq \phi(n) \leq n-1.$$

**Remark 9.** *Given a pair of nested ellipses the algebraic criterion for the existence of $N-$ periodic orbits was obtained by A. Cayley. For a modern exposition see [9, Chapter IV], [32, Chapter V], [53]. See also [76].*

**Remark 10.** *Various examples showing that the Poncelet theorem is not valid for general pair of convex algebraic curves can be found in [20]. In this paper is also derived a proof of Poncelet's theorem.*

**Remark 11.** *For other measure properties of billiards and of the Poncelet map see [17], [27], [55] [71], [73], [78], [112].*

## 5 Elliptic billiards

In this section it will be introduced basic properties of billiards in the region bounded by an ellipse. This is a rich example of the interplay of simple objects (conics, polygons) with more complex ones (integrable systems, variational calculus, caustics, etc). The systematic study started with Birkhoff [13].

**Proposition 21.** *Consider an ellipse and two tangent lines passing through an exterior point as shown in Fig. 19. Let also the two lines passing through the foci. Then we have that $\theta_1 = \theta_2$.*

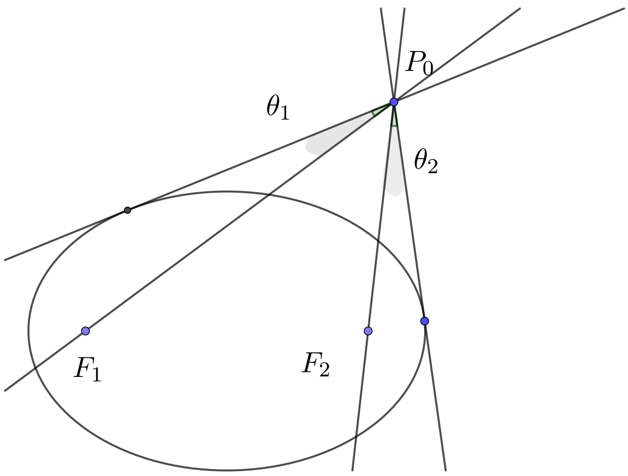


Figure 19: Angles $\theta_1$ and $\theta_2$ are equal

**Proposition 22.** *Consider an ellipse $\mathcal{E}$ with foci $F_1$ and $F_2$. Let $P$ be a point in the region exterior to the ellipse $\mathcal{E}$. Consider the two lines passing through $P$ and tangent to $\mathcal{E}$ at the two points $P_1$ and $P_2$. Then the bisectors of the angles $\widehat{P_1PP_2}$ and $\widehat{F_1PF_2}$ are coincident. When $P \in \mathcal{E}$ the tangent and the normal lines are bisectors of the lines passing through $P$ and the foci $F_1$ and $F_2$. See Figure 20.*

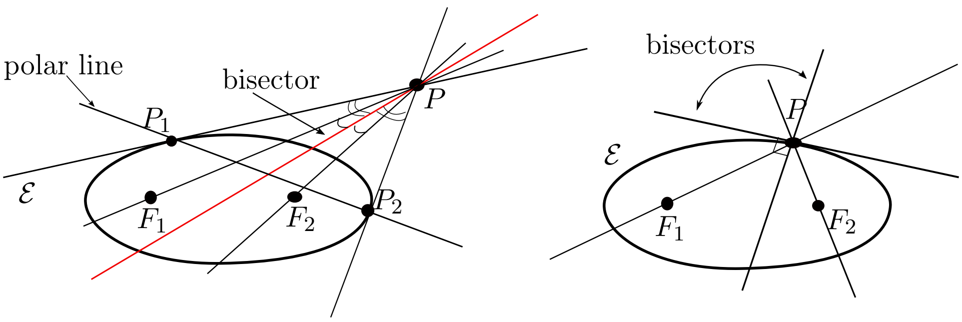


Figure 20: Bisectors of angles $\widehat{P_1PP_2}$ and $\widehat{F_1PF_2}$ are coincident.

*Proof.* See [76, Chapitre I] and [107, Chapter 4].

□

**Proposition 23.** *Consider an ellipse and two tangent lines passing through an exterior point as shown in Fig. 21. Let also a line passing through the foci. Then it follows that that $\alpha_1 = \alpha_2$.*

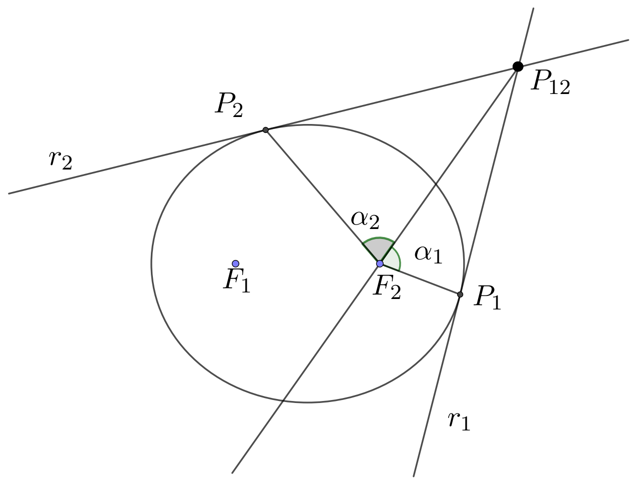


Figure 21: Angles $\alpha_1$ and $\alpha_2$ are equal

*Proof.* An analytic proof can be derived as follows. Consider a parametrization $\Gamma(t) = [a\cos t, b\sin t]$ of the ellipse and consider two points $P_1 = \Gamma(t_1)$ and $P_2 = \Gamma(t_2)$. The tangent line through $P_i$ is given by:

$$r_i : \frac{x\cos t_i}{a} + \frac{y\sin t_i}{b} = 1$$

Therefore $r_1 \cap r_2$ is given by

$$P_{12} = \left[\frac{a(\sin t_1 - \sin t_2)}{\sin(t_1 - t_2)}, \frac{b(\cos t_2 - \cos t_1)}{\sin(t_1 - t_2)}\right]$$

Let $F_2 = [\sqrt{a^2 - b^2}, 0]$. It is straightforward to verify that

$$\cos\alpha_1 = \frac{\langle P_1 - F_2, P_{12} - F_2\rangle}{|P_1 - F_2||P_{12} - F_2|} = \frac{\langle P_2 - F_2, P_{12} - F_2\rangle}{|P_2 - F_2||P_{12} - F_2|} = \cos\alpha_2$$

□

**Proposition 24** ([23, Chapitre III]). *Consider two confocal ellipses $\mathcal{E}$ and $\mathcal{E}_1$ and a point $M \in \mathcal{E}$. Consider the two tangents $\ell_P$ and $\ell_Q$, as shown in Figure 22, intersecting $\mathcal{E}_1$ in $P$ and $Q$. Then $|MP|+|MQ|-arc(P,Q) = cte$, where $arc(P,Q)$ is the length of the elliptic arc with extremal points $P$ and $Q$. In particular, in a billiard triangle $conv[P_1,P_2,P_3]$, $|P_1P_2| + |P_2P_3| + |P_3P_1| - L(\mathcal{E}_1) = c_1$, where $L(\mathcal{E}_1)$ is the length of $\mathcal{E}_1$, and all the billiard triangles have the same perimeter.*

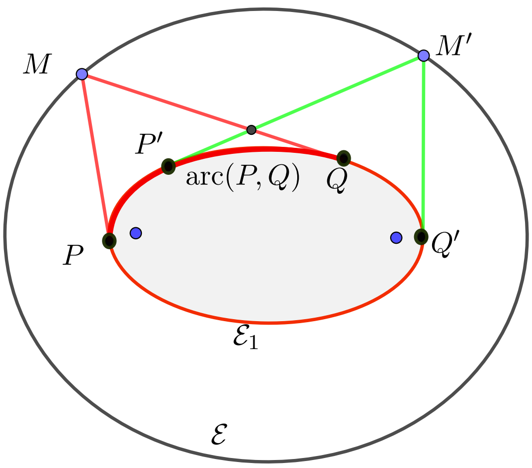


Figure 22: Tangents to a confocal ellipse $\mathcal{E}_1$ and invariance of the length of chords.

*Proof.* See [28], [23, pp. 283-284] and [90, pp. 115-116]. It would be useful to obtain a proof using only the properties of the confocal pair of ellipses. □

The above result is valid for any billiard in a convex curve having caustics.

**Proposition 25.** *Consider a billiard in a region with boundary a convex curve $\Gamma$. Let $\gamma$ be the caustic of a family of orbits as shown in Fig. 23. Then for any $x \in \Gamma$*

$$|x - y| + |x - z| - arc(y, z) = cte.$$

*Here $y, z \in \gamma$ are the points of tangency of the billiard orbit passing through $x$ with the caustic and $arc(x, z)$ is the length of caustic between $y$ and $z$.*

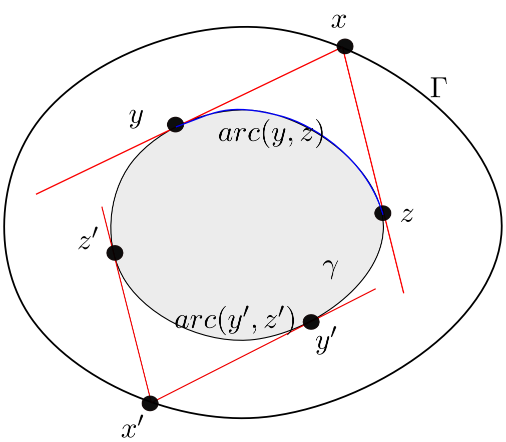


Figure 23: Tangents to a caustic and length of chords.

*Sketch of Proof.* Let $\Gamma(t)$ be a parametrization of the boundary. Consider also local parametrizations $\gamma_1(t)$ and $\gamma_2(t)$ of the caustic $\gamma$ with $\gamma_1(0) = z$, $\gamma_2(0) = y$, and $\Gamma(0) = x$. Suppose that all curves are counterclockwise oriented. Let also the caustic parametrized by natural parameter $s$. Then

$$\gamma(s) = \gamma_1(t) = \Gamma(t) + \lambda(t)d_1(t), \quad \gamma(s) = \gamma_2(t) = \Gamma(t) + \lambda(t)d_21(t).$$

Here $d_1$ and $d_2$ are the directions of the tangent lines $xy$ and $xz$ to the caustic.

Let $l_1(t) = |\Gamma(t) - \gamma_1(t)|$ with $l_1(0) = |x - z|$. Also define $l_2(t) = |\Gamma(t) - \gamma_2(t)|$ with $l_2(0) = |x - y|$.

By Proposition 9 it follows that

$$\begin{aligned} dl_1 &= \cos\eta|\Gamma'(t)|dt - |\gamma_1'(s)|ds \\ dl_2 &= |\gamma_2'(s)|ds - \cos\eta|\Gamma'(t)|dt \end{aligned}$$

Here we used the condition of billiard orbit at the point $x$ (angle of incidence is equal to angle of reflection) and that $\cos\theta_{1,2} = \pm 1$ (caustic is tangent to billiard orbits, taking into account the orientation). Therefore it follows that

$$d(l_1 + l_2) - |\gamma_2'(s)|ds + |\gamma_1'(s)|ds = 0.$$

Integrating it follows that

$$l_1(a) - l_1(0) + l_2(a) - l_2(0) = arc(\gamma_1(0), \gamma_1(a)) - arc(\gamma_2(0), \gamma_2(a))$$

Therefore,

$$l_1(a) + l_2(a) - arc(\gamma_1(a), \gamma_2(a)) = l_1(0) + l_2(0) - arc(\gamma_1(0), \gamma_2(0)).$$

□

**Proposition 26** ([23, Chapitre III])**.** *A billiard triangle has maximum perimeter among all triangles inscribed in $\mathcal{E}$ and the minimum perimeter among all the triangles circumscribed about $\mathcal{E}_c$. If the ellipse $\mathcal{E}$ has axes $a$ and $b$, $a > b$, then the perimeter of all billiard triangles is*

$$L = \frac{2\sqrt{2\,\delta - a^2 - b^2}\,\left(a^2 + b^2 + \delta\right)}{a^2 - b^2}, \quad \delta = \sqrt{a^4 - a^2b^2 + b^4}.$$

*Moreover, supposing $0 < b < a$, it follows that $4a < L < 3\sqrt{3}\,a$.*

Consider the ellipse $E(x, y) = x^2/a^2 + y^2/b^2 - 1 = 0$ be parametrized by $\gamma(u) = (x(u), y(u))$, where $u$ is the angle of the vector $\gamma(u) - 0$ with the $x$-axis. See [51]. Let $h(u) = |\gamma'(u)|$. It follows that

$$\begin{aligned}
\gamma(u) &= \left[\frac{ab\cos u}{\sqrt{a^2\sin^2 u + b^2\cos^2 u}}, \frac{ab\sin u}{\sqrt{a^2\sin^2 u + b^2\cos^2 u}}\right] \\
|\gamma'(u)| = h(u) &= \frac{ab\sqrt{a^4\sin^2 u + b^4 cos^2 u}}{\left(a^2\sin^2 u + b^2\cos^2 u\right)^{3/2}} \\
|N| = |\nabla E(\gamma(u))| &= \frac{2\sqrt{a^4\sin^2 u + b^4\cos^2 u}}{ab\sqrt{a^2\sin^2 u + b^2\cos^2 u}}
\end{aligned} \tag{10}$$

**Proposition 27.** *Consider an ellipse $\mathcal{E}$ and a line $\ell$ passing through $p_0 \in \mathcal{E}$ and making an angle $\theta$ counterclockwise with the tangent line to $\mathcal{E}$ at $p_0$. Then there exists an unique confocal ellipse tangent to $\ell$ and semi axes are given by $a'' = \sqrt{a^2 - r}$ and $b'' = \sqrt{b^2 - r}$ where*

$$r = \frac{\left(a^4\sin^2 u + b^4\cos^2 u\right)\sin^2\theta}{a^2\sin^2 u + b^2\cos^2 u} = M(u)\sin^2\theta$$

*The function $J(u, \theta) = b^2 - r = b^2 - M(u)\sin^2\theta$ is $T$-invariant.*

*Proof.* Straightforward calculations using that the straight line

$$\ell : \; \gamma(u) + s\left[\cos\theta\gamma'(u) + \sin\theta\gamma'(u)^{\perp}\right]$$

has quadratic contact with the confocal ellipse $x^2/(a^2 - r) + y^2/(b^2 - r) = 1$. □

**Proposition 28.** *Restrict to the regular level curves of $J = \lambda$ the billiard transformation $T_\lambda$ preserves the one differential form*

$$d\mu = -\frac{h(u)\sin\theta}{J_\theta}du = \frac{h(u)du}{2\sqrt{M(u)(M(u) - b^2 + \lambda)}}$$

*Proof.* The transformation $T$ preserves the two differential form defined by $\omega = \sin\theta\, h(u)\, du \wedge d\theta$. Here $ds = |\gamma'(u)|\, du = h(u)\, du$ is the arc length of the ellipse. The result follows performing the calculations as in the proof of Proposition 12. □

**Proposition 29.** *In the above conditions it follows that:*

$$d\mu = \frac{k_3(\lambda)\; du}{\sqrt{(1 - k_1(\lambda)\cos^2 u)\,(1 - k_2\cos^2 u)}}$$

$$\begin{aligned}
k_1(\lambda) &= \frac{(a^2 - b^2)(a^2 + 2b^2 - \lambda)}{a^2(a^2 + b^2 - \lambda)} \\
k_2 &= \frac{a^2 - b^2}{a^2} \\
k_3(\lambda) &= \frac{b}{a\sqrt{a^2 + b^2 - \lambda}}
\end{aligned}$$

*Proof.* Follows from direct symbolic computations.

In fact,

$$d\mu = \frac{ab\, du}{\sqrt{\left(b^2\,(2\,b^2-\lambda)\cos^2 u + a^2\,(a^2+b^2-\lambda)\sin^2 u\right)\left(a^2\sin^2 u + b^2\cos^2 u\right)}}$$

□

## 5.1 Joachimsthal integral

**Proposition 30.** *Consider an ellipse $\mathcal{E}$ defined by $\langle Ap, p\rangle = 1$. Let $u$ be an inward unit vector in the direction of the billiard orbit passing through the point $p_0 \in \mathcal{E}$ . Let $T(p_0, u) = (p_1, v)$. Then*

$$\langle Ap_0, u\rangle = -\langle Ap_1, u\rangle = \langle Ap_1, v\rangle$$

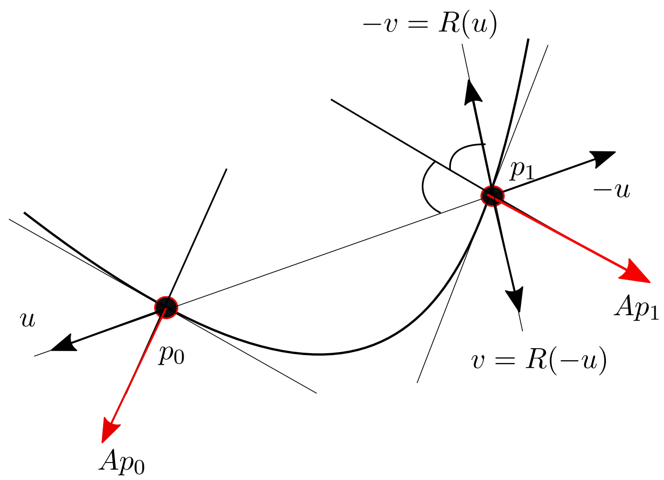


Figure 24: Joahimsthal first integral $\langle Ap_0, u\rangle$ is T-invariant

*Proof.* The tangent space $T_p\mathcal{E}$ is formed of the vectors $u$ such that $\langle Ap, u\rangle = 0$. Therefore $Ap$ is a normal vector to the ellipse at the point $p$. The vector $u$ is proportional to $p_1 - p_0$.

Therefore,

$$\begin{aligned}\langle Ap_0 + Ap_1, p_1 - p_0\rangle &= \langle Ap_0, p_1\rangle + \langle Ap_1, p_0\rangle - \langle Ap_0, p_0\rangle + \langle Ap_1, p_1\rangle \\ &= \langle p_0, Ap_1\rangle - \langle Ap_1, p_0\rangle = 0.\end{aligned}$$

Then,

$$\langle Ap_0, u\rangle = \langle Ap_1, -u\rangle = \langle Ap_1, r(-u)\rangle = \langle Ap_1, v\rangle$$

□

**Definition 1.** *A billiard is called algebraically integrable if there is a polynomial in velocity $u = (u_1, u_2)$ first integral $J(p, v)$ which is a non-constant function on the energy level $\{|u| = 1\}$.*

**Remark 12.** *In the elliptic billiard $x^2/a^2 + y^2/b^2 = 1, a > b > 0$, the function*

$$J(x, y, u_1, u_2) = b^2u_1^2 + a^2u_2^2 - (xu_2 - yu_1)^2$$

*is a first integral.*

We conclude this section with the following theorem. See [19, Chapter IV], [59, Chapter 6], [103, Lecture 10] and [107]. In the language of dynamical systems it can be expressed in terms of rotation numbers and properties of circle diffeomorphisms. See also [79].

**Theorem 6.** *Consider an elliptic billiard defined in the ellipse $\mathcal{E}$ given by $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$. Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ the foci of $\mathcal{E}$. Let $(P_n) = (P_n)_{n\in\mathbb{Z}}$ be a billiard orbit inscribed in $\mathcal{E}$. Then:*

- *i) If the segment of orbit $P_0P_1$ is outside the segment $F_1F_2$ then the caustic of the orbit $(P_n)$ is a confocal ellipse $\mathcal{E}_1$ and the orbit is periodic or dense in the annulus defined by the pair $\{\mathcal{E}, \mathcal{E}_1\}$.*
- *ii) If the segment of orbit $P_0P_1$ intersects the segment $F_1F_2$ then the caustic of the orbit is a confocal hyperbola $\mathcal{H}_1$ and the orbit is periodic or dense in the disk defined by the ellipse $\mathcal{E}$ and the caustic $\mathcal{H}_1$.*
- *iii) If the segment of orbit $P_0P_1$ pass through a focus then the orbit pass through the other focus and is asymptotic to the 2-periodic orbit (diameter of the ellipse $\mathcal{E}$) in the past (backward) and the future (forward).*

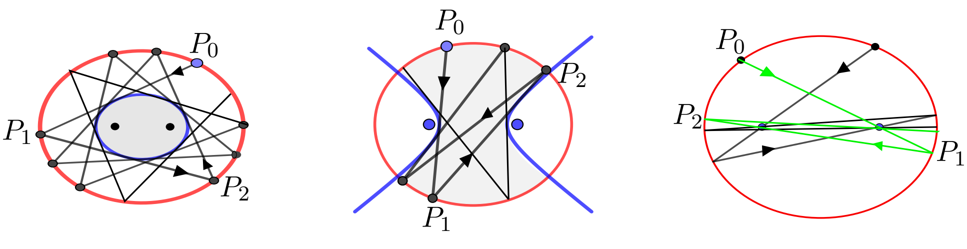


Figure 25: Three types of billiard orbits in the ellipse.

*Proof.* We follow [17] to obtain the billiard map as a composition of two deck transformations. Consider the pair of nested ellipses parametrized by

$$\mathcal{E}: \ f(z, w) = \frac{z^2}{a^2} + \frac{w^2}{b^2} - 1 = 0$$
$$\mathcal{E}_1: \ g(x, y) = \frac{x^2}{a_c^2} + \frac{y^2}{b_c^2} - 1 = 0.$$

A tangent (oriented) line to $\mathcal{E}_1$ (caustic), passing through $q_0 = (x, y)$ is given by

$$h(x, y, z, w) = \frac{xz}{a_c^2} + \frac{yw}{b_c^2} - 1 = 0.$$

Now consider the set $\Sigma = \{(x, y, z, w) : f(z, w) = g(x, y) = h(x, y, z, w) = 0\}$. The set $\Sigma$ is the union of two disjoint circles (curves diffeomorphic to circles) given by $\Sigma_+ = \{p \in \Sigma : xw - yz > 0\}$ and $\Sigma_- = \{p \in \Sigma : xw - yz < 0\}$. Given $q_0 \in \mathcal{E}_1$, let $p_0 = (z, w) \in \mathcal{E}$ such that $(q_0, p_0) \in \Sigma_+$. A line passing through $p_0$ and tangent to $\mathcal{E}_1$ passes through the point $q_1 = (u, v)$ and $(u, v, z, w) \in \Sigma_-$.

The projection $\pi_1 : \Sigma \to \mathcal{E}_1$ is a double cover. The same for the projection and $\pi : \Sigma \to \mathcal{E}$. Now we observe that there is a unique map $\tau : \Sigma_\pm \to \Sigma_\mp$ such that $\tau(x, y, z, w) = (x, y, \bar{z}, \bar{w})$. Here $(\bar{z}, \bar{w})$ is the other point of intersection of the tangent line passing $(x, y)$ with the outer ellipse $\mathcal{E}$.

Also there is a unique map $\sigma : \Sigma_\pm \to \Sigma_\mp$ such that $\tau(x, y, z, w) = (\bar{x}, \bar{y}, z, w)$. The point $q_1 = (\bar{x}, \bar{y}) \in \mathcal{E}_1$ is the in polar line of $p_0 = (z, w)$. Therefore the billiard orbit can be defined as follows. For each $q_i \in \mathcal{E}_1$, let $p_i \in \mathcal{E}$ the point of intersection of tangent line at $q_i$ to $\mathcal{E}_1$ meets $\mathcal{E}$ with $(q_i, p_i) \in \Sigma_+$. Now let $q_{i+1}$ the unique point on $\mathcal{E}_1$ such that $\{q_i, q_{i+1}\}$ are on the two tangent lines to $\mathcal{E}_1$ that pass through $p_i$. Therefore, the map $q_i \to q_{i+1}$ is given by $\sigma \circ \tau$ (resp. $p_i \to p_{i+2}$) is an orientation preserving diffeomorphism on $\mathcal{E}_1$ (resp. on $\mathcal{E}$).

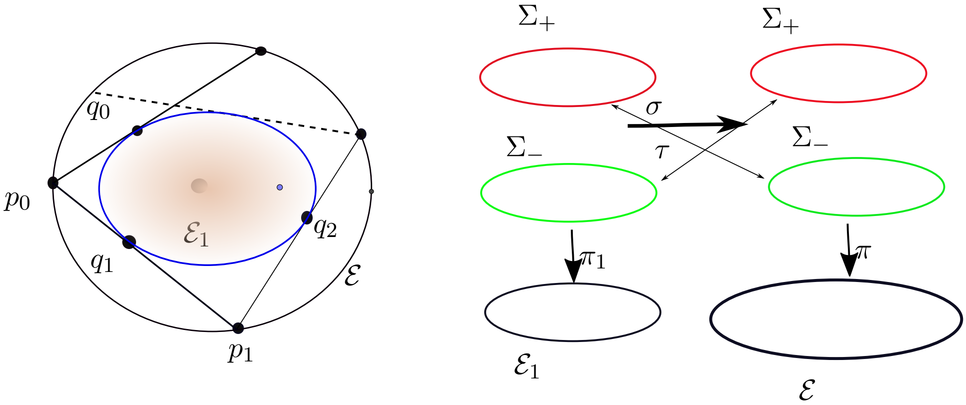


Figure 26: Three types of billiard orbits in the ellipse.

When the caustic is a hyperbola is necessary to consider the second iteration to obtain an orientation diffeomorphism. See [13] and [73].

Finally, when the orbit pass through a focus the billiard map is conjugated to a diffeomorphism of the circle having two hyperbolic fixed points. □

**Remark 13.** *The billiard orbits in the ellipse are "good" approximations of* *geodesics in the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,\ c \approx 0$. *See [13], [10, Chapter 10], [47, Chapter 7] and [72, Chapter 3].*

**Remark 14.** *The rotation number of an elliptic billiard map was obtained in [73]. For basic properties of rotation numbers of circle homeomorphisms see [60], [82], [106]. The phase space of the billiard map in the elliptic billiard is shown in Fig. 27.*

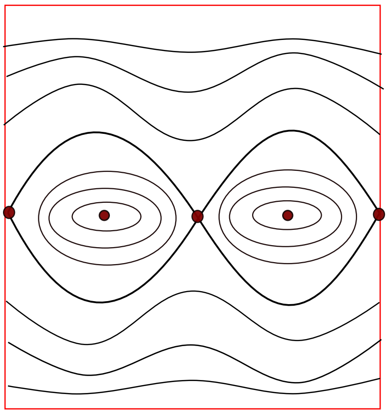

Figure 27: Phase space of the billiard map in the elliptic billiard. Each curve represents the family of lines tangent to a fixed confocal conic (ellipse or hyperbola). The $\infty-$curve represents the orbits that pass through the foci The singular points represent the two axes of the ellipse; see [39, Lecture 28].

**Problem 1.** *Consider two ellipses* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *with eccentricities* $e_1 = c_1/a_1$ *and* $e_2 = c_2/a_2$*. When the two associated elliptic billiard flows are* $C^0$*-equivalents? Is there a classification in terms of eccentricities? Recall that* $C^0$*-equivalence is the usual notion of topological equivalence of flows; see [82]. This means that there is a homeomorphism* $h$ *sending orbits of the first billiard system into orbits of the second one.*

## 6 Other objects associated to the elliptic billiard

In this section it will be considered only elliptic billiards defined in the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$. Consider a family of 3-periodic orbits parametrized by the vertices of the triangle $\Delta(t) = \{P_1(t), P_2(t), P_3(t)\}$ with $P_1(t) = [a\cos t, b\sin t]$. Explicit expressions of the vertices $P_2(t)$ and $P_3(t)$ are given by equations (16) and (17) in Appendix A. See also [42].

The caustic of the 3-periodic orbits is the ellipse $x^2/a_c^2 + y^2/b_c^2 = 1$.

$$a_c = \frac{a\left(\delta - b^2\right)}{c^2},\ b_c = \frac{b\left(a^2 - \delta\right)}{c^2},\ \delta = \sqrt{a^4 - a^2b^2 + b^4}.$$

Associated to a triangle we have the classical center points (incenter, barycenter, circumcenter, orthocenter any more). There are approximately 40000 catalogued triangular centers. These centers are denoted by $X_n$. See [70].

A basic question is about the locus of these triangular centers over the family of 3-periodic orbits. Consider the geometric locus $X_n = \{X_n(t) : t \in [0, 2\pi)\}$, where $X_n(t)$ is the triangular center of $\Delta(t)$.

The fact that $X_1$ is an ellipse was established by O. Romaskevich [93, 94], using techniques of complex algebraic geometry.

The following results were obtained in [42] and [46]. See also [43], [91] and [101].

**Theorem 7.** *The geometric locus $X_1$ is an ellipse of the form $x^2/a_1^2 + y^2/b_1^2 = 1$, where*

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}. \tag{11}$$

*The locus of the Excenters (triangle formed by the intersection of external bisectors) is an ellipse with axes:*

$$a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}$$

*Notice it is similar to the $X_1$ locus, i.e., $a_1/b_1 = b_e/a_e$.*

**Theorem 8.** *The geometric locus $X_2$ is an ellipse of the form $x^2/a_2^2 + y^2/b_2^2 = 1$, where*

$$(a_2, b_2) = k_2\,(a, b)\,, \text{ where } k_2 = \frac{2\delta - a^2 - b^2}{3c^2}$$

*The locus $X_2$ is similar to the billiard.*

**Theorem 9.** *The geometric locus $X_3$ is an ellipse of the form $x^2/a_3^2 + y^2/b_3^2 = 1$, where*

$$a_3 = \frac{a^2 - \delta}{2a}, \quad b_3 = \frac{\delta - b^2}{2b}. \tag{12}$$

*The foci of $X_3$ are $(0, \pm\frac{c^3}{2ab})$ and $\frac{b_c}{a_c} = \frac{a_3}{b_3}$; this means that $X_3$ and is similar to the rotated caustic $\mathcal{E}_c$.*

**Theorem 10.** *The geometric locus $X_4$ is an ellipse of the form $x^2/a_4^2 + y^2/b_4^2 = 1$, where*

$$(a_4, b_4) = \left(\frac{k_4}{a}, \frac{k_4}{b}\right), \; k_4 = \frac{(a^2 + b^2)\delta - 2\,a^2 b^2}{c^2}$$

*The locus $X_4$ is similar to the rotated billiard.*

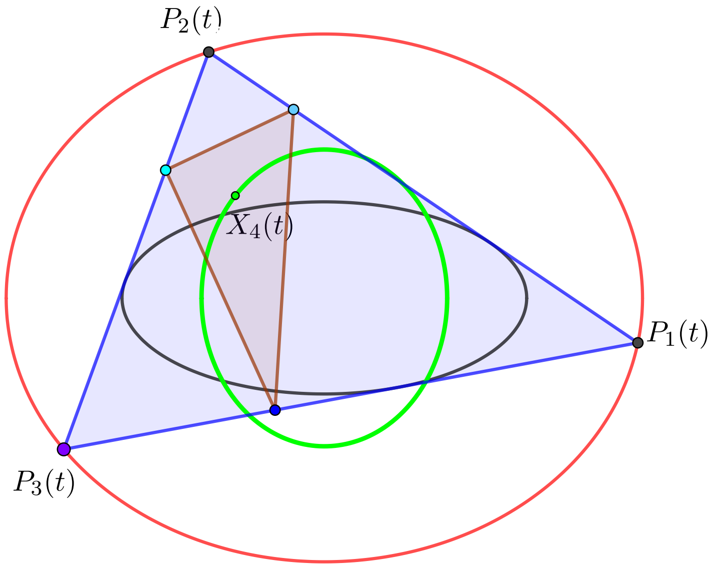


Figure 28: The locus of orthocenters $X_4$ (green) is an ellipse similar to the rotated billiard (red).

**Question 1.** *Let $X_n(t)$ be a triangular center of a 3-periodic orbit $\{P_1(t), P_2(t), P_3(t)\}$ on an elliptic billiard. For each values of $n$, the locus $X_n = \{X_n(t) : t \in [0, 2\pi\}$ is an ellipse? For more details see [45]. The first non elliptic locus is $X_6$ (symmedian points or Lemoine points) [70].*

*The symmedian point is the unique point in the interior of the triangle that minimize the sum of the squares distances from the point to the three sides of the triangle.*

Consider the space $\mathcal{P}_n$ of polygonal lines inscribed in an ellipse $\mathcal{E}_{a,b}$. By Poncelet's theorem, when we have a $n-$periodic elliptic billiard orbit all orbits are $n-$periodic and of the same type. A billiard $n$-periodic orbit will be denoted by the vertices $P_i(t) \in \mathcal{E}_{a,b}$. This defines a smooth path $\gamma : [0, 2\pi) \to \mathcal{P}_n$, $\gamma(t) = [P_1(t), \ldots, P_n(t)]$.

A function $\varphi : \mathcal{P}_n \to \mathbb{R}$ is called *invariant* if is constant in the path of billiard orbits, i.e., $\varphi(\gamma(t)) = \text{cte}$.

**Theorem 11.** *Let $r(t)$ and $R(t)$ be the variables radius of the incircle and circumcircle of $\Delta(t)$. Then $r(t)/R(t)$ is invariant over the 3-periodic billiard orbits and is given by:*

$$\frac{r(t)}{R(t)} = \frac{2(\delta - b^2)(a^2 - \delta)}{(a^2 - b^2)^2} = JL - 4. \tag{13}$$

*where $J = \sqrt{2\delta - a^2 - b^2}/(a^2 - b^2)$ is the Joachimstall constant of the 3-periodic orbit and $L = 2(\delta + a^2 + b^2)J$ is the perimeter of the 3-periodic orbit.*

*Proof.* See [46]. □

**Theorem 12.** *Let $\Delta(t) = \{P_1(t), P_2(t), P_3(t)\}$ be the family of 3-periodic orbits of an elliptic billiard. Let $\theta_i(t)$ the internal angles of $\Delta(t)$. Then the sum of cosines is invariant and*

$$\cos\theta_1(t) + \cos\theta_2(t) + \cos\theta_3(t) = JL - 3$$

*Proof.* See [46] □

**Remark 15.** *The above result was established for $N-$ periodic elliptic billiard orbits as*

$$\sum_{i=1}^{N} \cos\theta_i(t) = JL - N.$$

*Here $J$ is the Joachimstall constant of the $N$-periodic orbit and $L$ its length. For precise statement and proof see [2].*

The power of a point $Q$ with respect to a circle centered at $C_0 = (x_0, y_0)$ of radius $R$ is given by $|Q - C_0|^2 - R^2$. Let $\mathcal{C}(t)$ be the (moving) circumcircle to the 3-periodic billiard orbits.

**Theorem 13.** *The power of the billiard center $O = (0, 0)$ with respect to $\mathcal{C}(t)$ is invariant and equal to $-\delta$.*

*Proof.* See [46]. □

The next result gives a construction of 6-periodic orbits in a billiard pair from 3-periodic billiard orbits in another pair of confocal ellipses. For general constructions see [77], [100].

**Proposition 31.** *Consider a 3-periodic billiard orbit and its antipodal orbit. The six points of intersections of the two triangles are contained in a stationary confocal ellipse $\mathcal{E}_h$: $x^2/a_h^2 + y^2/b_h^2 = 1$ where:*

$$a_h = \frac{\left(\delta - b^2\right)\left(a^2 + b^2 + 2\,\delta\right)\sqrt{2\delta - a^2 - b^2}}{3\left(a^2 - b^2\right)^2}$$
$$b_h = \frac{\left(a^2 - \delta\right)\left(a^2 + b^2 + 2\,\delta\right)\sqrt{2\delta - a^2 - b^2}}{3\left(a^2 - b^2\right)^2}$$

*The pair of ellipses $\{\mathcal{E}_h, \mathcal{E}_1\}$ is a billiard pair having all orbits of period 6. Also the pair $\{\mathcal{E}, \mathcal{E}_h\}$ defines a zig-zag billiard. The orbits have period 12 and the perimeter is constant. See Fig. 29.*

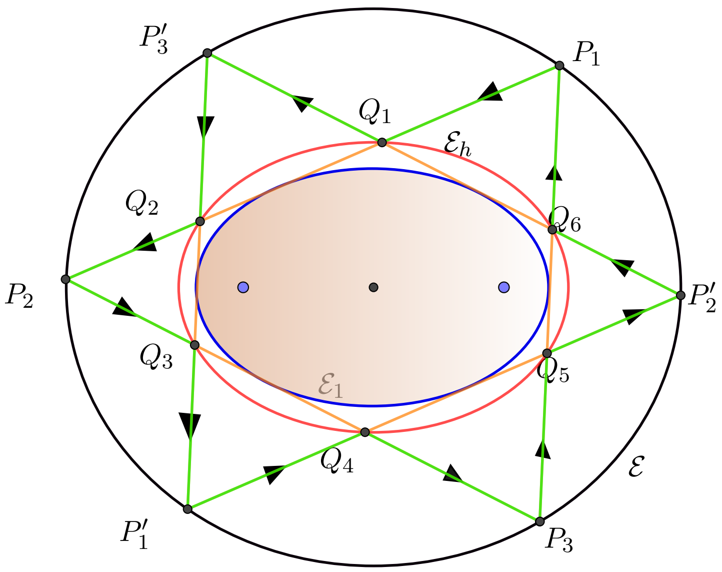


Figure 29: Stationary ellipse and an elliptic billiard pair $\{\mathcal{E}_h, \mathcal{E}_1\}$ with orbits of period 6. Also is shown a zig-zag billiard orbit $P_1 Q_1 P_3' Q_2 \dots P_2' Q_6 P_1$ of period 12 of the pair $\{\mathcal{E}, \mathcal{E}_h\}$.

*Proof.* Left to the reader. The easy part is to obtain the equation of the ellipse $\mathcal{E}_h$. This can be done using the explicit parametrization of 3-periodic orbits given in Appendix A. We need only to solve a linear system of equations. The more laborious part is to show that $\mathcal{E}_h$ is stationary. This can be justified using Poncelet threorem. □

**Remark 16.** *For a historical and projective properties of hexagons inscribed in conics see [26] and [30, Chapter 9]. For example, the Pascal's theorem tell us that an hexagon inscribed in a conic the three pairs of opposite (extended) sides meet in collinear points.*

**Proposition 32.** *Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$. For a 3-periodic billiard orbit with vertices $P_i = [x_i, y_i]$ (i=1,2,3) it follows that:*

$$(x_2\, y_3 - x_3\, y_2)\, x_1\, y_1 + (x_3\, y_1 - x_1\, y_3)\, x_2\, y_2 + (x_1\, y_2 - x_2\, y_1)\, x_3\, y_3 = 0$$

*Proof.* Left to the reader. □

Consider a 3-periodic billiard orbit given by $\{P_1(t), P_2(t), P_3(t)\}$. See Appendix A.

Consider the polygon $C_i(t) = [1/x_i(t), 1/y_i(t)]$. Here $P_i(t) = [x_i(t), y_i(t)]$.

**Lemma 2.** *The polygon $\{C_1(t), C_2(t), C_3(t)\}$ is a segment that can be bounded or unbounded.*

*Proof.* Follows directly from Proposition 32 and the explicit parametrization of this orbit (see Appendix A). □

**Proposition 33.** *The envelope of the family of lines defined by the degenerated polygon $C_i(t)$ (i=1,2,3) is the ellipse $\mathcal{B}$ given by*

$$\frac{a^6x^2}{\left(b^2+\sqrt{a^4-a^2b^2+b^4}\right)^2}+\frac{b^6y^2}{\left(a^2+\sqrt{a^4-a^2b^2+b^4}\right)^2}-1=0.$$

*Moreover, $\mathcal{B}$ is similar to the caustic rotated by $\frac{\pi}{2}$. See Fig. 30.*

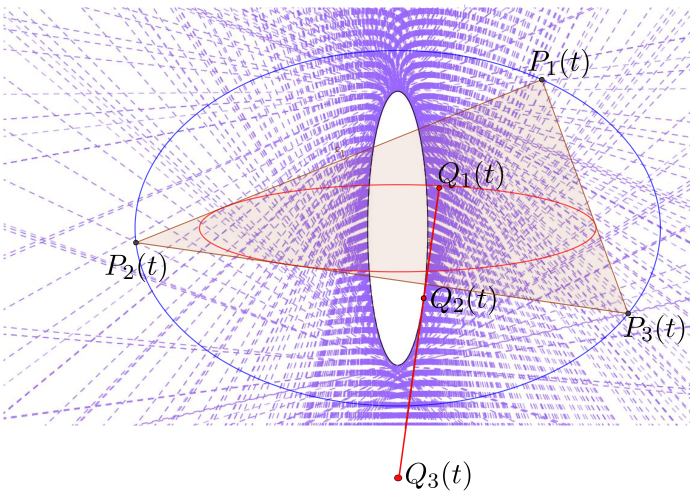


Figure 30: Envelope of the family of lines defined by the degenerated polygons $C_i(t)$.

*Proof.* This can be done from the definition of envelope of the family of lines containing the vertices $C_i(t)$. The calculations are long, but straightforward. It is helpful to use symbolic computations here. □

**Proposition 34.** *Let $\mathcal{E}$ be the circumellipse of a triangle $\Delta$ centered in the triangle center $X_9$ (mittenpunkt) of $\Delta$. Then $\mathcal{E}$ is an elliptic billiard having $\Delta$ as a 3-periodic orbit. The mittenpunkt is stationary over the family of 3-periodic orbits. See Fig. 31.*

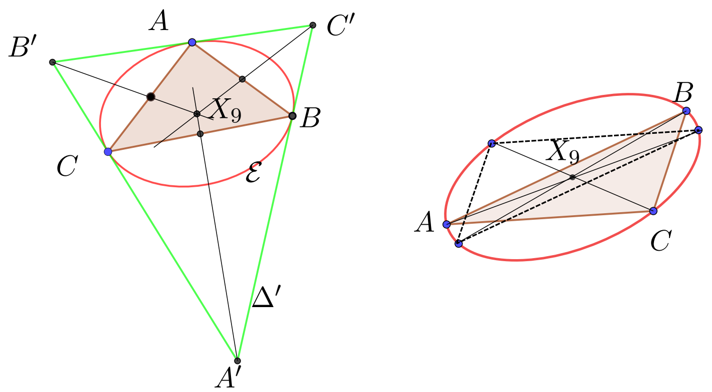


Figure 31: Triangles $\Delta = \{A, B, C\}$ and $\Delta' = \{A', B', C'\}$ (excentral triangle) and the elliptic billiard $\mathcal{E}$ centered at $X_9$.

*Proof.* See [43] and [70]. Recall that $X_9$ is the point of concurrence of lines drawn from each excenter (intersection of external bisectors of $\Delta$) to the midpoint of the corresponding side of the triangle. See Fig. 31. □

**Proposition 35.** *Let $\mathcal{E}_1$ be the inellipse of a triangle $\Delta$ centered in the triangle center $X_9$(mittenpunkt) of $\Delta$. Then $\mathcal{E}_1$ is the caustic of an elliptic billiard $\mathcal{E}$ having $\Delta$ as 3-periodic orbit. The mittenpunkt is stationary over the family of 3-periodic orbits. See Fig. 32.*

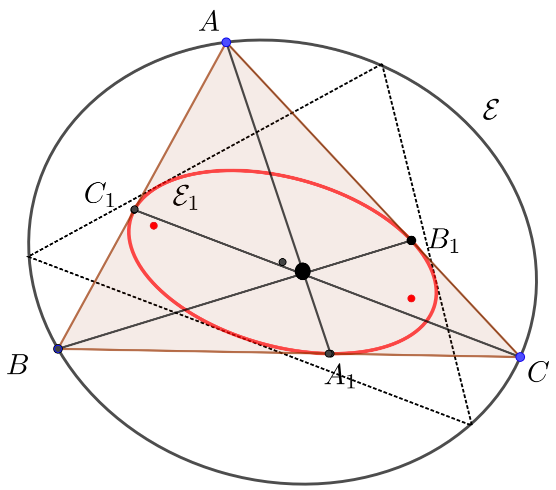


Figure 32: Triangle $\Delta = \{A, B, C\}$, inellipse $\mathcal{E}_1$ and the elliptic billiard $\mathcal{E}$ centered at $X_9$.

*Proof.* Left to the reader. We have that $\mathcal{E}_1$ is the unique ellipse inscribed in the triangle $\Delta$ and tangent to its sides at their extouch points (points of intersection of $\Delta$ with its excircles). The ellipse $\mathcal{E}_1$ is known as Mandart ellipse [113, Mandart Inellipse]. □

**Question 2.** *Which simple or self-intersected $N$- gon (closed polygon with $N$ vertices and $N$ sides) can be an orbit on an elliptic billiard?*

*For $N = 4$ only the parallelogram can be a non self-intersected orbit on an elliptic billiard, see [29]. For the analysis of self-intersected $4-$gons see [44].*

**Remark 17.** *For an extensive list of invariants and conjectures associated to $N-$ periodic elliptic billiard orbits see [92]. See also [44] for some explicit formulaes for small $N$.*

## 7 Caustics

In this section will be present some properties of the billiard caustics. Recall that is classically well known that the caustics of elliptic billiards are confocal ellipses (elliptic orbits) and confocal hyperbolas (billiard orbits pass between the two foci). See [9, Chapter XI], [59, Chapter 6].

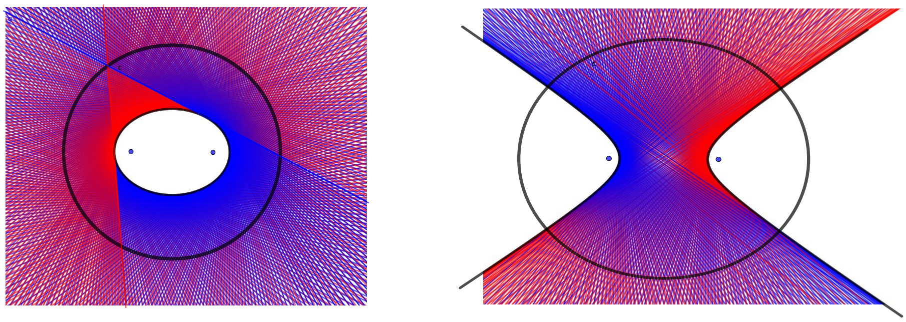

Figure 33: Envelope of billiard orbits: ellipse (left) and hyperbola(right).

Let $\Gamma$ be a smooth convex curve parametrized by arc length $s$, and consider a smooth family of oriented lines passing through the points of $\Gamma$. A parametric representation is given by

$$\ell(s,v) = \Gamma(s) + v d(s), \ |d(s)| = 1, \ |\Gamma'(s)| = 1$$

Denote by $\theta(s)$ the angle between $d(s)$ and $\Gamma'(s)$. Therefore we can write $d(s) = \cos\theta(s)T(s) + \sin\theta(s)N(s)$ where $\{T(s), N(s)\}$ is the Frenet frame of $\Gamma$.

**Lemma 3.** *The envelope of the family $\ell(s,v)$ is given by*

$$E(s) = \Gamma(s) - \frac{\langle \Gamma'(s), d'(s)\rangle}{\langle d'(s), d'(s)\rangle} d(s).$$

*Proof.* The envelope of a family of straight lines is the locus of singularities and is defined by the condition that the tangent vector of $E(s) = \ell(s, v(s))$ is parallel to $d'(s)$. Let $[.,.]$ denotes the determinant of two vectors (oriented area of the parallelogram). So it follows that

$$[E', d'] = [\Gamma'(s) + v(s)d'(s) + v'(s)d(s), d'(s)] = 0$$

Then,

$$v(s) = -\frac{\langle \Gamma'(s), d'(s)\rangle}{\langle d'(s), d'(s)\rangle}$$

□

Next, in other to apply to billiards, consider the reflected family of lines defined by

$$\ell_r(s, u) = \Gamma(s) + u d_r(s), \; |d_r(s)| = 1.$$

The envelope of $\ell_r(s, u)$ is defined by the function

$$u(s) = -\frac{\langle \Gamma'(s), d_r'(s)\rangle}{\langle d_r'(s), d_r'(s)\rangle}.$$

**Proposition 36.** *In the conditions above*

$$\frac{1}{|u(s)|} + \frac{1}{|v(s)|} = \frac{2k(s)}{sin\theta(s)} \tag{14}$$

*Proof.* We have that $d(s) = \cos\theta(s)T(s) + \sin\theta(s)N(s)$ and $d_r(s) = -\cos\theta(s)T(s) + \sin\theta(s)N(s)$.

Therefore,

$$\frac{1}{v(s)} = -\frac{\theta'(s) + k(s)}{\sin\theta(s)}, \quad \frac{1}{u(s)} = \frac{\theta'(s) - k(s)}{\sin\theta(s)}$$

This ends the proof. □

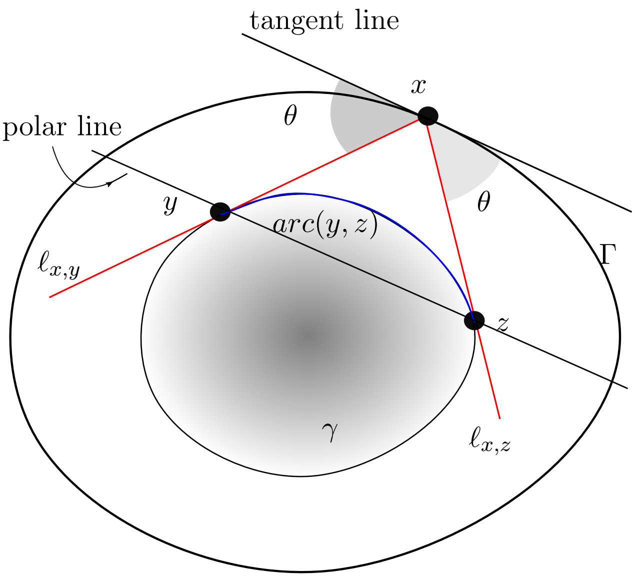


Figure 34: Caustic and length of chords.

The following is a direct consequence of Proposition 36. See also [57] and [114].

**Corollary 4.** *Consider a smooth convex curve $\Gamma$ with curvature $k > 0$. Let $\gamma$ be the caustic and the chords as shown in Fig. 34. Then*

$$\frac{1}{|x-y|} + \frac{1}{|x-z|} = \frac{2k(x)}{\sin\theta}.$$

When the curve is not strictly convex we have the following consequence. See [81] and [95, Chapter 3].

**Theorem 14.** *If the curvature of a convex smooth billiard curve vanishes at some point, then the associated billiard map has no invariant circles.*

**Theorem 15.** *Consider a convex billiard table with boundary $\Gamma$ of class $C^2$. Let $d$ be the diameter, $L$ be the length and $A$ be the area. Let $k = min\{k(s)\}$ and $K = max\{k(s)\}$ of the curvature $k$ of $\Gamma$. Suppose that $\lambda_1 = \sqrt{2}d^2kK \leq 1$ and let $\lambda_2 = 1-\sqrt{2}kLd^2/A$. Then the billiard table contains a convex region, free of caustics, whose area is at least $\lambda_2$.*

*Proof.* See [57]. □

The result below can be found in [103, Lecture 10]. See also [7], [39, Lecture 28] and [54].

**Proposition 37.** *Let $\gamma$ be a convex curve of length $l(\gamma) > 0$. For $p$ outside $\gamma$ let $L(p)$ the length of a string through $p$ stretched tightly around $\gamma$. For* *each $r > l(\gamma)$, let $C_r = \{p \in \mathbb{R}^2 : L(p) = r\}$. Then $C_r$ is the boundary of a convex body $\Gamma$ having $\gamma$ as a caustic.*

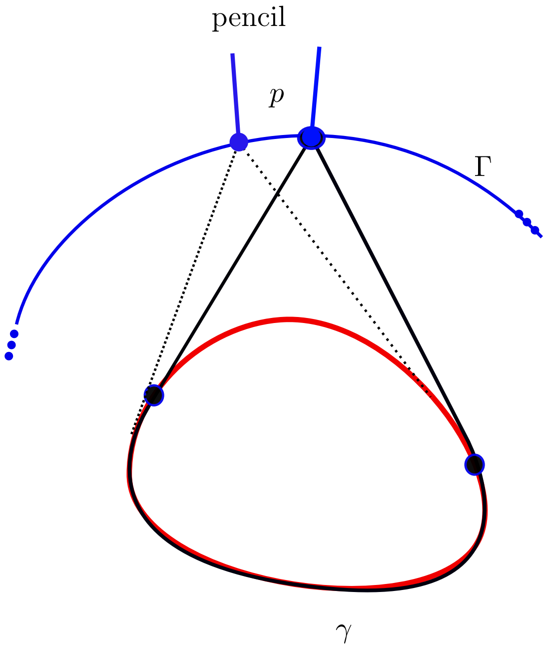


Figure 35: Caustic $\gamma$ and the string construction of the convex curve $\Gamma$.

**Problem 2.** *Consider an elliptic billiard $\mathcal{E}$ having a family of 3-periodic orbits given by $\Delta(t) = \{P_1(t), P_2(t), P_3(t)\}$. Denote by $\Delta_c(t) = \{Q_1(t), Q_2(t), Q_3(t)\}$ the vertices of the polar triangle (also it is the extouch triangle of $\Delta(t)$) inscribed in the caustic $\mathcal{E}_1$ (Mandart innelipse of $\Delta(t)$). Find explicitly the envelope $\mathcal{E}_2$ of the family of lines $\ell_{ij}(u,t) = uQ_i(t) + (1-u)Q_j(t)$ $(i \neq j = 1,2,3)$. Show that $\{\mathcal{E}_1, \mathcal{E}_2\}$ is a Poncelet pair.*

*In the three dimensional space of triangles inscribed in an ellipse analyze the properties of the curves $\Delta(t)$ and $\Delta_c(t)$.*

**Remark 18.** *The study of caustics of billiards on complex conics was developed in [36]. In this context it is used the complex reflection law to define the billiard orbit.*

## 8 Birkhoff Conjecture

The following is a classical open problem in theory of planar billiards. See [13], [55], [88] and [107, Chapter 2].

**Conjecture 1** ([13]). *Let $\Gamma$ be a strictly convex smooth billiard curve. If a neighborhood of $\Gamma$ is foliated by caustics (smooth simple closed curves), then $\Gamma$ is an ellipse.*

Partial deep results concerning the solution of this conjecture are the following works: [6], [12], [66].

Also it is worth to observe that billiards have a remarkable relation with the spectrum of the Laplace operator in the domain $\Omega$, see [51], [87] and references therein. Therefore, in some way from the billiard dynamical behavior in $\Omega$ it is possible to reconstruct the shape of the domain.

An outer billiard is polynomially integrable, if there exists a non-constant polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ that is invariant under the outer billiard mapping $T$, i.e., $f(T(x, y)) = f(x, y)$.

The following result is valid. See [52].

**Theorem 16.** *Let an outer billiard generated by a $C^4$-smooth strictly convex closed curve $\mathcal{C}$ be polynomially integrable. Then $\mathcal{C}$ is an ellipse.*

A dynamical version of the Birkhoff conjecture is the following.

**Conjecture 2** ([79]). *The elliptic billiards are the only ones whose associated billiard maps have polynomial entropy equal to 2.*

## 9 Other types of billiards: impact systems

In this section, we briefly describe a more general context of billiards.

Given any second order differential equation

$$p'' = f(p), \qquad p \in \Omega \subset \mathbb{R}^2 \tag{15}$$

in a connected region $\Omega$ of the plane, an extended solution (billiard orbit) is the union of orbits of (15) such that when the orbit meets the boundary it is reflected with equal angle of incidence and preserving the velocity in magnitude. An example considered in [63] is the following.

Let two disks $D_1 = \{p \in \mathbb{R}^2 : (x+1)^2 + (y+r)^2 < r^2\}$ and $D_2 = \{p \in \mathbb{R}^2 : (x-1)^2 + (y+r)^2 < r^2\}$ and $\Omega = \mathbb{R}^2 \setminus (D_1 \cup D_2)$. Consider a particle moving in $\Omega$ obeying the gravitational law, i.e., $x'' = 0, y'' = -g$.

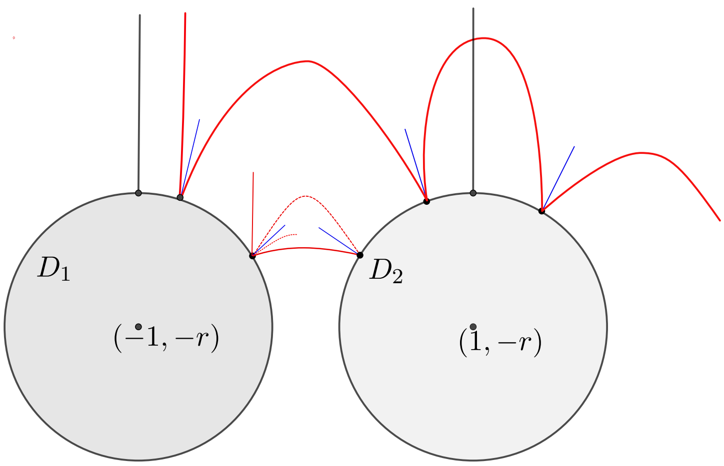


Figure 36: Gravitational inclined billiard.

All billiard orbits are vertical lines or union of lines and segments of parabolas. The analysis of this dynamical system is not immediate.

Another generalization of billiards is when the system is non-autonomous and/or the region is variable. For example consider a particle moving in the plane but meeting a boundary oscillating periodically. Let $\Omega_t = \{p = (x, y) : y \geq \sin \omega t\}$ and a particle moving according the gravitational law, hitting the boundary of $\Omega_t$ elastically (preserving moment of mass and angle of incidence equal to that of reflection). This rich dynamic was studied in [3], [64].

See [38], [67], [80] and [111] for other variations of billiards. Finally we mention the book by C. Coriolis [21] modelling a *billiard table*. This is another history!

Figure 37: A round billiard table https://images.app.goo.gl/sFMhHEw33XB3UgeC8.

# A Parametrization of 3-periodic orbits

The following explicit parametrization of the family of 3-periodic orbits in an elliptic billiard defined in the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found in [42].

A 3-periodic orbit is parametrized by $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$ with:

- $P_2 = (x_2, y_2)$, where $x_2 = \frac{p_{2x}}{q_2}$ and $y_2 = \frac{p_{2y}}{q_2}$ are given by

$$\begin{aligned}
p_{2x} =& -b^4\left(\left(a^2+b^2\right)\cos^2\alpha - a^2\right)x_1^3 - 2a^6\,\cos\alpha\sin\alpha\, y_1^3\\
&+a^4\left(\left(a^2-3\,b^2\right)\cos^2\alpha + b^2\right)x_1\,y_1^2 - 2\,a^4b^2\cos\alpha\,\sin\alpha x_1^2 y_1,\\
p_{2y} =& 2b^6\,\cos\alpha\sin\alpha\, x_1^3 - a^4\left(\left(a^2+b^2\right)\cos^2\alpha - b^2\right)y_1^3\\
&+2\,a^2b^4\cos\alpha\sin\alpha\; x_1y_1^2 + b^4\left(\left(b^2-3\,a^2\right)\cos^2\alpha + a^2\right)x_1^2y_1\\
q_2 =& b^4\left(a^2-\left(a^2-b^2\right)\cos^2\alpha\right)x_1^2 + a^4\left(b^2+\left(a^2-b^2\right)\cos^2\alpha\right)y_1^2\\
&-2\,a^2b^2\left(a^2-b^2\right)\cos\alpha\sin\alpha\; x_1\,y_1.
\end{aligned} \tag{16}$$

- $P_3 = (x_3, y_3)$, where $x_3 = \frac{p_{3x}}{q_3}$ and $y_3 = \frac{p_{3y}}{q_3}$ are given by

$$\begin{aligned}
p_{3x} =& b^4\left(a^2-\left(b^2+a^2\right)\right)\cos^2\alpha x_1^3 + 2\,a^6\cos\alpha\,\sin\alpha\, y_1^3\\
&+a^4\left(\cos^2\alpha\left(a^2-3\,b^2\right)+b^2\right)x_1\,y_1^2 + 2\,a^4b^2\cos\alpha\sin\alpha\, x_1^2y_1\\
p_{3y} =& -2\,b^6\cos\alpha\sin\alpha\, x_1^3 + a^4\left(b^2-\left(b^2+a^2\right)\cos^2\alpha\right)\; y_1^3\\
&-2\,a^2b^4\cos\alpha\sin\alpha\, x_1y_1^2 + b^4\left(a^2+\left(b^2-3\,a^2\right)\cos^2\alpha\right)x_1^2y_1,\\
q_3 =& b^4\left(a^2-\left(a^2-b^2\right)\cos^2\alpha\right)x_1^2 + a^4\left(b^2+\left(a^2-b^2\right)\cos^2\alpha\right)y_1^2\\
&+2\,a^2b^2\left(a^2-b^2\right)\cos\alpha\sin\alpha\, x_1\,y_1.
\end{aligned} \tag{17}$$

The angle $\alpha$ is given by

$$\cos\alpha = \frac{a^2 b\sqrt{-a^2-b^2+2\sqrt{a^4-b^2c^2}}}{c^2\sqrt{a^4-c^2x_1^2}}.$$

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## References

[1] Alkoumi, Naeem; Schlenk, Felix. Shortest closed billiard orbits on convex tables. Manuscripta Math. 147 (2015), no. 3–4, 365–380.

[2] Arseniy Akopyan and Richard Schwartz and Serge Tabachnikov, Billiards in Ellipses Revisited, Eur. J. Math., 9. 2020, http://doi.org/10.1007/s40879-020-00426-9

[3] Akhmet, Marat. Principles of discontinuous dynamical systems. Springer Verlag, New York, (2010) xii+176 pp.

[4] Amiran, Edoh Lazutkin coordinates and invariant curves for outer billiards. J. Math. Phys. 36 (1995), no. 3, 1232–1241.

[5] Arnold,V. I., Singularities of caustics and wave fronts, Math. Appl. 62, Kluwer, Dordrecht, 1990.

[6] Avila, A., Kaloshin, V., de Simoi, J. An integrable deformation of an ellipse of small eccentricity is an ellipse, Ann. of Math., Vol. 184 (2016), 527–558.

[7] Arnold, M. Bialy, M. Nonsmooth convex caustics for Birkhoff billiards. Pacific J. Math. 295 (2018), no. 2, 257–269.

[8] Barth, W. and Bauer, T. Poncelet Theorems, Exp. Math., 14 (1996), 125–144.

[9] Berger, Marcel, Geometry Revealed: a Jacob's Ladder to Higher Geometry. Springer Verlag, (2005) 831pp.

[10] Berger, Marcel, A panoramic view of Riemannian geometry. Springer Verlag, (2002).

[11] Bezdek, K., Connelly, R. Covering curves by translates of a convex set. Am. Math. Monthly 96(9),789–806 (1989).

[12] Bialy, M. and Mronovthe, E. The Birkhoff-Porstsky conjecture for centrally-symmetric billiard tables. arXiv:2008.03566v2 (2020).

[13] Birkhoff, G. On the periodic motions of dynamical systems. Acta Mathematica, 50(1):359–379 (1927).

[14] Birkhoff, George D. Dynamical systems. With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX American Mathematical Society, Providence, R.I. 1966 xii+305 pp.

[15] Brogliato, Bernard. Nonsmooth impact mechanics. Models, dynamics and control. Lecture Notes in Control and Information Sciences, 220. Springer-Verlag London, Ltd., London, 1996. xvi+400 pp.

[16] Boyland, P. Dual billiards, twist maps and impact oscillators Nonlinearity 9 (1996) 1411–1438.

[17] Bryant R. Poncelet's theorem. http://arimoto.lolipop.jp/PonceletforBMC.pdf

[18] Bunimovich, Leonid A. Mechanisms of chaos in billiards: dispersing, defocusing and nothing else. Nonlinearity 31 (2018), no. 2, R78–R92.

[19] Chernov, Nikolai; Markarian, Roberto Introduction to the ergodic theory of chaotic billiards. Monografías del Instituto de Matemática y Ciencias Afines IMCA, Lima; Pontificia Universidad Católica del Perú, Lima, 2001. 187 pp.

[20] Cima, Anna; Gasull, Armengol; Mañosa, Víctor. On Poncelet's maps. Comput. Math. Appl. 60 (2010), no. 5, 1457–1464.

[21] Coriolis, G.-G. Gaspard-Gustave de Coriolis,G.G. Théorie Mathématique des Effets du Jeu de Billiard, Carilian-Goeury, Paris (1835).

[22] H. T. Croft, H. T.; Swinnerton-Dyer, H. P. F. On the Steinhaus billiard table problem. Volume 59(1) (1963), pp. 37–41

[23] Darboux, G. *Principes de Géométrie Analytique.* Paris: Gauthier-Villars (1917).

[24] Del Centina, Andrea Poncelet's porism: a long story of renewed discoveries, I. Arch. Hist. Exact Sci. 70 (2016), no. 1, 1–122.

[25] Del Centina, Andrea Poncelet's porism: a long story of renewed discoveries, II. Arch. Hist. Exact Sci. 70 (2016), no. 2, 123–173.

[26] Del Centina, Andrea Pascal's mystichexagram, and a conjectural restoration of his lost treatise on conic sections. Arch. Hist. Exact Sci. 74 (2020), no. 5, 469–521.

[27] Cieślak, W. ; Mozgawa, W. In search of a measure in Poncelet's porism. Acta Math. Hungar. 149 (2016), no. 2, 338–345.

[28] Charles, M. Propriétés genérales des arcs d'une section conique dont la différence est rectifiable. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, T.17 (1843), 838–844.

[29] Connes, A., Zagier, D. A property of parallelograms inscribed in ellipses. The American Math. Monthly, 114(10): 909?914. (2007).

[30] Coxeter, H. Projective Geometry. Second Editon. Springer Verlag (1987).

[31] Dias Carneiro, Mário Jorge; Oliffson Kamphorst, Sylvie; Pinto de Carvalho, Sônia. Elliptic islands in strictly convex billiards. Ergodic Theory Dynam. Systems 23 (2003), no. 3, 799–812.

[32] Dragović, V. (2011). Poncelet-Darboux curves, their complete decomposition and Marden theorem. *Int. Math. Res. Not. IMRN* no. 15, 3502–3523.

[33] Dragović, V., Radnović, M. (2014). Bicentennial of the great Poncelet theorem (1813–2013): current advances. *Bull. Amer. Math. Soc. (N.S.)*, 51: 373–445.

[34] Dragović, Vladimir; Radnović, Milena Poncelet porisms and beyond. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, (2011). viii+293 pp.

[35] M. Farber and S. Tabachnikov, Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards, Topology 41 (2002), 553– 589.

[36] Fierobe, C. Complex caustics of the elliptic billiard. arXiv:1904.03706v6. Arnold Math J. (2020). https://doi.org/10.1007/s40598-020-00152-w

[37] Flatto, Leopold. Poncelet's theorem. Chapter 15 by S. Tabachnikov. American Mathematical Society, Providence, RI, (2009) xvi+240 pp.

[38] Fokicheva, V. V. Topological classification of billiards in locally planar domains bounded by arcs of confocal quadrics. Sb. Math. 206 (2015), no. 9–10, 1463–1507.

[39] Fuchs, Dmitry; Tabachnikov, Serge. Mathematical omnibus. Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, (2007) xvi+463 pp.

[40] Gaivão, José Pedro. Asymptotic periodicity in outer billiards with contraction. Ergodic Theory Dynam. Systems 40 (2020), no. 2, 402–417.

[41] G. Galperin, A. Stepin, and Y. Vorobets. Periodic orbits in polygons; generating mechanisms, Russian Math. Surueys 47 (1992), 5–80.

[42] Garcia, R., Elliptic Billiards and Ellipses Associated to the 3-Periodic Orbits, American Mathematical Monthly, 126(06), (2019) 491–504.

[43] Garcia, R., Reznik, D. and Koiller J. Loci of 3-periodics in Ell. Billiard: why so many ellipses? Arxiv. (2020).

[44] Garcia, R., Reznik, D. Invariants of Simple and Self-Intersected N-Periodics in the Elliptic Billiard. Arxiv (2020).

[45] Garcia, R., Reznik, D. and Koiller J. Loci of 3-periodics in elliptic billiard: why so many ellipses? Arxiv (2020).

[46] Garcia, R., Reznik, D. and Koiller J. New properties of elliptic billiards. Amer. Math. Monthly. To appear (2021).

[47] Garcia, Ronaldo; Sotomayor, Jorge Differential equations of classical geometry, a qualitative theory. Publicações Matemáticas do IMPA. $27^o$ Colóquio Brasileiro de Matemática. (IMPA), Rio de Janeiro, (2009) ii+256 pp.

[48] Genin, Daniel. Hyperbolic outer billiards: a first example. Nonlinearity 19 (2006), no. 6, 1403–1413.

[49] M. Ghomi, Shortest periodic billiard trajectories in convex bodies. Geom. Funct. Anal. 14(2), 295–302 (2004).

[50] Glaeser, G., Odehnal B., Stachel, H. (2016). *The Universe of Conics: From the Ancient Greeks to 21st Century Developments.* New York: Springer Verlag.

[51] Guillemin, Victor; Melrose, Richard An inverse spectral result for elliptical regions in $\mathbb{R}^2$. Adv. in Math. 32 (1979), no. 2, 128–148.

[52] Glutsyuk, A., Shustin, E. On polynomially integrable planar outer billiards and curves with symmetry property. Math. Ann. 372, 1481–1501 (2018).

[53] Griffiths, P., Harris, J. On Cayley's explicit solution to Poncelet's porism, Enseign. Math. 24(1-2): 31–40 (1978).

[54] Gruber, Peter M. Convex billiards. Geom. Dedicata 33 (1990), no. 2, 205–226.

[55] Gutkin, E. Billiard dynamics: a survey with the emphasis on open problems. Regul. Chaotic Dyn. 8(1):1–13 (2003).

[56] Gutkin, E. Billiards in Polygons. Physica D 19, 311–333, (1986).

[57] Gutkin, E. ; Katok, A. Caustics for inner and outer billiards. Communications in Mathematical Physics 173,(1995) pp. 101–133.

[58] Gutkin, E. Two Applications of Calculus to Triangular Billiards. Amer. Math. Monthly. 104(7) pp. 618–622 (1997).

[59] Hasselblatt, Boris; Katok, Anatole. A first course in dynamics. With a panorama of recent developments. Cambridge University Press, New York, (2003). x+424.

[60] Hassenblatt B. e Katok A. Introduction to the modern theory of dynamical systems, Cambridge University Press, Cambridge (1995).

[61] Halbeisen L., Hungerbühler N. (2015). A Simple Proof of Poncelet's Theorem (on the Occasion of Its Bicentennial). *Amer. Math. Monthly*, 122(6): 537–551.

[62] Halbeisen, Lorenz, Hungerbühler, Norbert. Closed chains of conics carrying Poncelet triangles. Beitr. Algebra Geom. 58 (2017), no. 2, 277–302.

[63] Hénon, M. Chaotic scattering modelled by an inclined billiard. Progress in chaotic dynamics. Phys. D 33 (1988), no. 1–3, 132–156.

[64] Holmes, P. J. The dynamics of repeated impacts with a sinusoidally vibrating table. J. Sound Vibration 84 (1982), no. 2, 173–189.

[65] Kaloshin V, Sorrentino A. On the integrability of Birkhoff billiards. Phil. Trans. R. Soc. A 376: 20170419 (2018)

[66] Kaloshin, Vadim; Sorrentino, Alfonso. On the local Birkhoff conjecture for convex billiards. Ann. of Math. (2) 188 (2018), no. 1, 315–380.

[67] Kozlova, T. V. Nonintegrability of a rotating elliptic billiard. J. Appl. Math. Mech. 62 (1998), no. 1, 81–85 .

[68] Kamphorst, Sylvie Oliffson; Pinto-de-Carvalho, Sônia. The first Birkhoff coefficient and the stability of 2-periodic orbits on billiards. Experiment. Math. 14(3): 299–306 (2005).

[69] Katok, A. B. Billiard table as a playground for a mathematician. In V. Prasolov & Y. Ilyashenko (Eds.), Surveys in modern mathematics, 216–242, London Math. Soc. Lecture Note Ser., 321, Cambridge Univ. Press, Cambridge, 2005.

[70] Kimberling, C.: Encyclopedia of triangle centers (2019). https://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[71] J. L. King. Three problems in search of a measure, Amer. Math. Monthly, 101 (1994), 609–628.

[72] Klingenberg, Wilhelm P. A. Riemannian geometry. Second edition. De Gruyter Studies in Mathematics, 1. Walter de Gruyter & Co., Berlin, (1995) x+409 pp.

[73] Kołodziej, Rafał. The rotation number of some transformation related to billiards in an ellipse. Studia Math. 81 (1985), no. 3, 293–302.

[74] Kozlov, Valeriĭ V.; Treshchëv, Dmitriĭ V. Billiards. A genetic introduction to the dynamics of systems with impacts. Translations of Mathematical Monographs, 89. American Mathematical Society, Providence, RI, (1991). viii+171 pp.

[75] Langevin, Remi ; Levitt, Gilbert ; Rosenberg, Harold. Classes d'homotopie de surfaces avec rebroussements et queues d'aronde dans $\mathbb{R}^3$. Canad. J. Math. 47 (1995), no. 3, 544–572.

[76] Lebesgue, H. Les coniques. Gauthier Villars. Paris. (1942).

[77] Levi, M., Tabachnikov, S. . The Poncelet Grid and Billiards in Ellipses. American Math. Monthly, 114:895–908 (2007).

[78] Lopes, Artur O. ; Sebastiani, Marcos. Poncelet pairs and the twist map associated to the Poncelet billiard. Real Anal. Exchange 35 (2010), no. 2, 355–374.

[79] Marco, Jean-Pierre. Entropy of billiard maps and a dynamical version of the Birkhoff conjecture. J. Geom. Phys. 124 (2018), 413–420.

[80] Marò, Stefano. Chaotic dynamics in an impact problem. Ann. Henri Poincaré 16 (2015), no. 7, 1633–1650.

[81] Mather, J. Non-existence of invariant circles. Ergod. Th. Dyn. Syst. 4 (1984), 301–309.

[82] Melo, W. and Palis, J. Introdução aos Sistemas Dinâmicos, Projeto Euclides, IMPA, 2a. edição. (2018).

[83] Mirman, Boris. Explicit solutions to Poncelet's porism. Linear Algebra Appl. 436 (2012), no. 9, 3531–3552.

[84] Moser, J.K. Is the solar system stable? Math. Intelligencer 1 no. 2, 65–71 (1978/79).

[85] Moser, J. K. On Invariant Curves of Area–Preserving Mappings of an Annulus , Nachr. Akad. Wiss. Göttingen, Math.–Phys. Kl.II, (1962), 1–20.

[86] Poncelet, J.-V. Traité des propriétés projectives des figures. Metz, Paris. (1822).

[87] Popov, Georgi. Invariants of the length spectrum and spectral invariants of planar convex domains. Comm. Math. Phys. 161 (1994), no. 2, 335–364.

[88] Poritsky, H. The billiard ball problem on a table with a convex boundary–an illustrative dynamical problem. Ann. of Math. (2) 51 (1950), 446–470.

[89] Previato, E. Poncelet's theorem in space. Proc. Amer. Math. Soc. 127(9) (1999), 2547–2556.

[90] Ragazzo, Clodoaldo; Dias Carneiro, Mário Jorge; Adda-Zanata, Salvador. Introdução à Dinâmica de Aplicações do Tipo Twist, 25º Colóquio Brasileiro de Matemática, IMPA, (2005).

[91] Reznik, Dan; Garcia, Ronaldo and Koiller, Jair. Can an elliptic billiard still surprise us? Math. Intelligencer vol.42, (2020) 6–17.

[92] Reznik, Dan; Garcia, Ronaldo and Koiller, Jair. Eighty New Invariants in the Elliptic Billiard. ArXiv:2004.12497v11.

[93] Romaskevich, O. On the incenters of triangular orbits on elliptic billiard. *Enseign. Math.* 60(3-4):247–255 (2014)

[94] Romaskevich, O. Dynamique des systèmes physiques, formes normales et chaînes de Markov. These de Doctarat de L'Université de Lyon. (2016).

[95] Rozikov, Utkir A. An Introduction to Mathematical Billiards. World Scientific Publishing Company, Singapore (2019)

[96] M. R. Rychlik, Periodic points of the billiard ball map in a convex domain. J. Differential Geom. 30 (1):91–205 (1989).

[97] Schwartz, Richard E. The plaid model. Annals of Mathematics Studies, 198. Princeton University Press, Princeton, NJ, (2019).

[98] R.E. Schwartz, R. Outer billiards on kites. Annals of Mathematics Studies. 171. Princeton University Press (2009).

[99] Schwartz, Richard E. Obtuse triangular billiards. II. One hundred degrees worth of periodic trajectories. Experiment. Math. 18 (2009), no. 2, 137–171.

[100] Schwartz, R. The Poncelet grid. *Adv. Geom.* 7(2): 157-175 (2007).

[101] Schwartz, R., Tabachnikov, S. Centers of Mass of Poncelet Polygons, 200 Years After. Math. Intelligencer 38(2): 29–34 (2016).

[102] I. J. Schoenberg, On Jacobi-Bertrand's proof of a theorem of Poncelet. Studies in pure mathematics, 623–627, Birkhäuser, Basel, (1983).

[103] Sinai, Ya. G. Introduction to ergodic theory. Princeton University Press, Princeton, NJ, (1977).

[104] Sinai, Y.G.: Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. Usp. Mat. Nauk 25(2 (152)), 141–192 (1970).

[105] Sorrentino, Alfonso. Mathematicians play... billiards! Mat. Cult. Soc. Riv. Unione Mat. Ital. (I) 4 (2019), no. 2, 131–144.

[106] Sotomayor, J. Lições de Equações Diferenciais Ordinárias. Projeto Euclides, IMPA (1979).

[107] Tabachnikov, S. Geometry and Billiards, vol. 30 of Student Mathematical Library. Providence, RI: American Mathematical Society (2005).

[108] Tabachnikov, S. Outer billiards. Russian Math. Surveys 48 (1993), no. 6, 81–109.

[109] Tabachnikov, Serge. A baker's dozen of problems. Arnold Math. J. 1 (2015), no. 1, 59–67.

[110] Tumanov, A. Scarcity of Periodic Orbits in Outer Billiards. J Geom Anal 30, 2479–2490 (2020).

[111] Vedyushkina, V. V. ; Fomenko, A. T. Integrable topological billiards and equivalent dynamical systems. Izv. Math. 81 (2017), no. 4, 688–733.

[112] Vorobets, Ya. B. On the measure of the set of periodic points of a billiard. Math. Notes 55 (1994), no. 5-6, 455–460

[113] Weisstein, E.: Mathworld (2019). http://mathworld.wolfram.com

[114] Wojtkowski, Maciej. Principles for the design of billiards with nonvanishing Lyapunov exponents. Comm. Math. Phys. 105 (1986), no. 3, 391–414.



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