MAT[2]
MATerials MATemàtics
Volum 2023, treball no. 4, 9 pp. ISSN: 1887-1097
Publicació electrònica de divulgació del Departament de Matemàtiques
de la Universitat Autònoma de Barcelona
www.mat.uab.cat/web/matmat

The golden ratio from a calculus point of view

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1 Introduction

The *golden number* or golden ratio ϕ, the positive number defined by the equation

$$\phi(\phi - 1) = 1, \ \phi = \frac{1+\sqrt{5}}{2} = 1,618033\ldots \qquad (1.1)$$

is widely known in relation with many different areas: mathematics and geometry, natural sciences, art, architecture, music, etc. The wikipedia article on the golden ratio is a good source of information on these many appearances of ϕ.

In this note we will present another look to the golden number from a calculus point of view. More precisely we will be dealing with the power functions x^{ϕ} and $x^{-\alpha}$, for $x > 0$, where $\alpha = \phi - 1 = \phi^{-1}$. The motivation comes from the so-called *power laws*. Power laws are ubiquitous in different areas of social sciences under different names, Zipf laws in linguistics, Pareto laws in economical sciences, etc. To the best of my knowledge, these laws play in social sciences a role similar to normal laws in statistics, but well funded mathematical statements to explain this are not so widely known.

When listening to a number of talks in linguistics in which power laws $x^{-\tau}$ appear modeling different aspects, I often ask the speaker about the exponent τ, to find out that values close to α are most common. Understanding whether the power law, with exponent α, has some intrinsic meaning seems to be an appealing research project, although not precisely defined.

In this note, as a first step and in a completely deterministic setting, first we ask ourselves whether the power functions x^{ϕ} and $x^{-\alpha}$ have some special property. In the second section we show that indeed they enjoy a characteristic property among all self-homeomorphisms of the positive real line. In the third section we show that in fact they are attractor points of a certain transformation T acting on self-homeomorphisms of $(0, +\infty)$. These two facts are just easy consequences of interpreting the defining equation (1.1) in terms of the power functions.

In the last section we will describe an integer sequence, that may be viewed as a discrete version of the power law x^ϕ. This integer sequence turns out to be a known sequence, the *Golomb sequence* quoted as sequence *A001462* in the *On-line encyclopedia of integer sequences*, `https://oeis.org`. We introduce a discrete analogue T_d of the transformation T, acting on integer sequences, and show that the Golomb sequence is a fixed point and an attractor for T_d. The transformation T_d is related to the operation of extracting frequencies. The fact that the golden ratio has some relation with frequencies was hinted to me some time ago by Alvaro Corral.

2 The golden diffeomorphism

We start by noticing that the defining equation (1.1) means that the derivative of the power function x^ϕ equals, up to a constant, its inverse. This turns out to be a characterization:

Theorem 2.1. *The function $f_0(x) = \alpha^\alpha x^\phi$, $x \geq 0$, is the unique solution of $f' = f^{-1}$, $f(0) = 0$. More precisely, if $f : [0, +\infty) \to [0, +\infty)$, $f(0) = 0$, $f(+\infty) = +\infty$ is strictly non-decreasing, differentiable and $f' = f^{-1}$, then $f = f_0$.*

As a restatement, the function $h_0(x) = (f_0)^{-1}(x) = f_0'(x) = \phi\, \alpha^\alpha x^\alpha = \phi^{1-\alpha} x^\alpha$ is then the unique self-diffeomorphism h of the positive real line such that $h = (h^{-1})'$. We call f_0, h_0 the *golden diffeomorphisms*.

Proof. Computation shows that f_0 is a solution, since $\alpha(\alpha + 1) = 1$. Now, if f is as in the statement, $(f(x) - x)' = f'(x) - 1 = f^{-1}(x) - 1$ grows from -1 to $+\infty$, whence $f(x) - x$ decreases from 0 to an absolute minima and increases thereafter. Therefore, the graph $y = f(x)$ meets $y = x$ at a unique point (a, a), and f maps $(0, a)$ onto $(0, a)$ and $(a, +\infty)$ onto $(a, +\infty)$. Now we argue separately in $(0, a)$ and $(a, +\infty)$.

In $(0, a)$ one has $f'(f(x)) = x \geq f(x)$, that is $f'(t) \geq t$ for $t \in (0, a)$, since $f(x)$ fills $(0, a)$. Thus, integrating we get $f(t) \geq \frac{t^2}{2}$, that is, $t \leq \sqrt{2\, f(t)}$. Then $f'(f(x)) = x \leq \sqrt{2}\,\sqrt{f(x)}$, that is, $f'(t) \leq \sqrt{2}\,\sqrt{t}$, implying $f(t) \leq \frac{2\sqrt{2}}{3} t^{\frac{3}{2}}$. Thus we get for some constants a_2, a_3

$$f(t) \leq t, f(t) \geq a_2 t^2, f(t) \leq a_3 t^{\frac{3}{2}}, \ldots$$

and so on. Assuming inductively that $f(t) \leq a_n\, t^{b_n}$, one has

$$f'(f(x)) = x \geq (\frac{1}{a_n})^{\frac{1}{b_n}} (f(x))^{\frac{1}{b_n}},$$

that is, $f'(t) = x \geq (\frac{1}{a_n})^{\frac{1}{b_n}} t^{\frac{1}{b_n}}$, implying $f(t) \geq a_{n+1} t^{b_{n+1}}$ with

$$a_{n+1} = \left(\frac{1}{a_n}\right)^{\frac{1}{b_n}} \frac{1}{1+\frac{1}{b_n}}, \quad b_{n+1} = 1 + \frac{1}{b_n}.$$

Similarly, if $f(t) \geq a_n t^{b_n}$ we get $f(t) \leq a_{n+1} t^{b_{n+1}}$. Now, the sequence defined inductively by $b_1 = 1$, $b_{n+1} = 1 + \frac{1}{b_n}$ has limit ϕ, because it is bounded and every limit L of a convergent subsequence satisfies $L(1+L) = 1$. Similarly, the sequence a_n has a limit L, with $L^{\phi} = \frac{1}{\phi}$, so $L = \alpha^{\alpha}$. Thus $f(x) = \alpha^{\alpha} x^{\phi}$ in $(0, a)$. Since $f(a) = a$, it follows that $a = \phi$.

For $x \in (\phi, +\infty)$, $f'(f(x)) = x \leq f(x)$, that is, $f'(t) \leq t$ for $t \geq \phi$, since $f(x)$ fills $(\phi, +\infty)$. This implies by integration

$$f(t) - \phi = f(t) - f(\phi) \leq \frac{1}{2}(t^2 - \phi^2), \;\; f(t) \leq \phi - \frac{1}{2}\phi^2 + \frac{1}{2}t^2.$$

Notice that $\phi - \frac{1}{2}\phi^2 = \frac{1}{2}(\phi - 1) > 0$. Here and later we will bound from above every expression $A + B\,t^{\delta}, A \geq 0, t \geq \phi$, with $A + B\phi^{\delta} = \phi$ as follows:

$$A + B\,t^{\delta} = \left(\frac{A}{t^{\delta}} + B\right) t^{\delta} \leq \left(\frac{A}{\phi^{\delta}} + B\right) t^{\delta} = \phi^{1-\delta} t^{\delta}. \tag{2.1}$$

Thus $f(t) \leq \phi^{-1} t^2$ if $t \geq \phi$, that is, $t \geq (\phi f(t))^{\frac{1}{2}}$. Then

$$f'(f(x)) = x \geq \phi^{\frac{1}{2}} f(x)^{\frac{1}{2}}, \;\; f'(t) \geq \gamma^{\frac{1}{2}}\, t^{\frac{1}{2}}, t \geq \phi.$$

This implies by integration

$$f(t) - \phi \geq \phi^{\frac{1}{2}}\, \frac{2}{3}\,(t^{\frac{3}{2}} - \phi^{\frac{3}{2}}), \;\; f(t) \geq \phi - \frac{2}{3}\,\phi^2 + \frac{2}{3}\,\phi^{\frac{1}{2}}\, t^{\frac{3}{2}}.$$

Notice that $\phi - \frac{2}{3}\phi^2 = \frac{1}{3}(\phi - 2) < 0$. Similarly, we bound from below every expression $A + B\,t^{\delta}, A \leq 0, t \geq \phi$, with $A + B\phi^{\delta} = \phi$ by

$$A + B\,t^{\delta} = \left(\frac{A}{t^{\delta}} + B\right) t^{\delta} \geq \left(\frac{A}{\phi^{\delta}} + B\right) t^{\delta} = \phi^{1-\delta}\, t^{\delta}. \tag{2.2}$$

Thus $f(t) \geq \phi^{-\frac{1}{2}}\, t^{\frac{3}{2}}, t \geq \phi$. This proceeds inductively as follows.

Assume $f(t) \leq \phi^{1-b_n}\, t^{b_n}$ for $t \geq \phi$, $b_n > \phi$, that is $t \geq \left(\phi^{b_n - 1} f(t)\right)^{\frac{1}{b_n}}$. Then

$$f'(f(x)) = x \geq \phi^{1-\frac{1}{b_n}} \left(f(x)\right)^{\frac{1}{b_n}}, \;\; f'(t) \geq \phi^{1-\frac{1}{b_n}}\, t^{\frac{1}{b_n}},$$

leading to

$$f(t) - \phi \geq \frac{\phi^{1-\frac{1}{b_n}}}{\frac{1}{1+\frac{1}{b_n}}} \left(t^{1+\frac{1}{b_n}} - \phi^{1+\frac{1}{b_n}}\right),$$

and

$$f(t) \geq \phi - \frac{\phi^2}{1+\frac{1}{b_n}} + \frac{\phi^{1-\frac{1}{b_n}}}{\frac{1}{1+\frac{1}{b_n}}} t^{1+\frac{1}{b_n}}.$$

Here

$$\phi - \frac{\phi^2}{1+\frac{1}{b_n}} = \frac{\phi - b_n}{1+b_n} < 0,$$

whence by (2.2) we get $f(t) \geq \phi^{1-b_{n+1}} t^{b_{n+1}}$ with $b_{n+1} = 1 + \frac{1}{b_n} < \phi$.

In the other direction, starting from $f(t) \geq \phi^{1-b_n} t^{b_n}$ for $t \geq \phi, b_n < \phi$, one obtains in the same way

$$f(t) \leq \frac{\phi - b_n}{1+b_n} + \frac{\phi^{1-\frac{1}{b_n}}}{\frac{1}{1+\frac{1}{b_n}}} t^{1+\frac{1}{b_n}},$$

which by (2.1) is bounded by $\phi^{1-b_{n+1}} t^{b_{n+1}}$.

Since $b_n \to \phi$ it follows that $f(t) = \phi^{1-\phi} t^{\phi} = \alpha^{\alpha} t^{\phi}$ too. □

In a similar way one can prove that if $f : [0,+\infty) \to [0,+\infty)$, $f(0) = +\infty$, $f(+\infty) = 0$, is strictly non-increasing, differentiable and $f' = -f^{-1}$, then $f(x) = \alpha^{\phi} x^{-\alpha}$. The derivative $-\alpha^{1+\phi} x^{-\phi}$ is then the unique strictly non-increasing h such that $h = -\left(h^{-1}\right)'$.

3 The golden homeomorphism as an attractor

In this section we seek for another description of the golden homeomorphisms within all non-decreasing homeomorphisms of the positive real line.

By theorem 2.1, the equation $f' = f^{-1}$ characterizes f_0 among all diffeomorphisms of the positive real line. Write I for the operation of taking inverses, $If = f^{-1}$, Df for f' and D^{-1} for the integration operator

$$D^{-1}f(y) = \int_0^y f(t)\,dt.$$

Indeed, $Df = g$ and $f = D^{-1}g$ are equivalent statements for functions vanishing at zero. From $Df_0 = If_0$ we may say that f_0 is a fix point of $S = ID$ and of $D^{-1}I$, or that $(f_0)^{-1}$ is a fix point of DI and ID^{-1}. Of those transformations, the ones involving D are not defined for all homeomorphisms and cannot be iterated. That's why we choose working instead say with ID^{-1}.

For a non-decreasing homeomorphism $h : [0,+\infty) \to [0,+\infty)$, $h(0) = 0$, $h(+\infty) = +\infty$, we define $Th = ID^{-1}h$, that is the map defined by the equation

$$\int_0^{Th(x)} h(t)\,dt = x.$$

Clearly Th is well-defined and it is a non-decreasing homeomorphism too, the inverse of $D^{-1}h$. We consider T as a self map in the class of all non-decreasing homeomorphisms of $[0,+\infty)$. In fact Th is differentiable and $(Th)'(x)\,h(Th(x))=1$, whence $(Th)'$ strictly decreases and so Th is strictly concave. Notice too that if $h_1(x)\le h_2(x)$, then $Th_1\ge Th_2$.

Theorem 3.1. *The golden homeomorphism $h_0(t)=\phi^{1-\alpha}t^\alpha$ is not only a fix point but an attractor for T, that is $T^k(h)\to h_0$ point-wise for all non-decreasing homeomorphisms h.*

Proof. Let us first estimate Th when $h=h_{a,b,c,d,e}$ has the specific form

$$h(t)=a\,t^b,\ t\le c;\ h(t)=dt^e,\ t\ge c,$$

with $a\,c^b=d\,c^e$. Then

$$\int_0^x h(t)\,dt=\frac{a}{1+b}\,x^{1+b},\ x\le c, \tag{3.1}$$

and

$$\int_0^x h(t)\,dt=\frac{d}{1+e}\,x^{1+e}+A,\ A=c^{1+e}\Big(\frac{a}{1+b}-\frac{d}{1+c}\Big),\ x\ge c.$$

If $A\le 0$, similarly as in (2.2), we bound the last quantity from below as follows

$$\int_0^x h(t)\,dt=x^{1+e}\left(\frac{d}{1+e}+\frac{A}{x^{1+e}}\right)$$
$$\ge x^{1+e}\left(\frac{d}{1+e}+\frac{A}{c^{1+e}}\right)=x^{1+e}\frac{d}{1+b}, \tag{3.2}$$

where we have used $a\,c^b=d\,c^e$. Now (3.1) says that $x=Th(s)$ for $s=\frac{a}{1+b}x^{1+b}\le\frac{a}{1+b}c^{1+b}$, that is

$$Th(s)=a'\,s^{b'},\ s\le c',\ a'=\left(\frac{1+b}{a}\right)^{\frac{1}{1+b}},\ b'=\frac{1}{1+b},\ c'=\frac{a}{1+b}c^{1+b},$$
$$Th(c')=c.$$

In the same way (3.2) implies that for $s\ge c'$, that is for $x=Th(s)\ge c$, one has

$$s=\int_0^x h(t)\,dt\ge x^{1+e}\frac{d}{1+b},$$

which amounts to

$$x=Th(s)\le d'\,s^{e'},\ d'=\left(\frac{1+b}{d}\right)^{\frac{1}{1+e}},\ e'=\frac{1}{1+e},\ s\ge c'.$$

Thus $Th_{a,b,c,d,e} \leq h_{a',b',c',d',e'}$ if $A \leq 0$.

In a similar way we see that $Th_{a,b,c,d,e} \geq h_{a',b',c',d',e'}$ if $A \geq 0$. Notice that $A = a\, c^{1+b}\left(\frac{1}{1+b} - \frac{1}{1+e}\right)$ has the sign of $e - b$.

Now we can prove that $T^k(h) \to h_0$ for an arbitrary h. By the remark before the statement we may assume that h is strictly concave. Then there is $c > 0$ such that $h(c) = c$ and therefore $h \geq h_1 = h_{1,1,c,1,0}$, defined $h_1(t) = t$ for $t \leq c$ and $h_1(t) = c$ for $t \geq c$. Let $h_n = h_{a_n,b_n,c_n,d_n,e_n}$ with $a_1 = 1$, $b_1 = 1$, $c_1 = c$, $d_1 = 1$, $e_1 = 0$ and the a_n, b_n, c_n, d_n, e_n recursively defined by the map $(a, b, c, d, e) \to (a', b', c', d', e')$, that is

$$a_{n+1} = \left(\frac{1+b_n}{a_n}\right)^{\frac{1}{1+b_n}}, \; b_{n+1} = \frac{1}{1+b_n}, \; c_{n+1} = \frac{a_n}{1+b_n} c^{1+b_n},$$

$$d_{n+1} = \left(\frac{1+b_n}{d_n}\right)^{\frac{1}{1+e_n}}, \; e_{n+1} = \frac{1}{1+e_n}.$$

Notice that the recursion formula for b_n, e_n is the same and $b_1 = 1$, $e_1 = 0$, $e_2 = 1 = b_1$ whence $e_{n+1} = b_n$. Since $Th_n - h_{n+1}$ has the sign of $e_n - b_n = e_n - e_{n+1}$ and this keeps alternating we see that

$$Th_1 \leq h_2, \; Th_2 \geq h_3, \; Th_3 \leq h_4, \cdots,$$

and so on. Then, $h \geq h_1$ implies $Th \leq Th_1 \leq h_2$, so $T^2h \geq Th_2 \geq h_3$, and in general

$$T^{2k}h \geq h_{2k+1}, \; T^{2k+1}h \leq h_{2k+2}.$$

Now, it is plain that $(e_n), (b_n)$ have limit $\alpha = \phi - 1$. Then (a_n), (d_n) are easily seen to converge to $\phi^{1-\alpha}$ and (c_n) to ϕ, and the theorem is proved. □

4 The Golomb sequence as a discrete analogue

Given a finite set M of non-negative integers, possibly with repetitions, we define its *frequency content* $F(M)$ as the set of observed frequencies, that is, the set consisting of the frequencies $f_n(M)$, the number of elements in M equal to n. For instance, M might consist of the observed frequencies of words in a book, and now we would be looking at frequencies of frequencies. We are just interested in the values $f_n(M)$ of the frequencies, and not in n or in general the objects having those frequencies.

One can visualize both M and $F(M)$ as monotone sequences. For instance in linguistics, frequencies are ordered in non-increasing order. It is intuitively clear that the size of $F(M)$ is generally much smaller than that of M and that iteration of F leads to a singleton in a fast way. For instance, if M consists of K numbers selected at random between 1 and N, with $K \gg N$, $F(M)$ will consist of N numbers from 1 to K, that most likely would be all different, so that $F^2(M)$ would consist in N ones and $F^3(M)$ is a singleton.

To avoid that one might consider countable sets M instead, presented as a non decreasing sequence by convenience, $M = (m_n)$. Of course, then $F(M)$ is not defined in general. If say $m_n \to +\infty$, then $F(M)$ is defined, but $F^2(M)$ is not in general. In fact, there is no natural choice of a sequence space $\mathbb{S}$ in which F acts.

For a sequence M, we can view its frequency sequence as

$$f_n(M) = \#\{j : m_j \leq n\} - \#\{j : m_j \leq n-1\}. \tag{4.1}$$

Now let us consider again the operator $T = ID^{-1}$ of the previous section. A formal inverse of T is then $T^{-1} = DI$. Now notice that at a formal level the operator F in (4.1) is the discrete analogue of $T^{-1} = DI$. Indeed, for a non-decreasing sequence $M = (m_n)$,

$$IM(n) = \#\{j : m_j \leq n\},$$

is a sort of inverse and

$$DM(n) = m_n - m_{n-1}$$

is the discrete derivative.

Therefore, at a formal level the discrete analogue of T is the inverse of F, that we may call the *deploying operator* defined as follows. Given a non-decreasing sequence $M : m_1 \leq m_2 \leq \cdots$, we produce another sequence E such that $F(E) = M$ by including m_1 terms equal to 1 in E, m_2 terms equal to 2 and so on. This inverse or *deploying* operator makes sense for infinite non-decreasing sequences. We denote by $\mathbb{S}$ the space of non-decreasing sequences of positive integers

$$M : 1 = m_1 \leq m_2 \leq \cdots \leq m_n \leq \cdots,$$

and by $T_d : \mathbb{S} \to \mathbb{S}$ the deploying operator just defined: starting from M, $T_d(M)$ is the non-decreasing sequence consisting in one 1 followed by $m_2\, 2's$ and so on.

Now think in a sequence $G = (G_n)$ such that $T_d(G) = G$, or $G = F(G)$, that is, for all n, G_n equals the number of n's in the sequence. This sequence is unique, namely

$$G : 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, \cdots$$

In fact, notice that since $G_1 = 1$, G_2 cannot be one, so $G_2 = 2$; this implies $G_3 = 2$ and so on. This is a known sequence, called the *Golomb sequence*, sequence *A001462* in the *On-line encyclopedia of integer sequences*, `https://oeis.org`. It is immediate to realize that G is an attractor for T_d, that is, starting from an arbitrary M the sequence $T_d^k(M)$ stabilizes to G. In fact the n-th term of the sequence stabilizes to G_n in at most n iterations. In the OEIS web page one can find references dealing with the asymptotic behavior of G, as stated in the next theorem. The result confirms that the Golomb sequence is the discrete analogue of the homeomorphism h_0 in theorem 2.1. We provide a proof for completeness (notice that the constant is the same as in theorem 2.1).

Theorem 4.1. *$G = (G_n)$ behaves like $\phi^{1-\alpha}n^\alpha$, that is, $\lim_{n\to\infty} G_n n^{-\alpha} = \phi^{1-\alpha}$.*

Proof. Set $c_n = G_n n^{-\alpha}$. We show first by induction that choosing a small enough and b big enough one has $a \le c_n \le b$. So we assume $a < 1 < b$. If $Y_n = \sum_{i=1}^{n} G_i = \#\{j : G_j \le n\}$, one has $G_n = m$ for $Y_{m-1} < n \le Y_m$. It is clear that $m < n$, that is, G_n depends just on G_i, $i < n$. Assume $a \le c_n \le b, n < N$. Then

$$Y_n \le b\sum_{i=1}^{n} i^\alpha \le \int_1^{n+1} t^\alpha\, dt \le \frac{b}{\alpha+1}(n+1)^{\alpha+1},$$

and similarly $Y_n \ge \frac{a}{\alpha+1}n^{\alpha+1}$. If $Y_{m-1} < N \le Y_m$, then

$$\frac{a}{\alpha+1}(m-1)^{\alpha+1} \le N \le \frac{b}{\alpha+1}(m+1)^{\alpha+1}.$$

Therefore

$$\left(\frac{a}{\alpha+1}\right)^\alpha (m-1) \le N^\alpha \le \left(\frac{b}{\alpha+1}\right)^\alpha (m+1),$$

$$\frac{2}{3}\left(\frac{\alpha+1}{b}\right)^\alpha N^\alpha \le m \le 2\left(\frac{\alpha+1}{a}\right)^\alpha N^\alpha,$$

and

$$\frac{2}{3}\left(\frac{\alpha+1}{b}\right)^\alpha \le c_N = \frac{m}{N^\alpha} \le 2\left(\frac{\alpha+1}{a}\right)^\alpha.$$

Thus it is enough to choose a, b such that $2\left(\frac{\alpha+1}{a}\right)^\alpha \le b$, $\frac{2}{3}\left(\frac{\alpha+1}{b}\right)^\alpha \ge a$, that is $k_1\, a^{-\alpha} \le b \le k_2\, a^{-\frac{1}{\alpha}}$, which is indeed possible for a small enough because $\frac{1}{\alpha} > \alpha$.

To prove the more precise statement about c_n we need exploiting the fact that in the above argument the tails are most important. Set $L = \limsup c_n = \lim_n d_n$, $d_n = \sup_{k\ge n} c_k$, $l = \liminf c_n = \lim_n e_n$, $e_n = \inf_{k\ge n} c_k$. If N, m are as above and $p = [m^\alpha] \ll m$ we use in the induction argument that $a \le c_n \le b$ for $n < p$ and $e_p \le c_n \le d_p$ for $p \le n < N$ to get

$$\frac{e_p}{\alpha+1}\left(1-\frac{1}{p}\right)m^{\alpha+1} - O(m) \le N \le \frac{d_p}{\alpha+1}\left(1+\frac{1}{p}\right)m^{\alpha+1} + O(m).$$

Therefore

$$\left(\frac{e_p}{\alpha+1}\left(1-\frac{1}{p}\right)\right)^{\alpha} m \le N^{\alpha}(1+o(1)),$$
$$\left(\frac{d_p}{\alpha+1}\left(1+\frac{1}{p}\right)\right)^{\alpha} m \ge N^{\alpha}(1-o(1)),$$

that is

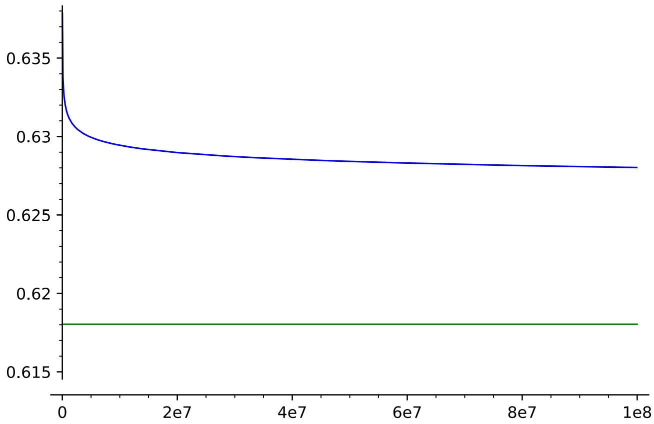
$$(1-o(1))\left(\frac{\alpha+1}{d_p}\right)^{\alpha}\left(1+\frac{1}{p}\right)^{-\alpha} \le c_N \le \left(\frac{\alpha+1}{e_p}\right)^{\alpha}\left(1-\frac{1}{p}\right)^{-\alpha}(1+o(1)).$$

Taking limit as $N \to +\infty$ (so $p \to \infty$ as well) gives

$$\left(\frac{\alpha+1}{L}\right)^{\alpha} \le l,\ L \le \left(\frac{\alpha+1}{l}\right)^{\alpha},$$

which implies $L = l = (1+\alpha)^{1-\alpha}$. □

Computer assisted generation of the sequence G_n shows that the convergence to α of $\frac{\log G_n}{\log n}$ is very slow, see next figure. In fact, after $n = 10^8$ terms, $\frac{\log G_n}{\log n} - \alpha \approx 0.00999$



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Publicat el 8 de novembre de 2023