MAT$^2$
**MATerials MATemàtics**
Volum 2022, treball no. 1, 10 pp. ISSN: 1887-1097
Publicació electrònica de divulgació del Departament de Matemàtiques
de la Universitat Autònoma de Barcelona
www.mat.uab.cat/web/matmat

# A curious infinite product

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## 1 The product

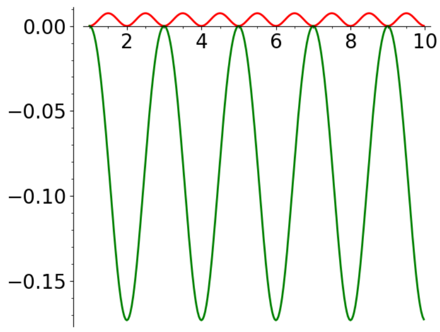
Behold the following result!

**Theorem 1.1.** *Let $B_1$ and $B_2$ be integers and let $a_1$ and $a_2$ be positive divisors of $B_1^2+1$ and $B_2^2+1$, respectively. Then*

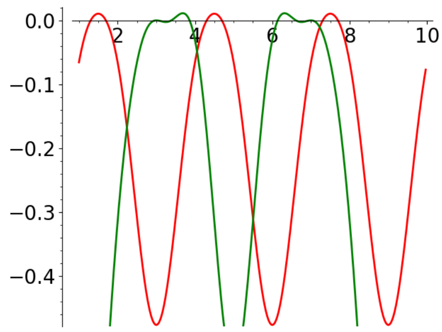
$$\prod_{n=1}^{\infty} \frac{e^{2\pi n/a_2}\big(\cosh\frac{2\pi n}{a_1} - \cos\frac{2\pi n B_1}{a_1}\big)}{e^{2\pi n/a_1}\big(\cosh\frac{2\pi n}{a_2} - \cos\frac{2\pi n B_2}{a_2}\big)} = \frac{e^{\pi/(6a_1)}\sqrt{a_1}}{e^{\pi/(6a_2)}\sqrt{a_2}}.$$

Note that each factor tends exponentially to 1, then the convergence is assured.

The curious point is that there is something arithmetic involved. It is in general false without the divisibility condition. The following figures show the difference between the left hand side and the right hand side of the formula for $a_2 = B_2 = 1$ and the values of $a_1$ indicated in the captions when $B_1$ takes real values in $[0, 10]$. In the first figure we see that the identity acts as a perfect detector of integers, $1 \mid B_1^2+1$, or odd integers, $2 \mid B_1^2+1$. The second figure reveals a more complicate truth because $3 \nmid B_1^2+1$ for $B_1 \in \mathbb{Z}$ but clearly the plot crosses the $OX$ axis for some non integral values of $B_1$. On the other hand, $10 \mid B_1^2+1$ implies $B_1 = 3, 7$ if $B_1 \in [0, 10] \cap \mathbb{Z}$ and we see also some other real zeros. Summing up, occasionally the identity can be also true "by chance" for some real values.




$a_1 = 1$, $a_1 = 2$, $B_1 \in [0, 10]$




$a_1 = 3$, $a_1 = 10$, $B_1 \in [0, 10]$

If you are a modular person, after reading the following lines showing the relation with $\eta$, you will be able to find a quick proof by yourself, perhaps adding a teaspoon of class number one or a grain of complex multiplication. If you are modular but not to the bone, you will have a chance of reading §3. Anyway, the challenge here is to provide a proof simpler enough to fit in a lecture, only one, of an undergraduate course. This is done in §2 assuming an analytic result known as *Kronecker limit formula* which is proved in [2] with little more than the residue theorem. The proof is reproduced in §4 adapted to a special case, for the sake of clarity, and with complementary comments to convince you that it generalizes finely.

We start defining the Dedekind $\eta$ function on the upper half complex plane as

$$\eta(z) = \mathrm{e}^{\pi i z/12} \prod_{n=1}^{\infty} \big(1 - \mathrm{e}^{2\pi i n z}\big).$$

It converges quickly if $z$ is far apart from the real axis because $2\pi i n z$ amplifies the imaginary part of $z$ giving a negative exponential.

We have $\big|1 - \mathrm{e}^{u+iv}\big|^2 = \mathrm{e}^u \big|\mathrm{e}^{-u/2} - \mathrm{e}^{(u/2)+iv}\big|^2 = 2\mathrm{e}^u(\cosh u - \cos v)$, for $u, v \in \mathbb{R}$. Taking $u + iv = 2\pi i(B_j + i)/a_j$, we see that the infinite product in Theorem 1.1 is

$$\prod_{n=1}^{\infty} \frac{\big|1 - \mathrm{e}^{2\pi i(B_1+i)/a_1}\big|^2}{\big|1 - \mathrm{e}^{2\pi i(B_2+i)/a_2}\big|^2} = \frac{\mathrm{e}^{\pi/6a_1}\big|\eta\big((B_1+i)/a_1\big)\big|^2}{\mathrm{e}^{\pi/6a_2}\big|\eta\big((B_2+i)/a_2\big)\big|^2}$$

Then Theorem 1.1 is equivalent to say that $\big|\eta\big((B_j+i)/a_j\big)\big|^2/\sqrt{a_j}$ is constant. It does not depend on the choice of $a_j$ and $B_j$ fulfilling the hypotheses.

Modular people know how to relate the values of $\eta(z)$ at different points connected by some symmetries and then they may find the previous claim fairly easy. We pedestrians aspire for a proof not requiring any knowledge about those relations and symmetries. At the same time, we can learn a formula, the aforementioned Kronecker limit formula, which plays a role in some explicit evaluations.

## 2 A proof for everybody (summoning Kronecker)

The binary quadratic forms $Q(x,y) = a\,x^2 + b\,x\,y + c\,y^2$ may seem very simple for a linear algebra student but they become a deep and classic topic in number theory when we impose $a,b,c \in \mathbb{Z}$ and we want to determine $Q(\mathbb{Z}^2)$ or the prime numbers in this image [3]. Here we do not dwell on these difficult topics and indeed in the first part of this section we only need $Q$ to be a real form i.e., $a,b,c \in \mathbb{R}$. On the other hand, we assume all the time $Q$ to be positive definite, equivalently $a > 0$ and $4\,a\,c - b^2 > 0$.

The Riemann zeta function and the Epstein zeta function $\zeta(s,Q)$, where $Q$ is a positive definite binary quadratic form, are defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \zeta(s,Q) = \sum_{\vec{n}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \big(Q(\vec{n})\big)^{-s}.$$

Both definitions can be extended analytically to real and complex values beyond $s > 1$. It is well known that for the Riemann zeta function there is an obstacle at $s = 1$. Some insight about this point comes from the identity $(1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$, which reduces to multiplication term by term. Recalling $\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1} = \log 2$, we have

$$\lim_{s\to 1^+}(s-1)\zeta(s) = (\log 2)\lim_{s\to 1^+}\frac{s-1}{1-2^{1-s}} = 1 \tag{1}$$

by L'Hôpital's rule. This means that $\zeta(s)$ is approximately $(s-1)^{-1}$ for $s > 1$ close to 1. The Kronecker limit formula implies that $\zeta(s,Q)$ is approximately $\frac{2\pi}{\sqrt{D}}(s-1)^{-1}$ near 1 (this is not a big deal and it can be done with fairly elementary methods) and shows that the difference tends to a constant that can be expressed in terms of the Dedekind $\eta$ function (this is the difficult part). Kronecker show yourself, we beckon you!

**Proposition 2.1** (Kronecker limit formula)**.** *Let $Q(x,y) = ax^2 + bxy + cy^2$ be a real quadratic form with $D = 4ac - b^2 > 0$ and $a > 0$. Then*

$$\lim_{s\to 1^+}\Big(\frac{\sqrt{D}}{4\pi}\zeta(s,Q) - \zeta(2s-1)\Big) = \log\frac{\sqrt{a/D}}{|\eta(z_Q)|^2} \quad \textit{with } z_Q = \frac{-b+i\sqrt{D}}{2a}.$$

I have downgraded this theorem to proposition to emphasize that it is not so hard to prove. In [2] there is a proof that requires little more than the residue theorem. To not repeat myself, if you are interested I have adapted it in §4 to $Q(x,y) = x^2+y^2$ which allows more reductions and any hard working reader should be able to obtain the general case from it, perhaps following the hints included there. A last comment is that if you look up authorized sources (for instance [7] or [11, §1.1]) Proposition 2.1 does not seem like the standard Kronecker limit formula. Take my word, it is a compact equivalent version.

Proposition 2.1 is purely analytic, a limit, while Theorem 1.1 is somehow arithmetic. The integers enter into the game through the humble but important group of matrices

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} : m_{jk} \in \mathbb{Z},\ \det(M) = 1 \right\}.$$

It is a group, with the product of matrices, because the inverse of an integral matrix of determinant 1 is also a matrix of the same type.

The key result to deduce Theorem 1.1 from the Kronecker limit formula is that for the integral case with $D = 4$ there is only a possible Epstein zeta function!

**Lemma 2.2.** *If $Q(x,y) = ax^2 + bxy + cy^2$ is a quadratic form with $b, c \in \mathbb{Z}$, $a \in \mathbb{Z}^+$ and $4ac - b^2 = 4$ then there exists $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q(M\vec{v}) = x^2 + y^2$ where $\vec{v} = (x,y)$. In particular, for any of these forms we have $\zeta(s,Q) = \zeta(s, x^2 + y^2)$.*

Of course, here it is in use the typical typographical abuse in linear algebra: We have to think $\vec{v}$ as a vertical vector to multiply $M\vec{v}$. This lemma is based on an elementary reduction algorithm due to Lagrange and Gauss for general binary quadratic forms with $a, b, c \in \mathbb{Z}$. If you want to trumpet proudly "I read Gauss", go to his masterpiece [4, Art.171].

*Proof.* Note that the last part follows from the first part because $M \in \mathrm{SL}_2(\mathbb{Z})$ only rearranges the elements of $\mathbb{Z}^2$. In other words, $M$ defines a bijective map $\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$.

If $b = 0$ then clearly the result is true with $M$ the identity matrix. If $b \neq 0$ we are going to show that there is a "reduction matrix" $R \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q'(\vec{v}) = Q(R\vec{v})$ has a smaller value of $|b|$. Repeating the process a number of times we get $x^2 + y^2 = Q(M\vec{v})$ with $M = R_n R_{n-1} \cdots R_1$ and we are done.

Let us see how to construct $R$. If $\langle x \rangle$ is the *nearest integer function* (define it as you want at half-integers), we choose $R$ as

$$R = \begin{pmatrix} \langle b/(2a) \rangle & 1 \\ -1 & 0 \end{pmatrix} \quad \text{if } a < c \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & \langle b/(2c) \rangle \end{pmatrix} \quad \text{if } a > c.$$

There is not an $a = c$ case with $b \neq 0$ because $4 = 4a^2 - b^2 = (2a - b)(2a + b)$ implies $2a - b = 2a + b = 2$. Both cases are similar changing the role of the variables. Let us check for instance the second one:

$$\begin{aligned} Q(R\vec{v}) &= a\,y^2 - b\,y\left(x + \left\langle \frac{b}{2\,c} \right\rangle y\right) + c\left(x + \left\langle \frac{b}{2\,c} \right\rangle y\right)^2 \\ &= A\,x^2 + \left(2\,c\left\langle \frac{b}{2\,c} \right\rangle - b\right) x\,y + C\,y^2. \end{aligned}$$

The absolute value of the new $xy$ coefficient is clearly less than $|b|$ when $|b| > c$ and $|b| \le c$ is impossible because it would imply $4ac - b^2 \ge 4(c+1)c - c^2 > 4$. $\square$

*Proof of Theorem 1.1.* Consider the quadratic forms $Q_j = a_j x^2 - 2B_j xy + (B_j^2+1)y^2/a_j$ for $j = 1, 2$. By the last part of Lemma 2.2 the limits in Proposition 2.1 corresponding to both quadratic forms are identical. Then we conclude

$$\log \frac{\sqrt{a_1}/2}{\big|\eta\big((B_1+i)/a_1\big)\big|^2} = \log \frac{\sqrt{a_2}/2}{\big|\eta\big((B_2+i)/a_2\big)\big|^2}$$

and, as mentioned before, the constancy of $\big|\eta\big((B_j+i)/a_j\big)\big|^2/\sqrt{a_j}$ establishes the result. $\square$

## 3 The quick proof for modular people

Even if you are not a modular person surely you have heard about modular forms by their relation with the proof of Fermat's last theorem (by the way, [6] is an excellent reference if you do not dare to face the readings for the experts). Roughly speaking, modularity implies a kind of symmetry under the changes $z \mapsto z+1$ and $z \mapsto -1/z$ of a holomorphic function defined on the upper half plane. The essence of the modular proof below is that if $a_j \mid B_j^2 + 1$ for $j = 1, 2$ then $(B_1 + i)/a_1$ can be transformed into $(B_2 + i)/a_2$ by successive applications of these changes and the symmetries provide the needed cancellation between the numerator and the denominator of the product.

In the case of the Dedekind $\eta$ function, the modularity means

$$\eta(z+1) = \mathrm{e}^{\pi i/12}\eta(z) \qquad \text{and} \qquad \eta(-1/z) = \sqrt{-iz}\,\eta(z). \tag{2}$$

Of course, the first formula is trivial from the definition. Absolutely, the second is not. To my knowledge the simplest proof is still one due to Siegel [10] (see also [7, §9.2]) based on the residue theorem. Let us go fancy proclaiming that $|\Im(z)|^{1/2}|\eta(z)|^2$ is invariant under $z \mapsto z+1$ and $z \mapsto -1/z$, where $\Im(z)$ is the imaginary part of $z$. This follows immediately from (2) using $\Im(z+1) = \Im(z)$ and $\Im(-1/z) = |z|^{-2}\Im(z)$.

If you are really a modular person you know that $z \mapsto z+1$ and $z \mapsto -1/z$ generate all the maps $z \mapsto (m_{11}z + m_{12})/(m_{21}z + m_{22})$ with $M = (m_{jk}) \in \mathrm{SL}_2(\mathbb{Z})$, these maps are the *modular group*. Hence $|\Im(z)|^{1/2}|\eta(z)|^2$ is also invariant by them. In the particular case $z = i$ we get

$$\big|\eta(i)\big|^2 = \big|\Im\big(\gamma_M(i)\big)\big|^{1/2}\big|\eta\big(\gamma_M(i)\big)\big|^2 \qquad \text{with} \quad \gamma_M(i) = \frac{m_{11}i + m_{12}}{m_{21}i + m_{22}}.$$

It only remains to check that if $0 < a \mid B^2+1$ then there exists a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_M(i) = (B+i)/a$. Actually, we have already done it because taking $Q(x,y) = ax^2 - 2Bxy + (B^2+1)y^2/a$ in Lemma 2.2, as before, and choosing $\vec{v} = (i,1)$, we have $0 = Q(M\vec{v}) = \big(m_{21}i + m_{22}\big)^2 Q\big(\gamma_M(i), 1\big)$. The roots of $Q(z,1) = 0$ are $(B \pm i)/a$, just solving the second degree equation, and $\Im\big(\gamma_M(i)\big) > 0$ (check it!), therefore necessarily $\gamma_M(i) = (B+i)/a$, as expected.

## 4 Who fears the Kronecker limit formula?

The case $Q(x,y) = x^2+y^2$ of Proposition 2.1 reads

$$\lim_{s\to 1^+} \Big(\frac{1}{2\pi}\zeta(s, x^2+y^2) - \zeta(2s-1)\Big) = -\log\big(2|\eta(i)|^2\big). \tag{3}$$

Let us see how to get it using only undergraduate tools. After it, there are some indications about the variations to obtain the full proof of the Kronecker limit formula.

*Proof of Proposition 2.1 for $Q(x,y) = x^2+y^2$.* Define $g_s(x) = 2(x^2+1)^{-s}$ and $G(s) = -\int_{-\infty}^{\infty} g_s(x)\,dx$. The limit in (3) equals $L_1 - L_2$ with

$$L_1 = \lim_{s\to 1^+} \frac{1}{2\pi}\Big(\zeta(s,Q) + \zeta(2s-1)G(s)\Big), \qquad L_2 = \lim_{s\to 1^+} \zeta(2s-1)\Big(\frac{1}{2\pi}G(s) + 1\Big).$$

By a direct computation, $G(1) = -2\pi$ and L'Hôpital's rule shows $L_2 = (4\pi)^{-1}G'(1)$ because, by (1), $(2s-2)\zeta(2s-1) \to 1$. Then the result follows if we prove

$$L_1 = -\log|\eta(i)|^2 \text{ and } G'(1) = 4\pi\log 2 \text{ with } G'(1) = 2\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx. \tag{4}$$

To compute this integral the easy way is to look up a table (e.g. [5, 4.295.1]). If you want to be fully in charge, check the following formula performing the change of variables $x = \tan(t/2)$ and the application of Cauchy's integral formula on the unit circle $C$ parametrized as $z = \mathrm{e}^{it}$

$$G'(1) = -\int_{-\pi}^{\pi} \log|\cos(t/2)|^2\,dt = -\Re\int_C \log\Big(\frac{1+z}{2}\Big)\frac{dz}{iz} = 4\pi\log 2.$$

For the first formula in (4) we separate from $\zeta(s, x^2+y^2) = \sum_{m,n}(m^2+n^2)^{-s}$ the terms with $n = 0$ which contribute $2\zeta(2s)$. By the residue theorem in the band $B_\epsilon = \{|\Im z| < \epsilon\}$ with $0 < \epsilon < 1$,

$$\zeta(s, x^2+y^2) = 2\zeta(2s) + \sum_{n=1}^\infty \frac{1}{n^{2s}} \sum_{m\in\mathbb{Z}} g_s\big(\frac{m}{n}\big)$$
$$= 2\zeta(2s) + \sum_{n=1}^\infty \frac{-1}{2n^{2s-1}} \int_{\partial B_\epsilon} g_s(z) i\cot(\pi n z)\, dz,$$

because $2\pi\, i\, n \operatorname{Res}\big(i\cot(\pi n z), m/n\big) = -2$. As $g_s$ is even, $\int_{\partial B_\epsilon} = -2\int_{L_\epsilon}$ with $L_\epsilon = \{\Im z = \epsilon\}$ oriented to the right and the sum is $\sum_n n^{1-2s}\int_{L_\epsilon}$. Note that $\int_{L_\epsilon} g_s = \int_{L_0} g_s = -G(s)$. Then adding $\zeta(2s-1)G(s)$ is equivalent to replace $i\cot(\pi n z)$ by $i\cot(\pi n z) - 1$ in $\int_{L_\epsilon}$. The expansion $i\cot w - 1 = 2\mathrm{e}^{2iw}/(1-\mathrm{e}^{2iw}) = 2(\mathrm{e}^{2iw} + \mathrm{e}^{4iw} + \dots)$ assures an exponential decay and we have, substituting $\zeta(2) = \pi^2/6$,

$$L_1 = \frac{1}{2\pi}\Big(\frac{\pi^2}{3} + \sum_{n,k=1}^\infty \frac{2}{n}\int_{L_\epsilon} g_1(z)\mathrm{e}^{2\pi i n k z}\, dz\Big).$$

As an aside, a harmonic analyst might prefer to arrive to this formula using Fourier series or the Poisson summation formula. We continue with the complex analysis approach. The residue theorem in $\{\Im z > \epsilon\}$ gives promptly, noting $(z-i)(z+i)g_1(z) = 2$,

$$L_1 = \frac{\pi}{6} + \sum_{n,k=1}^\infty \frac{2}{n}\mathrm{e}^{-2\pi n k} = \frac{\pi}{6} - \sum_{k=1}^\infty \log\big(1-\mathrm{e}^{-2\pi k}\big)^2$$

where we have employed the Taylor expansion $\log(1-x)^2 = -2(x/1 + x^2/2 + \dots)$. The sum is $\log\big(\mathrm{e}^{\pi/6}|\eta(i)|^2\big)$ and the proof of (4) is complete. $\square$

The question is how close is this to a full proof of Proposition 2.1. Actually, it is quite close. Essentially, the whole point is to replace $x^2+1$ by $Q(x,1) = ax^2+bx+c$, restoring the constants coming from Proposition 2.1. Read [2, §3] for the full details. Here there are some hints for an intermediate level of details. In the general case, $G(s) = -2\int_{-\infty}^{\infty} Q(x,1)^{-s}\, dx$,

$$L_1 = \lim_{s\to 1^+} \frac{\sqrt{D}}{4\pi}\big(\zeta(s,Q) + \zeta(2s-1)G(s)\big)$$

and

$$L_2 = \lim_{s\to 1^+} \zeta(2s-1)\Big(\frac{\sqrt{D}}{4\pi}G(s) + 1\Big).$$

Again the limit in the statement is $L_1 - L_2$. The computation of $G'(1)$ to evaluate $L_2$ is as before because we can transform $Q(x,1)$ into a multiple of $x^2+1$ completing squares. This leads to

$$\frac{\sqrt{D}}{4\pi}G'(1) = \log\sqrt{\frac{D}{a}}.$$

The evaluation of $L_1$ follows the same lines. The only noticeable issue is that at some point we used that $g_s$ was even and $Q(x,1)$ is not in general. The simple solution is to substitute $g_s(x)$ by $Q(x,1)^{-1} + Q(x,-1)^{-1}$. With this change, we get

$$L_1 = -\log\eta(z_Q) - \log\eta(-\bar{z}_Q) = -\log\big|\eta(z_Q)\big|^2.$$

The values $z_Q$ and $-\bar{z}_Q$ come from the fact that $g_s(z)$ has simple poles at these points in the upper half plane.

## 5 A sharper result

Theorem 1.1 is a direct consequence of the stronger less symmetric result:

**Theorem 5.1.** *Let $a$ be a positive divisor of $B^2+1$, $B \in \mathbb{Z}$. Then*

$$\prod_{n=1}^{\infty} 2\mathrm{e}^{-2\pi n/a}\Big(\cosh\frac{2\pi n}{a} - \cos\frac{2\pi n B}{a}\Big) = \frac{1}{4}\Gamma^2\Big(\frac{1}{4}\Big)\mathrm{e}^{\pi/(6a)}\sqrt{\frac{a}{\pi^3}}$$

*where $\Gamma$ indicates the classical Gamma function.*

The last sentence is not very informative if you have not heard about the Gamma function. In this case, you only need to learn that

$$\Gamma\Big(\frac{1}{4}\Big) = 4\int_0^\infty \mathrm{e}^{-t^4}\,dt = 3.6256099\ldots$$

and it is not known a closed expression for this constant in term of high school mathematical constants.

Dividing the formula of Theorem 5.1 for two choices of the parameters, we get Theorem 1.1. Then both results become equivalent if we assume Theorem 5.1 for a single couple $(a,B)$. For instance $(1,0)$, which gives

$$\prod_{n=1}^{\infty} 2\mathrm{e}^{-2\pi n}\big(\cosh(2\pi n) - 1\big) = \frac{\mathrm{e}^{\pi/6}}{4\pi^{3/2}}\Gamma^2\Big(\frac{1}{4}\Big).$$

This follows immediately squaring the evaluation

$$\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}. \tag{5}$$

An strategy to get it (see [2] and [7]) is to use the nontrivial factorization

$$\zeta(s,x^2+y^2)=4\zeta(s)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},$$

which allows to compute the limit in the Kronecker limit formula in an alternative way.

The evaluation (5) relates to the classical problem of the inversion of elliptic integrals with theta functions [1] led by Jacobi and preceded by Gauss [8]. Even if you do not know what I am talking about, you will enjoy the impressive and highly nontrivial equalities

$$\sqrt{2}\,\eta(i)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\pi n^2}=\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\sqrt{2}\,dt}{\sqrt{2-\sin^2 t}}\right)^{1/2}.$$

The identity (5) is also a special case of the *Chowla-Selberg formula* [9]. This is a curious formula evaluating a product of $\eta$ at several quadratic values. In some cases, including (5), these values reduce to only one producing an individual evaluation. It was announced by its authors almost 20 years before they published the proof. The Fields medalist Selberg did not like to collaborate with other colleagues. In the nowadays ultra-connected scientific world, it sounds astonishing that Chowla-Selberg formula was the only joint work that Selberg published during his long and fruitful mathematical life.

**Acknowledgments.** I appreciate the suggestions provided by the reviewer. I am indebted to E. Valenti for the conversations and the endurance in these hard times.

## References

[1] J. V. Armitage and W. F. Eberlein. *Elliptic functions*, volume 67 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006.

[2] F. Chamizo. A simple evaluation of a theta value and the Kronecker limit formula. `arXiv:2107.07245 [math.NT]`, 2021.

[3] D. A. Cox. *Primes of the form* $x^2+ny^2$. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989. Fermat, class field theory and complex multiplication.

[4] C. F. Gauss. *Disquisitiones arithmeticae*. Springer-Verlag, New York, 1986. Translated and with a preface by Arthur A. Clarke, Revised by W. C. Waterhouse, C. Greither and A. W. Grootendorst and with a preface by Waterhouse.

[5] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian. Translation edited and with a preface by A. Jeffrey and D. Zwillinger.

[6] Y. Hellegouarch. *Invitation to the mathematics of Fermat-Wiles*. Academic Press, Inc., San Diego, CA, 2002. Translated from the second (2001) French edition by L. Schneps.

[7] N. Kurokawa, M. Kurihara, and T. Saito. *Number theory. 3*, volume 242 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2012. Iwasawa theory and modular forms, Translated from the Japanese by M. Kuwata, Iwanami Series in Modern Mathematics.

[8] B. L. Laptev, B. A. Rozenfel′d, and A. I. Markushevich. *Mathematics of the 19th century*. Birkhäuser Verlag, Basel, 1996. Geometry, analytic function theory, With a bibliography by F. A. Medvedev, Edited and with a preface by A. N. Kolmogorov and A. P. Yushkevich, Translated from the 1981 Russian original by R. Cooke.

[9] A. Selberg and S. Chowla. On Epstein's zeta-function. *J. Reine Angew. Math.*, 227:86–110, 1967.

[10] C. L. Siegel. A simple proof of $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/i}$. *Mathematika*, 1:4, 1954.

[11] C. L. Siegel. *Advanced analytic number theory*, volume 9 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay, second edition, 1980.

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*Publicat el 14 de gener de 2022*