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Instructions for carpenters on how to cut a tetrahedron

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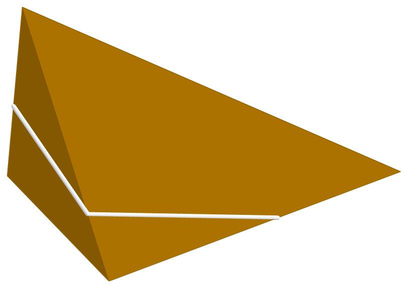

It is actually an easily formulated task. Write instructions for carpenters on how to make a cut of a wooden tetrahedron; of course not a regular one, but a quite general one. A cut by a plane in such a way that the new face which arises after cutting is a parallelogram with the maximum possible area. (The author found inspiration for this problem in the Exercise 342 of the high school problem book [1] written in Russian, where the stereometric chapter was written by the world-famous mathematician Boris Nikolayevich Delone. However, that was only the first inspiration; Delone does not address the question of the maximality of a parallelogram, nor does he address the question of providing easily understandable instructions to a non-mathematician.)

1 The normal vector of the cutting plane

For a tetrahedron $ABCD$, we take

$$\overrightarrow{n_1} = \overrightarrow{AB} \times \overrightarrow{CD}.$$

A plane ρ_1 with this normal vector intersects faces ABC and ABD in parallel lines (with the direction vector $\overrightarrow{AB}$) and it also intersects faces ACD and BCD in parallel lines (with the direction vector $\overrightarrow{CD}$).

Analogously, we consider a plane ρ_2 with the normal vector

$$\overrightarrow{n_2} = \overrightarrow{AC} \times \overrightarrow{BD}$$

and a plane ρ_3 with the normal vector

$$\overrightarrow{n_3} = \overrightarrow{AD} \times \overrightarrow{BC}.$$

To summarize, we have three one-parameter systems of planes giving a parallelogram in their intersection with the tetrahedron.

2 The area of the parallelogram

Without loss of generality, we can write coordinates of $ABCD$ as

$$A = [0, 0, 0]\,,\ B = [b_1, 0, 0]\,,\ C = [c_1, c_2, 0]\,,\ D = [d_1, d_2, d_3]\,,$$

where $b_1 > 0$, $c_2 > 0$, $d_3 > 0$. We take the plane ρ_1 with the normal vector $\overrightarrow{n_1}$. Further, we denote vertices of the parallelogram arising by a cut of this plane by P, Q, R and S, see Figure 1 (clicking on it you can experiment with a dynamic GeoGebra construction).

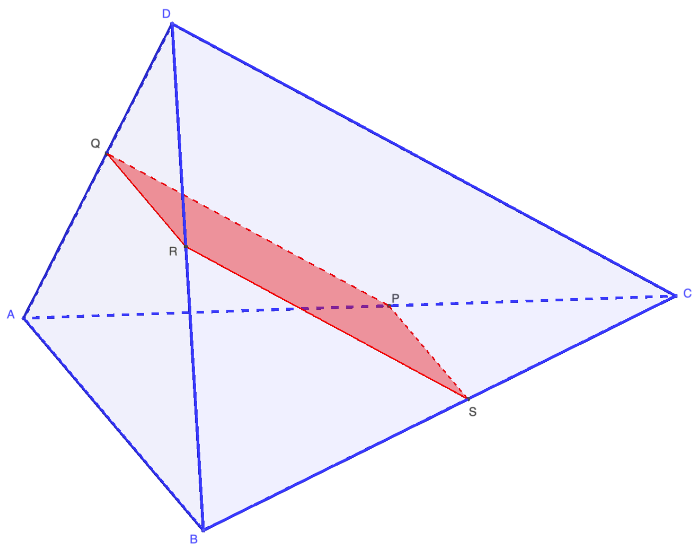

Figure 1: The intersection of the tetrahedron $ABCD$ with a plane with the normal vector $\overrightarrow{n_1} = \overrightarrow{AB} \times \overrightarrow{CD}$.

As P is on the edge AC with $A = [0, 0, 0]$ and $C = [c_1, c_2, 0]$, then $P = [c_1 t, c_2 t, 0]$ for $t \in [0, 1]$. We proceed in the same way for the other points of the parallelogram: therefore, we express

$$\begin{aligned} P &= [c_1 t, c_2 t, 0]\,,\\ Q &= [d_1 t, d_2 t, d_3 t]\,,\\ R &= [b_1 + (d_1 - b_1)t, d_2 t, d_3 t]\,,\\ S &= [b_1 + (c_1 - b_1)t, c_2 t, 0]\,, \end{aligned}$$

where $t \in [0, 1]$. The area of the space parallelogram given by vectors $\overrightarrow{PQ}$,

$\overrightarrow{PS}$ is

$$\sqrt{\begin{aligned}&\big((q_1-p_1)(s_2-p_2)-(q_2-p_2)(s_1-p_1)\big)^2\\&\qquad+\big((q_2-p_2)(s_3-p_3)-(q_3-p_3)(s_2-p_2)\big)^2\\&\qquad\quad+\big((q_3-p_3)(s_1-p_1)-(q_1-p_1)(s_3-p_3)\big)^2\end{aligned}}$$

which for our particular points gives the function with variable t

$$\mathcal{G}(t) = b_1 t(1-t)\sqrt{(c_2-d_2)^2+d_3^2}\,.$$

By standard calculus, we find the function $\mathcal{G}(t)$ has its maximum for $t=\frac{1}{2}$. Hence

$$\mathcal{G}_{\max} = \frac{1}{4} b_1\sqrt{(c_2-d_2)^2+d_3^2}\,. \tag{1}$$

We remark that $\sqrt{(c_2-d_2)^2+d_3^2}$ expresses the length of the projection of the edge CD onto a plane perpendicular to the x-axis.

3 A note on rotations

The order of the group of rotations of a regular tetrahedron is 12. Of course, it is a subgroup of symmetric group S_4 with the order 24. However, we already know that it is enough to consider three different cuts. So a subgroup of the group of rotations in which we keep (for example) the vertex D always as the top vertex is sufficient for us.

So, if the edges of the tetrahedron $ABCD$ be of lengths

$$|AB| = u_1,\ |AC| = u_2,\ |AD| = u_3,\ |BC| = u_4,\ |BD| = u_5,\ |CD| = u_6,$$

we take this identical sextuple $(u_1, u_2, u_3, u_4, u_5, u_6)$ and together with it sextuples $(u_4, u_1, u_5, u_2, u_6, u_3)$ and $(u_2, u_4, u_6, u_1, u_3, u_5)$. Indeed, note that these permutations are group forming.

4 The instructions

- Mark the vertex D on the tetrahedron. It is fixed and it will always be at the top. Names of vertices on the base will be working and will change.

- Draw perpendicular x and y axes on paper. Put the selected edge of the tetrahedron in the positive x-semiaxis: in particular, put A at the intersection of the two axes, B on the positive x-semiaxis and C so that its y-coordinate is positive.

- Measure the edge AB. Shine a flashlight on the tetrahedron, from the direction of the positive x-semiaxis. On a plane that is behind the tetrahedron and that is perpendicular to the x-axis, measure the shadow cast by edge CD. Multiply these two numbers and write down the product carefully.

- Now repeat this by putting another edge on the positive side, so the vertices A, B and C will be renamed. And then repeat once more for the remaining edge.

- From the three results, determine the maximum and return to the configuration for which it was achieved.

- Make the cut as shown in the Figure 1, going through centers of edges. Use a pencil to connect these centers to see exactly where the cut will go.

5 Edge lengths

It is evident that the edge lengths cannot be arbitrary to form a tetrahedron. The paper [2] discusses the limitations for edges nicely. Here we write only the main result.

The edges on all six faces must satisfy triangular inequalities and, moreover, the Calyer-Menger determinant

$$\begin{vmatrix} 0 & u_1^2 & u_2^2 & u_3^2 & 1 \\ u_1^2 & 0 & u_4^2 & u_5^2 & 1 \\ u_2^2 & u_4^2 & 0 & u_6^2 & 1 \\ u_3^2 & u_5^2 & u_6^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

must be positive. We remark that, in our notation, u_6 is the opposite edge to u_1, u_5 is the opposite edge to u_2 and u_4 is the opposite edge to u_3. By "opposite" we mean we mean not sharing any vertex. A sextuple $(u_1, u_2, u_3, u_4, u_5, u_6)$ that meets the conditions above is then called the *tetrahedral sextuple.*

6 The formula using edge lengths

We can express b_1, c_1, c_2, d_1, d_2 and d_3 by u_i, $i = 1, \ldots, 6$ as well. Using this and the permutations in question, we can rewrite (1) into the effective formula

$$\mathcal{G}_{\max} = \frac{1}{8}\max\left\{\sqrt{4u_1^2u_6^2 - \left(u_2^2 - u_3^2 - u_4^2 + u_5^2\right)^2},\right. \sqrt{4u_4^2u_3^2 - \left(u_1^2 - u_5^2 - u_2^2 + u_6^2\right)^2}, \left.\sqrt{4u_2^2u_5^2 - \left(u_4^2 - u_6^2 - u_1^2 + u_3^2\right)^2}\right\} \tag{2}$$

which can be applied directly for a given tetrahedral sextuple $(u_1, u_2, u_3, u_4, u_5, u_6)$.

7 The numerical example

For instance, let the edges of the tetrahedron $ABCD$ be of lengths

$$|AB| = 16,\ |AC| = 20,\ |AD| = 25,\ |BC| = 24,\ |BD| = 32,\ |CD| = 29.$$

After checking that the lengths form a tetrahedral sextuple, we can use the formula (2) and obtain the area of the largest parallelogram as $7\sqrt{519} \approx 159.471$.

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