

Numbers that commute under exponentiation in complex plane

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The raising to the a -th power and the exponential function with base a are defined for all positive arguments x , if a is any positive real number.

For various a we calculate the coordinates of the

non-trivial common point of the graphs of the above mentioned functions.

We also discuss a similar problem for complex numbers defining a power dual number and show that there exist infinitely many numbers commuting with a given one under the exponentiation operation.

$$x^a = a^x$$

1 Introduction

Each of us has —while learning elementary arithmetic— noticed that $2^4 = 4^2 = 16$ and probably got curious whether there exist other pairs of integers (a, x) , $a \neq x$ commuting this way: $x^a = a^x$. Obviously, each such pair corresponds to a non-trivial common point $(x, y) \neq (a, a^a)$ of the graphs of functions $y = x^a$ and $y = a^x$.

On Fig. 1(a) the graphs of functions $y = x^2$ and $y = 2^x$ are presented, passing through common points $(2, 4) = (2, 2^2)$ and $(4, 16)$, the last of which is non-trivial. During the last three centuries a lot of investigations have been done on this topics, mainly independent on each other. We refer the interested reader to [Knoebel \(1981\)](#) who attached to his article an impressive list of over 120 collected references. Our intention here is to briefly summarize the results previously obtained for positive real numbers and then extend them to the field of complex numbers. For the last case the mentioned problem was not much examined earlier due to limited use of numerical methods in the pre-computer age.

Definition 1. Let $a \neq b$ be two positive real numbers. We call b a power dual number of a when $a^b = b^a$.

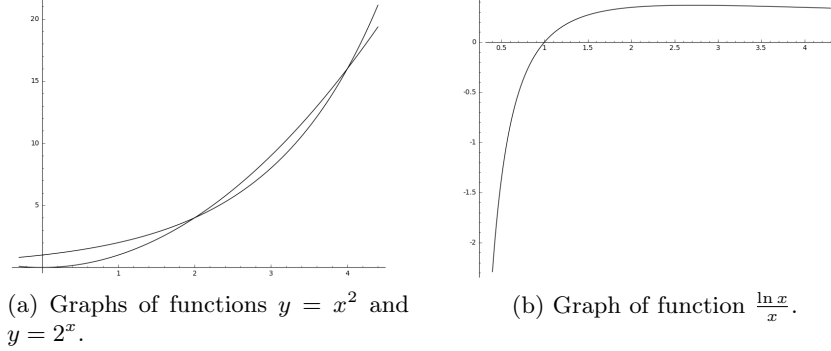


Figure 1: Some preliminary examples.

Obviously, the numbers 0 and 1 have no real power duals: for any $x \neq 0$ we have $0^x = 0 \neq 1 = x^0$ and from $1 = 1^x = x^1 = x$ it follows that $x = 1$. In this paper we show that for any non-zero complex number infinitely many power duals exist in the complex plain and we derive equations from which they (their approximate values) may be calculated.

We start with positive numbers a, x and take the logarithms of both sides of equality $x^a = a^x$ to obtain the equation $a \ln x = x \ln a$ or $\frac{\ln x}{x} = \frac{\ln a}{a}$ suggesting the study of mapping $x \mapsto \frac{\ln x}{x}$. It will occur in this paper several times so let us denote it by l . As $l'(x) = \frac{1-\ln x}{x^2}$ the graph of this function is increasing from minus infinity to the maximum point $(e, \frac{1}{e})$ and then decreasing towards x -axis (Fig. 1(b)).

Therefore we have the following result.

Theorem 1. *For each $a > 1$, $a \neq e$ there exists a unique positive power dual number $b = b(a)$. While a belongs to the interval $(1, e)$ we have $b \in (e, \infty)$ and vice versa. By additionally defining $b(e) = e$ we obtain an decreasing function b of argument a on the interval $(1, \infty)$ whose graph is presented on Fig. 2(a). On the other hand, the power $a^{b(a)}$ is an increasing function of a on the interval (e, ∞) with the image (e^e, ∞) (Fig. 2(b)). Finally, we find out that the numbers from the interval $(0, 1)$ have no positive power duals.*

Proof. We have already checked the content of the first two sentences. Next, let us consider b as a function of $a \in (1, \infty)$. As we assume that $b \ln a = a \ln b$ deriving by variable a gives us $b' \ln a + \frac{b}{a} = \ln b + \frac{a}{b} b'$. Thus $b' = \frac{\ln b - \frac{b}{a}}{\ln a - \frac{a}{b}} = \frac{b(a \ln b - b)}{a(b \ln a - a)} = \frac{b^2(\ln a - 1)}{a^2(\ln b - 1)} < 0$, for all $a \neq e$. And for the derivative of the function $p(a) = \ln a^b = b \ln a$ we obtain $p' = b' \ln a + \frac{b}{a} = \frac{b(\ln a \ln b - 1)}{a(\ln b - 1)}$. As the logarithm function is a 1-1 mapping this yields that the power $a^b = b^a$ will reach its extreme value exactly when $\ln b = \frac{1}{\ln a}$ or, equivalently, $b = e^{\frac{1}{\ln a}}$. But in this case $\frac{a}{\ln a} = \frac{b}{\ln b} = b \ln a = e^{\frac{1}{\ln a}} \ln a$, so the variable a must satisfy

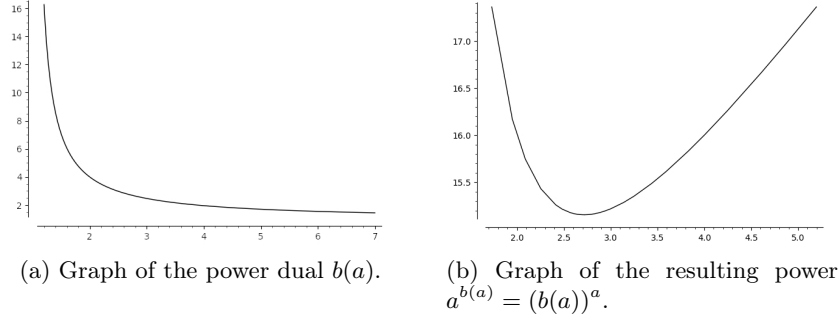


Figure 2: Power dual of any number greater than 1 and the related power.

the condition $a e^{-\frac{1}{\ln a}} - (\ln a)^2 = 0$. The only solutions of this equation are $a = e$, yielding $b = e$, and $a = \frac{1}{e} \notin (1, \infty)$. So the minimal value of a^b is $e^e \approx 15.1543$. For $a > e$ we have $\ln b < \ln e = 1$ and $\ln a \ln b < 1$ yielding $p' > 0$, so the function $p(a) = \ln a^{b(a)}$ is increasing and so are the values of our powers $a^{b(a)}$ (Fig. 2(b)).

Finally, let $a \in (0, 1)$. Suppose that its power dual number $b > a$ is greater than 1. Then, according to the properties of the power function, $a^b < a < 1$ while $b^a > 1$. Therefore, $a^b \neq b^a$. Similarly, we get a contradiction supposing that $b \in (a, 1)$, as then the properties of the power and exponential functions yield inequalities $a^b < b^b < b^a$. $\square$

Corollary 1. *The only pairs of integers $m \neq n$ satisfying the condition $m^n = n^m$ are $\{2, 4\}$ and $\{-2, -4\}$ for which the value of the power m^n is equal to 16 and $\frac{1}{16}$, respectively.*

Slobin (1931) gave a complete description of all pairs of positive numbers $x \neq y$ for which $x^y = y^x$, admitting between them the pairs of rational numbers. Let $y = tx$ where $t = 1 + \frac{1}{u}$, $u > 0$. Then $x^y = y^x$ or $y \ln x = x \ln y$ yields $t \ln x = \ln t + \ln x$. So $\ln x = \frac{\ln t}{t-1} = \frac{\ln(1+\frac{1}{u})}{\frac{1}{u}}$. Thus $x = (1 + \frac{1}{u})^u$ and $y = (1 + \frac{1}{u})^{u+1}$. If u runs over all positive integers we get all pairs of rational solutions of our problem: $(2, 4), (\frac{9}{4}, \frac{27}{8}), (\frac{64}{27}, \frac{256}{81}), (\frac{625}{256}, \frac{3125}{1024}), \dots$ Later, in Sato (1972), it was proved that both components of the solution will be algebraic numbers iff the parameter u is a rational number. Examples of such pairs are, for instance, $(\sqrt{3}, 3\sqrt{3}), (\frac{5}{9}\sqrt{15}, \frac{25}{27}\sqrt{15}), (\sqrt[3]{4}, 4\sqrt[3]{4})$ and $(\frac{100}{343}\sqrt[3]{490}, \frac{1000}{2401}\sqrt[3]{490})$. The solutions which we obtain while parameter u runs over positive numbers are presented on Fig. 2(a).

In Table 1 we give some examples of number a from the interval $(1, e)$ with (approximate) values of its power dual number $b \in (e, \infty)$ and the resulting power $a^b = b^a$.

The next problem consists in finding a power dual of a given real number $x > 1$. As our equation $x^y = y^x$, $x, y > 0$ is equal to $x^{\frac{1}{x}} = y^{\frac{1}{y}}$, to find

Table 1: Power duals of some $a > 1$ and the values of resulting powers.

a	1.15	1.3	1.45938	$\sqrt{3}$	2	$\frac{9}{4}$	2.38218	2.47805
b	27.1710	12.5246	8.05408	$3\sqrt{3}$	4	$\frac{27}{8}$	π	3
a^b	44.588	26.736	21	17.362	16	15.439	15.286	15.217

the power dual of a number $x > e$ it is sufficient to have inverse function $y^{\frac{1}{y}} \mapsto y$, mapping the interval $[1, e^{\frac{1}{e}}]$ onto $[1, e]$. [Knoebel \(1981\)](#) proved that such partial inverse exists and it is given as infinitely iterated exponentiation $y \mapsto \lim_{n \rightarrow \infty} \underbrace{(f_y \circ f_y \circ \dots \circ f_y)}_{n\text{-times}}(y)$ where $f_y(x) = y^x$ for all $x > 0$.

On Fig. 3(a) the graphs of function $x \mapsto x^{\frac{1}{x}}$ and its partial inverse are presented. In the last section we suggest an efficient algorithm for finding all power dual numbers of an arbitrary non-zero complex number z .

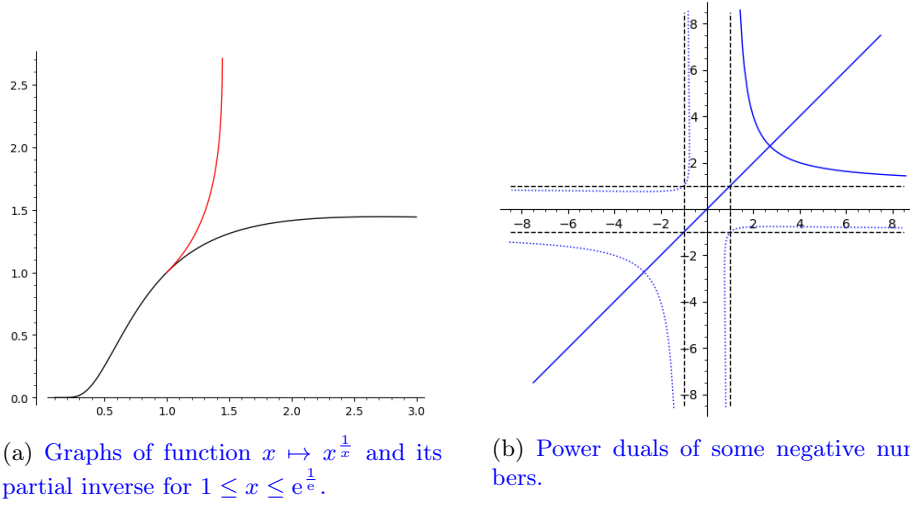


Figure 3: Real power duals of real numbers.

2 First generalization

To extend the above results to all real numbers we need to switch to complex arithmetic. By definition of the exponential function for every real φ we have $e^{i\varphi} = \cos \varphi + i \sin \varphi$, so it is more convenient to work with complex numbers written in polar form. Let $z = r e^{i\varphi}$ where $r = |z|$ is a non-negative number and φ is the so called principal argument of the given complex number, for instance, belonging to the interval $(-\pi, \pi]$. Then the complex logarithm function defined as $\ln z = \ln(r e^{i\varphi}) = \ln r + i \varphi$ has many properties similar to

those of real-valued logarithms. Basic facts and several examples of complex exponential and logarithm functions can be found in [Sochi \(2021\)](#).

For every real s we have $z^s = r^s e^{is\varphi}$. But due to the periodicity of sine and cosine functions the value of a power is now not necessarily uniquely determined. For instance, $(-1)^{\frac{1}{3}}$ can be -1 , $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ or $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ and for $16^{\frac{3}{4}}$ we obtain four different values: 8 , -8 , $8i$ and $-8i$. More than that, $2^{\sqrt{3}} = e^{\sqrt{3}(\ln 2 + 2k\pi i)}$, $k \in \mathbb{Z}$ has infinitely many different values in the complex plane. To distinguish between them we will (when necessary) add to the power a lower index related to the number of multiples of period 2π added to the principal argument of the base of the given power. For instance, $2^{\sqrt{3}}_0 \approx 3.322$, $2^{\sqrt{3}}_3 \approx 1.103 + 3.134i$ and $2^{\sqrt{3}}_{-1} \approx -0.374 + 3.301i$.

Definition 2. Let $z \neq w$ be two complex numbers. We will call w a power dual number of z (and vice versa) when one of the powers z^w matches with one of the powers w^z , i.e. there exist such $k, m \in \mathbb{Z}$ that $z^{w_k} = w^{z_m}$. When $z^{w_0} = w^{z_0}$ we call w the main power dual of z .

As we will see, a number can have several power duals in the complex plane. Figure 4 shows some points on few rays running from the coordinate beginning (left pictures) and their main power duals (right pictures). The rays presented are chosen in the first two quadrants as choosing them in the lower complex half plane we would obtain symmetric pictures with respect to the real axes.

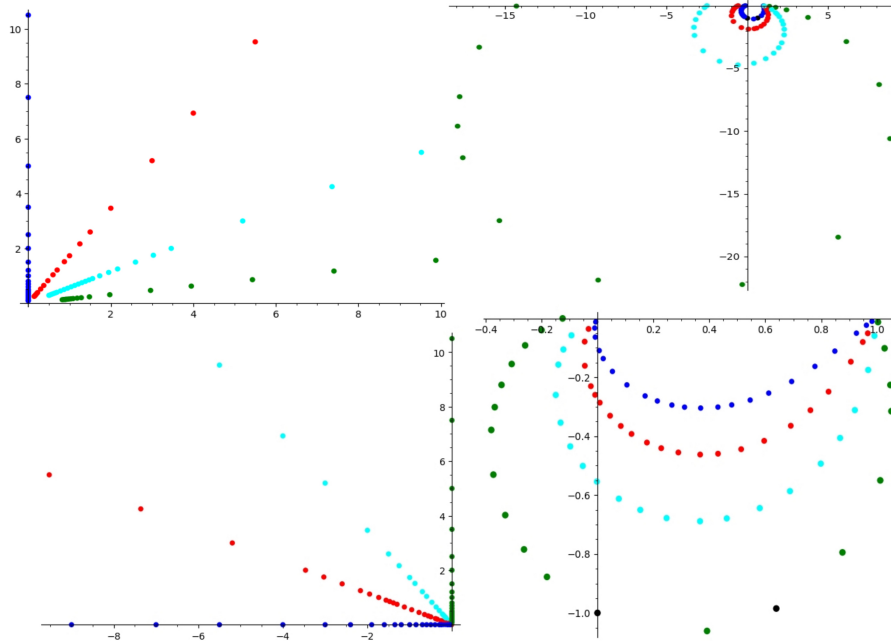


Figure 4: Some complex numbers and their main power duals.

The rest of this section is dedicated to the description of all pairs of real numbers (x, y) where y is a power dual of x (and vice versa). Let us start in the third quadrant, taking $x < 0$ and $y = tx$ with $t > 1$. Suppose that $x^y = e^{y \ln x} = e^{x \ln y} = y^x$. As $y \ln x - x \ln y = tx(\ln |x| + (2k+1)\pi i) - x(\ln |x| + \ln t + (2m+1)\pi i)$, $k, m \in \mathbb{Z}$, this yields $(t-1)x \ln |x| - x \ln t = 0$ and $(2m+1)x - (2k+1)y \in 2\mathbb{Z}$. From $\ln |x| = \frac{\ln t}{t-1}$ we obtain the solutions $(x, y) = (-t^{\frac{1}{t-1}}, -t^{\frac{t}{t-1}})$ with t satisfying $t^{\frac{1}{t-1}}(2m+1 - (2k+1)t) = 2n$ for some integer n . This is clearly true for all rational numbers t of the form $\frac{2m+1}{2k+1}$, $m > k > 0$ (in this case $n=0$). But there are also infinitely many other pairs of negative numbers that can be obtained this way. A few examples are given in Table 2. By substitution $t = \frac{u+1}{u}$ we get $x = -(1 + \frac{1}{u})^u$ and $y = -(1 + \frac{1}{u})^{u+1}$, so our solutions lay on the curve symmetric to the one obtained in the first quadrant (Fig. 3(b)). But now not every point of this curve represents a solution of our problem. For instance, $(-\frac{9}{4})^{-\frac{27}{8}} = (\frac{2}{3})^{\frac{27}{4}} \sqrt[4]{-1} \neq (-\frac{27}{8})^{-\frac{9}{4}} = (\frac{2}{3})^{\frac{27}{4}} \sqrt[4]{-1}$. This is the answer to one of the questions from Moulton (1916) and shows that the right choice of the parameter t is essential here.

Table 2: Some negative numbers with their negative power duals.

k	m	n	t	x	y	$x^y_k = y^x_m$
0	2	1	3.762	-1.6156	-6.0780	$0.0526 - 0.0131i$
1	2	1	1.381	-2.3333	-3.2222	$0.0326 + 0.0565i$
0	3	2	4.411	-1.5451	-6.8156	$-0.0431 - 0.0282i$
2	3	2	1.093	-2.6023	-2.8432	$0.0513 - 0.0414i$
0	4	4	3.971	-1.5907	-6.3163	$0.0291 - 0.0447i$
1	4	0	3	$-\sqrt{3}$	$-3\sqrt{3}$	$0.0158 + 0.0554i$
1	4	4	1.739	-2.1144	-3.6765	$-0.0635 + 0.00593i$

It remains to examine the situation in the even quadrants. We take $x < 0$, $y = -tx$, where $t > 0$. Then $x^y = y^x$ iff $(t+1) \ln |x| + \ln t = 0$ and $(2k+1)tx - 2mx \in 2\mathbb{Z}$. From the first condition we have $(x, y) = (-t^{-\frac{1}{t+1}}, t^{\frac{t}{t+1}})$. The last condition is obviously satisfied if t is a rational number of the form $\frac{2m}{2k+1}$, $m > 0$, $k \geq 0$, while in general the parameter t must satisfy the condition $2nt^{\frac{1}{t+1}} + (2k+1)t - 2m = 0$ for some integers k, m, n .

Let Γ be the set $\{(x, y) : x^y = y^x, xy < 0\}$. The points (x, y) that we just obtained lay above the bisector of even quadrants ($|x| < |y|$) iff $t > 1$. Replacing any $t > 1$ with its inverse $t' = \frac{1}{t} < 1$ we obtain a symmetric point $(-t'^{-\frac{1}{t'+1}}, t'^{\frac{t'}{t'+1}}) = (-t^{\frac{t}{t+1}}, t^{-\frac{1}{t+1}}) = (-y, -x)$. This means that $(-y, -x) \in \Gamma$ iff the equality $2nt'^{\frac{1}{t'+1}} + (2k+1)t' - 2m = 0$ holds for some integers k, m, n . But the condition that $(x, y) \in \Gamma$ was $2nt^{\frac{1}{t+1}} + (2k+1)t - 2m = 0$,

for some integers k, m, n , yielding $2nt^{\frac{1}{t'+1}} + 2k + 1 - 2mt' = 0$. Thus the bisector of even quadrants is not the line of symmetry of the graph Γ .

As the binary relation $x \sim y$ iff $x^y = y^x$ is clearly symmetric, the point (y, x) in the fourth quadrant belongs to graph Γ at the same time as (x, y) in the second quadrant. So the bisector of the odd quadrants is a line of symmetry of Γ . Clearly, $(1, -1)$ and $(-1, 1)$ are not elements of Γ as $1^{-1} = 1 \neq -1 = (-1)^1$. So we can extend the graph from Fig. 2(a) to the whole real plane, obtaining Fig. 3(b). A sketch of this curves was already given in Moulton (1916) but he did not mention their parametric equations.

The upper left curve on Fig. 3(b) has a local minimum. As $\frac{dy}{dt} = y^{\frac{t+1+\ln t}{(t+1)^2}}$, it is reached when $\ln t = -t - 1$ or $t = t_0 \approx 0.2785$. Then $x = -t_0^{\frac{-1}{t_0+1}} = -e^{\frac{-\ln t_0}{-t_0}} = -e$ and $y = t_0 e \approx 0.7569$. But the points $(-e, t_0 e)$ and $(-t_0 e, e)$ do not belong to graph Γ since the equality $(-e)^{t_0 e} = (t_0 e)^{-e}$ would yield $(2k+1)t_0 = 2m$ for some integers k, m . As we know that $t_0 = e^{-t_0-1}$, supposing that $t_0 = \frac{p}{q}$ is a rational number we would get that $e = \left(\frac{q}{p}\right)^{\frac{q}{p+q}}$ is an algebraic number what is a contradiction.

Thus we obtained the following theorem.

Theorem 2. *For real numbers with absolute value less or equal to $t_0 e \approx 0.757$ no real power dual number exist. If $t_0 e < |x| < 1$ number x has at most one real power dual while for $|x| > 1$ up to two real power duals may exist.*

Some examples of points on the left (upper) branch of Γ are listed in Table 3.

Table 3: Positive power duals of some negative numbers.

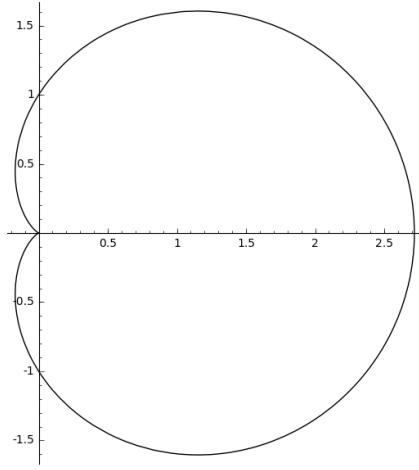
k	m	n	t	x	y	$x^y_k = y^x_m$
0	1	0	2	$-\frac{1}{2}\sqrt[3]{4}$	$\sqrt[3]{4}$	$0.188 - 0.667i$
0	4	0	8	$-\frac{1}{2}\sqrt[3]{4}$	$4\sqrt[3]{4}$	$0.105 + 0.205i$
1	1	0	$\frac{2}{3}$	$-\frac{1}{2}\sqrt[5]{108}$	$\frac{1}{3}\sqrt[5]{108}$	$-0.196 + 1.214i$
3	2	0	$\frac{4}{7}$	$-\frac{1}{4}\sqrt[11]{2^8 7^7}$	$\frac{1}{7}\sqrt[11]{2^8 7^7}$	$0.823 - 1.054i$
0	2	1	6.565	-0.77978	5.1191	$-0.2605 - 0.1023i$
2	2	1	4.411	-0.90833	1.1267	$0.3649 - 0.8198i$
11	4	1	0.3922	-1.9586	0.7682	$0.8494 - 1.445i$

3 Numbers with conjugated power duals

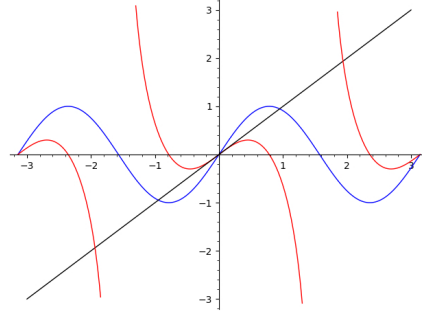
In the first section we found out that every real number greater then e^e can be represented as $a^b = b^a$ for some positive numbers $a \neq b$. The use of

complex numbers leads to a similar representation of real numbers from the interval $(1, e^e)$.

Let $z = r e^{i\varphi}$ and $w = s e^{i\psi}$ where r, s are positive numbers and φ, ψ arguments of the given complex numbers, belonging to the interval $(-\pi, \pi]$. Then $l(z) = \frac{\ln z}{z} = \frac{\ln r + i\varphi}{r(\cos \varphi + i \sin \varphi)} = \frac{1}{r}(\varphi \sin \varphi + \ln r \cos \varphi + i(\varphi \cos \varphi - \ln r \sin \varphi))$ and $l(w) = \frac{1}{s}(\psi \sin \psi + \ln s \cos \psi + i(\psi \cos \psi - \ln s \sin \psi))$. Suppose that these values are real numbers. Then their imaginary parts are 0 and we get $\ln r = \frac{\varphi}{\tan \varphi}$, $\ln s = \frac{\psi}{\tan \psi}$. Thus the numbers z, w lay on the curve $\mathcal{C}$ given in polar coordinates by equation $r = e^{\frac{\varphi}{\tan \varphi}}$.



(a) Curve $\mathcal{C}$ with equation $r = e^{\frac{\varphi}{\tan \varphi}}$.



(b) Solutions of equations $\sin 2x = x$ and $\cos 2x \tan x = x$.

Figure 5: On complex numbers with conjugated power duals.

The curve $\mathcal{C}$ is continuous if we additionally define $r(0) = e$, $r(\pm \frac{\pi}{2}) = \pm 1$, $r(\pi) = 0$ and is presented on Fig. 5(a). As $y(\varphi) = r \sin \varphi = e^{\frac{\varphi}{\tan \varphi}} \sin \varphi$, we get the extremal values of the imaginary part of number z when $\frac{dy}{d\varphi} = e^{\frac{\varphi}{\tan \varphi}} (\frac{\tan \varphi - \varphi \cos^{-2} \varphi}{\tan^2 \varphi} \sin \varphi + \cos \varphi) = 0$ or $\sin 2\varphi = \varphi$. Non-trivial solutions of this equation $\varphi \approx \pm 0.9477$ (Fig. 5(b)) result in $z \approx 1.1529 \pm 1.6046i$. Similarly we get the left most points $z \approx -0.17036 \pm 0.44397i$ on the curve $\mathcal{C}$ when $\cos 2\varphi \tan \varphi = \varphi$ or $\varphi \approx \pm 1.937$ (Fig. 5(b)).

The equality $z^w = w^z$ holds iff the real parts of the logarithms of these powers also match, thus $s(\varphi \sin \varphi + \frac{\varphi}{\tan \varphi} \cos \varphi) = r(\psi \sin \psi + \frac{\psi}{\tan \psi} \cos \psi)$ or $\frac{s \sin \psi}{\psi} = \frac{r \sin \varphi}{\varphi}$. So we are searching for the points on curve $\mathcal{C}$ determined by angles φ, ψ for which $f(\varphi) = f(\psi)$ where the function f is defined as $f(\varphi) = \frac{\sin \varphi}{\varphi} e^{\frac{\varphi}{\tan \varphi}}$. The function f is clearly even and its derivative $f'(\varphi) = \frac{e^{\frac{\varphi}{\tan \varphi}}}{\varphi^2 \sin \varphi} ((2 \cos \varphi - \frac{\varphi}{\sin \varphi})\varphi - \sin \varphi) = \frac{e^{\frac{\varphi}{\tan \varphi}}}{\varphi^2 \sin \varphi} (2\varphi \sin \varphi \cos \varphi - \varphi^2 - \sin^2 \varphi) = -\frac{e^{\frac{\varphi}{\tan \varphi}}}{\varphi^2 \sin \varphi} ((\varphi - \sin \varphi)^2 + 2\varphi \sin \varphi (1 - \cos \varphi))$ is negative on the inter-

val $(0, \pi)$. The graph of the function f on interval $(-\pi, \pi)$ with additional points $(-\pi, 0)$, $(0, e)$ and $(\pi, 0)$ is shown on Fig. 6(a).

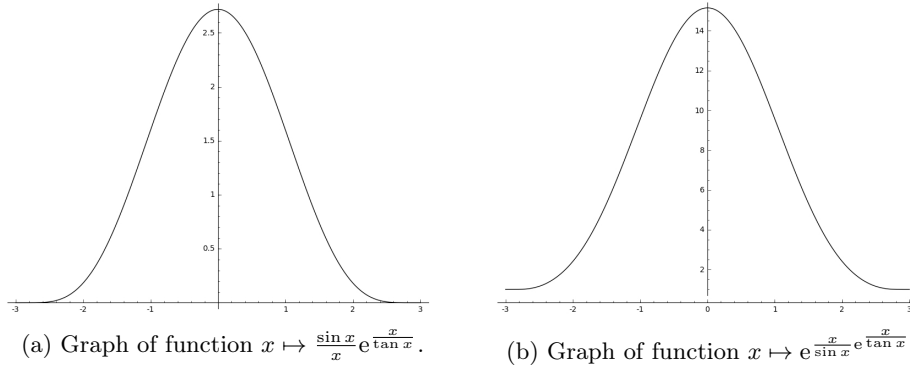


Figure 6: Two auxiliary functions.

We found that $f(\varphi) = f(\psi)$ iff $\psi = \pm\varphi$ yielding that the numbers $z \neq w$ are conjugated. In this case our powers equal to $z^{\bar{z}} = e^{r(\cos \varphi - i \sin \varphi)(\ln r + i\varphi)}$

and substituting $\ln r = \frac{\varphi}{\tan \varphi}$ we obtain $\bar{z}^z = z^{\bar{z}} = e^{\frac{r\varphi}{\sin \varphi}} = e^{\frac{\varphi e^{\frac{\varphi}{\tan \varphi}}}{\sin \varphi}} = h(\varphi)$. For $\varphi \in (0, \pi)$ we have $h(\varphi) \in (1, e^e)$ and these were precisely the values of our powers missing in the previous section. The graph of function h with additional points $(-\pi, 1)$, $(0, e^e)$, $(\pi, 1)$ is presented on Fig. 6(b). On the other hand, $r = e^{\frac{\varphi}{\tan \varphi}}$ is the necessary condition that the power dual of a complex number $z = r e^{i\varphi}$ is its conjugate $\bar{z}$. As $\bar{z}^z = z^{\bar{z}}$ from $\bar{z}^z = z^{\bar{z}}$ it follows that the power, obtained in this case, must be some real number. But $\text{Im}(r e^{i\varphi})^{r e^{-i\varphi}} = \text{Im} e^{(\ln r + i\varphi)(r \cos \varphi - i r \sin \varphi)} = r(\varphi \cos \varphi - \ln r \sin \varphi) = 0$ yields $\ln r = \frac{\varphi}{\tan \varphi}$. Thus we obtained the following two statements.

Theorem 3. *Complex numbers z of the form $e^{\frac{\varphi}{\tan \varphi} + i\varphi}$ are the only ones for which their power dual number is the conjugate number $\bar{z}$. In this case the obtained power $z^{\bar{z}} = e^{\frac{\varphi e^{\frac{\varphi}{\tan \varphi}}}{\sin \varphi}}$ is a real number from the interval $(1, e^e)$.*

Corollary 2. *Each number $t \neq e^e$ from the interval $(1, \infty)$ can be in a unique way presented as $x^y = y^x$, $x \neq y$ where x, y are either real numbers or $y = \bar{x}$.*

Corollary 3. *The binary relation $x \sim y$ iff $x^y = y^x$ is an equivalence relation on the set $(1, \infty) \cup \mathbb{C} \setminus \{0, e\}$, breaking it into equivalence classes consisting of two elements: a number and its power dual number (the curve $\mathcal{C}$ was defined at the beginning of this section).*

Proof. The relation $\sim$ is clearly reflexive and symmetric by its definition, but it is also transitive as $u^x = x^u$ and $x^y = y^x$ yields $x \ln u = u \ln x$ and

$y \ln x = x \ln y$. So $\frac{\ln u}{u} = \frac{\ln x}{x} = \frac{\ln y}{y}$ yielding $y \ln u = u \ln y$ and $u^y = y^u$. The second half of our statement follows from the Theorems 1 and 3 obtained above. $\square$

Remark 1. As we will see in the next section the above binary relation on the set of complex numbers is, in general, not transitive (Theorem 3).

Some examples illustrating the results of this section are presented in Table 4.

Table 4: Examples of complex numbers with conjugated power duals.

φ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = e^{\frac{\varphi}{\tan \varphi}}$	2.4766	2.1933	1.8305	1
$z = r e^{i\varphi}$	$2.1448 + 1.2383 i$	$1.5509 + 1.5509 i$	$0.9153 + 1.5853 i$	i
$\bar{z}^z = z^{\bar{z}}$	13.3768	11.4286	9.1473	$e^{\frac{\pi}{2}}$

φ	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$r = e^{\frac{\varphi}{\tan \varphi}}$	0.2984	0.0948	0.0107
$z = r e^{i\varphi}$	$-0.1492 + 0.2585 i$	$0.06702 + 0.06702 i$	$-0.0093 + 0.0054 i$
$\bar{z}^z = z^{\bar{z}}$	2.0580	1.3714	1.0578

4 On power duals of complex numbers

An early attempt to describe general situation in the complex plane was made in Schwering (1878). Replacing the equation $z^w = w^z$ with $\frac{\ln z}{z} = \frac{\ln w}{w}$ and showing that for any (non-zero) complex number C there exists an infinite number of solutions of the equation $l(z) = C$, one made the conclusion that for a given $z \neq 0$ there are infinitely many $w \in \mathbb{C}$ satisfying the condition $z^w = w^z$ and so being power duals of z . For this purpose one gave a qualitative description of two curves in real plane to the equations of which reduces the condition $l(z) = C$. Below we derive the equations of these curves in polar coordinates and show the resulting sets of complex numbers for two typical cases. If $z = r e^{i\psi}$, $C = a e^{i\alpha}$, $\alpha \geq 0$, we have $\ln r + (\psi + 2m\pi)i = a r e^{i(\alpha + \psi)}$ for some $m \in \mathbb{Z}$. Thus $\psi + 2m\pi = a r \sin(\alpha + \psi)$ and $\ln r = a r \cos(\alpha + \psi)$. Due to the periodicity of trigonometric functions we can replace $\psi \in (-\pi, \pi]$ with $\varphi \in \mathbb{R}$ to obtain the system

$$\begin{cases} \varphi = a r \sin(\varphi + \alpha) \\ \ln r = a r \cos(\varphi + \alpha) \end{cases}$$

or

$$\begin{cases} r(\varphi) = \frac{\varphi}{a \sin(\varphi + \alpha)} \\ \varphi(r) = \arccos \frac{\ln r}{ar} - \alpha \end{cases}.$$

As $r = |z|$ is positive, φ and $\sin(\varphi + \alpha)$ must have the same sign, so we obtain the condition $\varphi \in \{((2k-1)\pi - \alpha, 2k\pi - \alpha); k \leq 0\} \cup (0, \pi - \alpha) \cup \{(2k\pi - \alpha, (2k+1)\pi - \alpha); k \geq 1\}$, for the parameter φ in the first case, while the parameter r in the second case must satisfy the condition $\frac{|\ln r|}{r} \leq a$. If $a \geq \frac{1}{e}$ the last inequality holds for all positive r . On Fig. 7 two examples of the infinite number of connected sets of complex numbers of the form $r(\varphi)e^{i\varphi}$ intersected by the set $\{re^{i\varphi(r)} : |\ln r| \leq ar\}$ are presented.

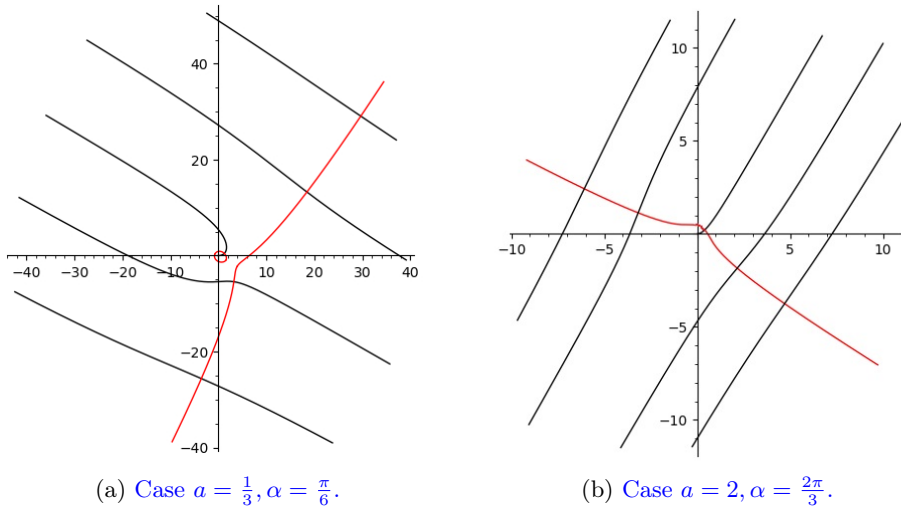


Figure 7: Intersections of curves with equations $r(\varphi) = \frac{\varphi}{a \sin(\varphi + \alpha)}$ and $\varphi(r) = \arccos \frac{\ln r}{ar} - \alpha$.

If we replace in the equation $\cos(\varphi + \alpha) - \frac{\ln r}{ar} = 0$ variable r with the fraction $\frac{\varphi}{a \sin(\varphi + \alpha)}$ we obtain the condition $e^{\frac{\varphi}{\tan(\varphi + \alpha)}} - \frac{\varphi}{a \sin(\varphi + \alpha)} = 0$, so we can find values of the angle φ and the corresponding points of intersection of our curves, representing the solutions of our problem.

In computer algebra program [SageMath](#) (see, for instance, [Bard \(2015\)](#)) which we have used, the necessary numeric calculation is produced by executing the command `find_root(expr.subs(a==..., alpha==...), Low, Upp)` where `expr` is the left side of the given equation, `Low` and `Upp` are the lower and upper boundaries of the interval on which we are searching for the solution and the appropriate values of variables `a`, `alpha` are inserted instead of the dots. All the figures in this paper were also produced by [SageMath](#) software.

In the first of the above two cases (when $C = \frac{\sqrt{3}}{6} + \frac{i}{6}$) in this way we obtain the following values of parameter φ : ..., $\varphi_{-2} \approx -7.991$, $\varphi_{-1} \approx$

-1.033 , $\varphi_0 \approx 0.338$, $\varphi_1 \approx 6.906$, $\varphi_2 \approx 13.341, \dots$ and the corresponding power dual numbers are $\dots, z_{-2} \approx -3.533 - 25.641i$, $z_{-1} \approx 3.255 - 5.458i$, $z_0 \approx 1.261 + 0.443i$, $z_1 \approx 18.475 + 13.255i$, $z_2 \approx 29.695 + 29.071i, \dots$. In Table 5 we list also the powers determined by the pairs of these numbers: $z_k^{z_m} = z_m^{z_k}$.

Table 5: Powers obtained by different numbers that all commute under exponentiation.

$k \backslash m$	-1	0	1	2
-2	$(4.45 + 0.98i)10^{-15}$	$-1469 - 1488i$	$(-8.60 + 9.43i)10^{71}$	$(-1.66 + 6.83i)10^{142}$
-1		$14.41 - 7.55i$	$(3.97 - 4.60i)10^{20}$	$(-3.60 - 6.88i)10^{36}$
0			$-1.89 - 1.49i$	$0.276 - 0.109i$
1				$(1.07 + 0.68i)10^{-47}$

At the end of the first section we mentioned the algorithm suggested in Knoebel (1981) for finding the unique real power dual number of an arbitrary positive number greater than e . Here we suggest an efficient algorithm for calculating an arbitrary power dual of a given non-zero complex number $w = s e^{i\psi}$. In this case $l(w) = \frac{\ln s + i(\psi + 2k\pi)}{s(\cos \psi + i \sin \psi)} = \frac{1}{s}(\ln s \cos \psi + (\psi + 2k\pi) \sin \psi + i((\psi + 2k\pi) \cos \psi - \ln s \sin \psi)) = C_k(w)$.

Defining

$$a_k = \frac{\sqrt{\ln^2 s + (\psi + 2k\pi)^2}}{s}$$

and

$$\beta_k = \begin{cases} \arctan \frac{\psi + 2k\pi}{\ln s} & , \text{ if } \frac{\psi + 2k\pi}{\ln s} > 0 \\ \pi + \arctan \frac{\psi + 2k\pi}{\ln s} & , \text{ if } \frac{\psi + 2k\pi}{\ln s} < 0 \end{cases}$$

we translate our problem to previously discussed search of numbers z for which $l(z) = C_k(w) = a_k e^{i(\beta_k - \psi)}$ if $s \neq 1$, while for power duals of a number w from the unit circle we have the condition $l(z) = |\psi + 2k\pi| e^{i(\frac{\pi}{2} - \psi)}$. We already know that for each $k \in \mathbb{Z}$ there exist infinitely many complex numbers $z_{k,m}$, $m \in \mathbb{Z}$ for which $l(z_{k,m}) = a_k e^{i(\beta_k - \psi)}$. Therefore we always obtain a 2-parametric family of power duals for a given non-zero complex number w . (When $k = m$ the result of this algorithm is the number w itself.)

As an example let us take $w = \frac{e}{2}(1 + i\sqrt{3})$. Then $a_k = \frac{1}{e}\sqrt{1 + (\frac{\pi}{3} + 2k\pi)^2}$ and $\beta_k = \arctan(\frac{\pi}{3} + 2k\pi)$. A few power duals of the number w and the resulting powers are listed in tables 6 and 7.

From the equality $\overline{z^w} = \bar{z}^{\bar{w}}$ it follows that the condition $z^w = w^z$ yields $\bar{z}^{\bar{w}} = \bar{w}^{\bar{z}}$, therefore power duals of a real number w always appear in pairs: z and $\bar{z}$. In Table 8 we present some power duals of numbers $\frac{1}{2}$ and -1 .

As for every w we have $w^1 = w$ power duals of number 1 are the solutions of the equation $1^w = w$. Examples of such complex numbers are: $-0.5293 \pm 0.09854i$, $-0.2769 \pm 0.09752i$, $0.04444 \pm 0.091084i$, $0.08313 \pm 0.1431i$ and $0.1516 \pm 0.2133i$

Table 6: Some $z_{k,m} \in \mathbb{C}$ commuting with $\frac{e}{2}(1 + i\sqrt{3})$ under the exponentiation operation.

$k \backslash m$	-1	0	1	2
-1	w	$-1.5380 - 1.2478i$	$-4.0118 - 3.3549i$	$-6.3021 - 5.6325i$
0	$1.2407 - 1.0960i$	w	$1.5497 + 15.3576i$	$-0.2503 + 27.271i$
1	$-0.3779 - 0.6723i$	$0.03158 + 0.5893i$	w	$2.4452 + 4.4111i$
2	$-0.3312 - 0.3538i$	$-0.03013 + 0.3795i$	$0.7057 + 1.2849i$	w

Table 7: The resulting powers $z_{k,m}^w = w^{z_{k,m}}$.

$k \backslash m$	-1	0	1	2
-1	$(4.38 + 87.67i)10^4$	$(2.71 + 1.56i)10^{-4}$	$(1.55 - 3.96i)10^{-10}$	$(-1.75 + 2.25i)10^{-16}$
0	$10.67 + 2.20i$	$-0.266 - 0.197i$	$(-1.44 - 4.67i)10^{-7}$	$(-0.927 + 2.97i)10^{-13}$
1	$-90.4 + 28.1i$	$(9.36 + 10.04i)10^{-3}$	$(12.09 - 3.08i)10^{-8}$	$(-9.84 - 3.53i)10^{-14}$
2	$13.30 + 87.65i$	$(5.53 - 0.170i)10^{-3}$	$(-0.527 - 5.10i)10^{-8}$	$(-1.99 + 4.26i)10^{-14}$

Table 8: Some power duals $z = z_{k,m}$ of numbers $\frac{1}{2}$ and -1 with the resulting powers $z^w = w^z$.

$w = \frac{1}{2}; z$	$-0.8665 \pm 3.209i$	$-1.494 \pm 7.795i$	$-1.821 \pm 12.36i$	$-2.045 \pm 16.91i$
$\sqrt{z} = (\frac{1}{2})^z$	$1.108 \mp 1.447i$	$1.795 \pm 2.172i$	$-2.319 \mp 2.675i$	$2.737 \pm 3.089i$
$w = \frac{1}{2}; z$	$-2.215 \pm 21.45i$	$-2.353 \pm 26.00i$	$-2.469 \pm 30.54i$	$-2.568 \pm 35.07i$
$\sqrt{z} = (\frac{1}{2})^z$	$3.111 \mp 3.445i$	$3.446 \pm 3.772i$	$-3.753 \mp 4.068i$	$4.037 \pm 4.344i$
$w = -1; z$	$-6.9718 \pm 0.6194i$	$-4.9673 \pm 0.5119i$	$-2.9628 \pm 0.3479i$	$0.2660 \pm 0.2943i$
$\frac{1}{z} = (-1)^z$	$-0.1423 \mp 0.0126i$	$0.1992 \mp 0.0205i$	$-0.3329 \mp 0.0391i$	$1.6904 \mp 1.8699i$
$w = -1; z$	$1.9650 \pm 0.2169i$	$3.9648 \pm 0.4404i$	$5.9697 \pm 0.5702i$	$7.974 \pm 0.6619i$
$\frac{1}{z} = (-1)^z$	$0.5028 \mp 0.0555i$	$0.2491 \mp 0.0277i$	$0.1660 \mp 0.0159i$	$0.1246 \mp 0.0103i$

5 Conclusion

Through the article we intended to show that the exponentiation function over the field of complex numbers behaves quite differently than in the real domain. Summarizing the above results we come to the following statement.

Theorem 4. *For each number $w \in \mathbb{C} \setminus \{0\}$ there exists an infinite set of complex numbers $z_{k,m}(w)$, $k, m \in \mathbb{Z}$ such that $(z_{k,m}(w))^w = w^{z_{k,m}(w)}$ in the sense of Definition 2 from Section 2. In general, if $k \neq n$, we have $(z_{k,m}(w))^{z_{n,q}(w)} \neq (z_{n,q}(w))^{z_{k,m}(w)}$ for any values m, q , so the relation ‘is a power dual of’ is not transitive in the complex plane.*

Proof. To verify the content of the last sentence let $z_{k,m}(w) = r e^{i\varphi}$ and $w = s e^{i\psi}$. Comparing the modules and arguments of the powers $(z_{k,m}(w))^w$ and $w^{z_{k,m}(w)}$ results in the system of equations:

$$\begin{cases} s(\ln r \cos \psi - \varphi \sin \psi) = r(\ln s \cos \varphi - \psi \sin \varphi) \\ s(\varphi \cos \psi + \ln r \sin \psi) = r(\psi \cos \varphi + \ln s \sin \varphi) + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

Suppose that we also have $(z_{n,q}(w))^w = w^{z_{n,q}(w)}$ where $z_{n,q}(w) = t e^{i\xi}$, yielding

$$\begin{cases} s(\ln t \cos \psi - \xi \sin \psi) = t(\ln s \cos \xi - \psi \sin \xi) \\ s(\xi \cos \psi + \ln t \sin \psi) = t(\psi \cos \xi + \ln s \sin \xi) + 2n\pi, \quad n \in \mathbb{Z}. \end{cases}$$

Then from

$$\frac{s}{r} = \frac{\ln s \cos \varphi - \psi \sin \varphi}{\ln r \cos \psi - \varphi \sin \psi}, \quad \frac{t}{s} = \frac{\ln t \cos \psi - \xi \sin \psi}{\ln s \cos \xi - \psi \sin \xi}$$

we get

$$\frac{t}{r} = \frac{(\ln t \cos \psi - \xi \sin \psi)(\ln s \cos \varphi - \psi \sin \varphi)}{(\ln r \cos \psi - \varphi \sin \psi)(\ln s \cos \xi - \psi \sin \xi)}.$$

But $|(z_{k,m}(w))^{z_{n,q}(w)}| = |(z_{n,q}(w))^{z_{k,m}(w)}|$ iff $t(\ln r \cos \xi - \psi \sin \xi) = r(\ln t \cos \varphi - \xi \sin \varphi)$. So the modules of these powers in general differ and the relation ‘is a power dual of’ is not transitive on the set of complex numbers.

An exception represent some power duals of positive real numbers. Let $w = s > 0, z_{k,m}(w) = r > 0$. Then $\varphi = \psi = 0$ and we have $\frac{\ln r}{r} = \frac{\ln s}{s} = \frac{\ln t}{t \cos \xi}$, so $|(z_{k,m}(w))^{z_{n,q}(w)}| = |(z_{n,q}(w))^{z_{k,m}(w)}|$ and comparing the expressions for the arguments of these powers we see that $\frac{\ln s}{s} t \sin \xi + q \frac{2\pi}{s} = \xi = \frac{\ln r}{r} t \sin \xi + \frac{qr}{s} \frac{2\pi}{r}$. So $\frac{qr}{s}$ must be an integer. For instance, all power duals of number 2 are also power duals of 4, while only the ‘even’ duals of number 4 (obtained by adding in the second equation an integer multiple of 4π) are power duals of 2. $\square$

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