A CLOSURE OPERATION IN RINGS

VERSION 1.7

PERE ARA, GERT K. PEDERSEN
and FRANCESC PERERA

November 2000

ABSTRACT. We study the operation $E \to \mathrm{cl}(E)$ defined on subsets E of a unital ring R, where $x \in \mathrm{cl}(E)$ if $(x + Rb) \cap E \neq \emptyset$ for each b in R such that $Rx + Rb = R$. This operation, which strongly resembles a closure, originates in algebraic K-theory. For any left ideal L we show that $\mathrm{cl}(L)$ equals the intersection of the maximal left ideals of R containing L. Moreover, $\mathrm{cl}(Re) = Re + \mathrm{rad}(R)$ if e is an idempotent in R, and $\mathrm{cl}(I) = I$ for a two-sided ideal I precisely when I is semi-primitive in R (i.e. $\mathrm{rad}(R/I) = 0$).

We then explore a special class of von Neumann regular elements in R, called persistently regular and characterized by forming an "open" subset R^{pr} in R, i.e. $\mathrm{cl}(R \setminus R^{pr}) = R \setminus R^{pr}$. In fact, $R \setminus R^{pr} = \mathrm{cl}(R \setminus R^r)$, so that R^{pr} is the "algebraic interior" of the set R^r of regular elements. We show that a regular element x with partial inverse y is persistently regular, if and only if the skew corner $(1-xy)R(1-yx)$ is contained in R^r. If $I_{\mathrm{reg}}(R)$ denotes the maximal regular ideal in R and R_q^{-1} the set of quasi-invertible elements, defined and studied in [**6**], we prove that $R_q^{-1} + I_{\mathrm{reg}}(R) \subset R^{pr}$.

Specializing to C^*-algebras we prove that $\mathrm{cl}(E)$ coincides with the norm closure of E, when E is one of the five interesting sets R^{-1}, R_ℓ^{-1}, R_r^{-1}, R_q^{-1} and R_{sa}^{-1}, and that R^{pr} coincides with the topological interior of R^r. We also show that the operation cl respects boundedness, self-adjointness and positivity.

1. Prerequisites

1.1. Recall from [**6**, §3] that for any subset E of a unital ring R we define $\mathrm{cl}(E)$ to be the set of elements x in R such that, whenever $ax + b = 1$ for some a, b in R, there is an element y in R such that $x + yb \in E$. Equivalently, if $Rx + Rb = R$ then $(x + Rb) \cap E \neq \emptyset$.

1.2. The following elementary properties of the operation cl were proved in [**6**, Lemma 3.2] and we list them here again for easy reference:

(i) $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(R) = R$.

1991 *Mathematics Subject Classification.* Primary 16A12; Secondary 16A30, 16A32.

Key words and phrases. Bass stable rank, exchange ring, extremally rich C^*-algebras, Jacobson radical, von Neumann regularity, semi-primitive ring, $QB-$ring, quasi-invertible element.

Research supported by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya, the DGESIC (Spain) and the Danish Research Council (SNF).

(ii) $E \subset F$ implies $\mathrm{cl}(E) \subset \mathrm{cl}(F)$.
(iii) $E \subset \mathrm{cl}(E) = \mathrm{cl}(\mathrm{cl}(E))$.
(iv) If $E \neq \emptyset$, then $\mathrm{rad}(R) \subset \mathrm{cl}(E)$, where $\mathrm{rad}(R)$ is the Jacobson radical of R.
(v) $\mathrm{rad}(R) = \mathrm{cl}(0)$.
(vi) $\mathrm{cl}(E) \cap R_\ell^{-1} = E \cap R_\ell^{-1}$.
(vii) $F\,\mathrm{cl}(E) \subset \mathrm{cl}(FE)$.
(viii) If $F \subset R^{-1}$, then $\mathrm{cl}(E)F \subset \mathrm{cl}(EF)$.
(ix) $\mathrm{cl}(E) + F \subset \mathrm{cl}(E + RF)$.
(x) If $RF \subset F$ and $E + F \subset \mathrm{cl}(E)$, then $\mathrm{cl}(E) + F \subset \mathrm{cl}(E)$.
(xi) If $\pi\colon R \to S$ is a surjective morphism, then $\pi(\mathrm{cl}(E)) \subset \mathrm{cl}(\pi(E))$.

Here and in the following we use the symbols R_ℓ^{-1}, R_r^{-1} and R^{-1} to designate the sets of left, right and two-sided invertible elements in R, respectively.

Since our work is focused on non-commutative rings, it is perhaps worth mentioning that condition (viii) can be strengthened to

(viii') If $F \subset R_r^{-1}$, then $\mathrm{cl}(E)F \subset \mathrm{cl}(EF)$.

To see this take x in $\mathrm{cl}(E)$ and y in F and consider an equation $axy + b = 1$. If z is a right inverse for y, then by multiplying left and right with y and z, respectively, we get $yax + ybz = 1$. Since $x \in \mathrm{cl}(E)$ this means that $x + tybz \in E$ for some t in R, whence

$$xy + tybzy \in Ey \subset EF\,.$$

However,

$$tybzy = ty(1 - axy)zy = ty - tyaxy = tyb\,,$$

so that $xy + tyb \in EF$, proving that $xy \in \mathrm{cl}(EF)$.

1.3. For non-commutative rings the equation $\mathrm{cl}(E \cup F) = \mathrm{cl}(E) \cup \mathrm{cl}(F)$ is not true in general (cf. Example 1.10), so that cl is not a closure operation in the sense of Kuratowski, even though the conditions (i)–(iii) strongly suggest that. However, these conditions imply that sets of the form $\mathrm{cl}(E)$ have one property in common with closed sets in a topology. We list it for handy reference and leave the easy verification to the reader:

(xii) If $\{F_i \mid i \in I\}$ is a family of subsets of R such that $\mathrm{cl}(F_i) = F_i$ for all i, then $\mathrm{cl}(\cap F_i) = \cap F_i$.

1.4. It is evident from condition (vi) that $\mathrm{cl}(R_r^{-1}) \cap R_\ell^{-1} = R^{-1}$. Less evident, but more useful, is the relation:

$$(*) \qquad R_r^{-1} \cap \mathrm{cl}(R_\ell^{-1}) = R^{-1}\,.$$

To prove the non-trivial inclusion $\subset$, take an element x in R_r^{-1} and choose a in R with $xa = 1$. If now also $x \in \mathrm{cl}(R_\ell^{-1})$, then from the trivial equation $ax + (1 - ax) = 1$ we obtain an element $x_0 = x + y(1 - ax)$ in R_ℓ^{-1} for some y in R. However, $x_0 a = xa = 1$, so $x_0 \in R_\ell^{-1} \cap R_r^{-1} = R^{-1}$. But then also $a \in R^{-1}$, and thus $x \in R^{-1}$.

It is immediate from $(*)$ that if $\mathrm{cl}(R_\ell^{-1}) = R$, which is the formal definition of having Bass stable rank one, then $R_r^{-1} = R_\ell^{-1} = R^{-1}$ (as already noticed in [**22**, Theorem 2.6]). Of course, it is this "finiteness" property that makes Bass' first stable rank so much more important than the higher stable ranks.

1.5. The reader will have noticed that the definition of cl is asymmetric and favours the sinister multiplication. To balance this we defined in [**6**, §3] the dextrous version cr, where $x \in \mathrm{cr}(E)$ if $xR + bR = R$ implies that $(x + bR) \cap E \neq \emptyset$. Evidently there is a parallel theory for the operation cr, but, more importantly, for sufficiently symmetric subsets E the two sets $\mathrm{cl}(E)$ and $\mathrm{cr}(E)$ are related. Thus $\mathrm{cl}(R^{-1}) = R$ if and only if $\mathrm{cr}(R^{-1}) = R$ (which happens precisely when R is a Bass ring), and $\mathrm{cl}(R_q^{-1}) = R$ if and only if $\mathrm{cr}(R_q^{-1}) = R$ (which is the defining property for a $QB-$ring). Further examples occur in Theorem 3.2 and throughout §4.

1.6. For any subset E of a unital ring R we define

$$\mathrm{inl}(E) = R \setminus \mathrm{cl}(R \setminus E)\,.$$

Insofar as $\mathrm{cl}(E)$ resembles "the closure" of E, the set $\mathrm{inl}(E)$ resembles "the interior" of E. Thus, inl will be a monotone decreasing and idempotent operation on subsets of R, and the class of sets for which $\mathrm{inl}(E) = E$ will be stable under arbitrary unions by condition (xii) above, cf. Lemma 1.8.

If $x \in R$ and $Rx + Rb = R$ for some b in R, we set $U_x(b) = x + Rb$, cf. [**6**, Remark 3.3]. For a commutative ring, where cl is an honest closure operation, the sets $U_x(b)$ will be a neighbourhood basis for x in the ensuing topology. But also in the general case these subsets strive to fulfill this task as the next result testifies.

1.7. Proposition. *For each subset $U_x(b)$ of a unital ring R we have* $\mathrm{inl}(U_x(b)) = U_x(b)$. *Conversely, a non-empty subset E of R will satisfy* $\mathrm{inl}(E) = E$ *if and only if*

$$E = \bigcup_{x \in E} U_x(b_x)\,.$$

Proof. Assume that $ax + b = 1$ and consider an element $z = x + yb$ in $U_x(b)$. Then $az + (1 - ay)b = 1$, so if $z \in \mathrm{cl}(R \setminus U_x(b))$ we have $z + s(1 - ay)b \in R \setminus U_x(b)$ for some s in R. However,

$$z + s(1 - ay)b = x + (y + s(1 - ay))b \in U_x(b)\,,$$

a contradiction. Thus $z \in R \setminus \mathrm{cl}(R \setminus U_x(b)) = \mathrm{inl}(U_x(b))$, as desired.

In the converse direction we have already noticed that each subset E which can be written as a union of sets U for which $\mathrm{inl}(U) = U$ will satisfy $\mathrm{inl}(E) = E$.

Assume now that $\mathrm{inl}(E) = E$ and take any x in E. Then $x \notin \mathrm{cl}(R \setminus E)$, so there must exist some $b = b_x$ in R with $Rx + Rb = R$, such that $(x + Rb) \cap (R \setminus E) = \emptyset$. But this means precisely that $U_x(b) \subset E$, as claimed. □

1.8. Lemma. *The operation* inl *defined 1.6 has the following properties relative to any subsets E and F of a unital ring R:*

- **(i)** $\mathrm{inl}(\emptyset) = \emptyset$ *and* $\mathrm{inl}(R) = R$.
- **(ii)** $E \subset F$ *implies* $\mathrm{inl}(E) \subset \mathrm{inl}(F)$.
- **(iii)** $E \supset \mathrm{inl}(E) = \mathrm{inl}(\mathrm{inl}(E))$.
- **(iv)** *If* $E \neq R$, *then* $\mathrm{rad}(R) \cap \mathrm{inl}(E) = \emptyset$.
- **(v)** $R \setminus \mathrm{rad}(R) = \mathrm{inl}(R \setminus \{0\})$.
- **(vi)** $E \cap R_\ell^{-1} \subset \mathrm{inl}(E)$.

(vii) *If* $F \subset R^{-1}$*, then* $F\,\mathrm{inl}(E) \subset \mathrm{inl}(FE)$.
(viii) *If* $F \subset R_\ell^{-1}$*, then* $\mathrm{inl}(E)F \subset \mathrm{inl}(EF)$.
(ix) $\mathrm{inl}(E) + F \subset \mathrm{inl}(E + RF)$.
(x) *If* $RF \subset F$ *and* $E + F \subset E$*, then* $\mathrm{inl}(E) + F \subset \mathrm{inl}(E)$.
(xi) *If* $\pi\colon R \to S$ *is a surjective morphism and* $E \subset S$*, then* $\pi^{-1}(\mathrm{inl}(E)) = \mathrm{inl}(\pi^{-1}(E))$.
(xii) *If* $\{E_i \mid i \in I\}$ *is a family of subsets of* R *such that* $\mathrm{inl}(E_i) = E_i$ *for all* i*, then* $\mathrm{inl}(\cup E_i) = \cup E_i$.

Proof. Properties (i)–(v) and (xii) follow immediately from the corresponding properties of cl listed in 1.2 and 1.3 by taking complements. To verify (vi), note that if $x \in E \cap R_\ell^{-1}$, then $Rx + R0 = R$, so $U_x(0) = \{x\}$; and $x \in \mathrm{inl}(E)$.

(vii) & (viii) If $x \in \mathrm{inl}(E)$, then $ax + b = 1$ and $U_x(b) \subset E$ for some a, b in R. If now $y \in F \subset R^{-1}$, then $ay^{-1}(yx) + b = 1$ and $yx + Rb = yU_x(b) \subset yE$, so $yx \in \mathrm{inl}(FE)$. If instead $y \in R_\ell^{-1}$ and $zy = 1$, then $zaxy + zby = 1$ and $xy + Rzby \subset U_x(b)y \subset Ey$, so $xy \in \mathrm{inl}(EF)$.

(ix) Again, if $x \in \mathrm{inl}(E)$, so that $ax + b = 1$ and $U_x(b) \subset E$ for some a, b in R, then for each y in F we have $a(x + y) + (b - ay) = 1$ and $x + y + R(b - ay) \subset U_x(b) + Ry \subset E + RF$, whence $x + y \in \mathrm{inl}(E + RF)$ by definition. From (ix) we immediately get (x).

(xi) Suppose that $x \in R$ such that $\pi(x) \in \mathrm{inl}(E)$. Since π is surjective, this means that for some a, b in R we have $\pi(ax + b) = 1_S$ and $\pi(x) + S\pi(b) \subset E$. Perturbing if necessary b with an element from $\ker \pi$ we may assume that $ax + b = 1_R$, and evidently $\pi(U_x(b)) \subset E$, whence $U_x(b) \subset \pi^{-1}(E)$, and $x \in \mathrm{inl}(\pi^{-1}(E))$.

Conversely, if $x \in \mathrm{inl}(\pi^{-1}(E))$ we have $ax + b = 1_R$ and $U_x(b) \subset \pi^{-1}(E)$. But then $\pi(ax + b) = 1_S$ and $\pi(x) + S\pi(b) \subset E$, so that $\pi(x) \in \mathrm{inl}(E)$. □

1.9. Remarks. a. In connection with condition (vi), note that if E is "bounded" in the primitive sense that it contains no "rays" of the form $x + \mathbb{N}b$, for $b \neq 0$, then $\mathrm{inl}(E) = E \cap R_\ell^{-1}$. Because then $ax + b = 1$ and $U_x(b) \subset E$ only if $b = 0$, which implies that $x \in E \cap R_\ell^{-1}$.
b. If L is a left ideal in R, then $\mathrm{cl}(L)$ is an interesting object of study, as we shall see in §2. By contrast, $\mathrm{inl}(L) = \emptyset$ whenever $L \neq R$. Indeed, if $x \in \mathrm{inl}(L)$, then $x + Rb \subset L$ for some b in R with $Rx + Rb = R$. Since $x \in L$ this implies that also $b \in L$, whence $L = R$.
c. By conditions (vii) and (viii) we see that for any subset E of R such that $R^{-1}E = ER^{-1} = E$ we also have $R^{-1}\,\mathrm{inl}(E) = \mathrm{inl}(E)R^{-1} = \mathrm{inl}(E)$.
d. Condition (ix) shows that for every left ideal L in a unital ring R and any subset E such that $E = \mathrm{inl}(E)$ we also have $E + L = \mathrm{inl}(E + L)$
e. We finally note that condition (xi) implies that if $E = \mathrm{inl}(E)$ in S, then $\pi^{-1}(E) = \mathrm{inl}(\pi^{-1}(E))$ in R, so that π is "continuous". Since evidently $\pi(U_x(b)) = U_{\pi(x)}(\pi(b))$ we also have $\pi(\mathrm{inl}(E)) \subset \mathrm{inl}(\pi(E))$. It follows that $\mathrm{inl}(E) = E$ in R implies that $\mathrm{inl}(\pi(E)) = \pi(E)$ in S, so that π is an "open" map as well.

1.10. Example. In the ring $R = \mathbb{M}_2(\mathbb{R})$ we consider the three proper idempotents

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad q = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Since both $e+p$ and $e+q$ are invertible we may consider the two subsets

$$U_e(p) = e + Rp \quad \text{and} \quad U_e(q) = e + Rq\,.$$

However, $U_e(p) \cap U_e(q) = \{e\}$, which contains no subsets of the form $U_e(b)$. Thus, if we put $E = R \setminus U_e(p)$ and $F = R \setminus U_e(q)$, then $\mathrm{cl}(E) = E$ and $\mathrm{cl}(F) = F$; but

$$\begin{aligned}\mathrm{cl}(E \cup F) &= R \setminus \mathrm{inl}(U_e(p) \cap U_e(q)) = R \setminus \mathrm{inl}(\{e\}) \\ &= R \neq R \setminus \{e\} = E \cup F = \mathrm{cl}(E) \cup \mathrm{cl}(F)\,.\end{aligned}$$

These sets therefore provide a concrete counterexample for cl to satisfy Kuratowski's fourth closure axiom.

2. Closures of Left Ideals and Idempotents

2.1. For a left ideal L in a unital ring R we define $\ker\mathrm{hull}\,(L)$ to be the intersection of all maximal left ideals of R containing L. If $\pi\colon R \to R/L$ denotes the quotient map, regarded as a module morphism between the left R−modules R and R/L, then π induces a bijection between the maximal left ideals L_m of R containing L and the maximal left submodules of R/L, these having the form $\ker\rho$, where $\rho\colon R/L \to R/L_m$ is the quotient map into the simple R−module R/L_m. The intersection of all maximal submodules of a given left R−module M is the radical of M, denoted by $\mathrm{rad}(M)$, cf. [**12**, §5]. It follows that we have the equation

$$(*) \qquad \ker\mathrm{hull}\,(L) = \pi^{-1}(\mathrm{rad}(R/L))\,.$$

In particular, $\ker\mathrm{hull}\,\{0\}$ is the Jacobson radical of R, and $\ker\mathrm{hull}\,(L) = L$ if and only if $\mathrm{rad}(R/L) = 0$.

2.2. Theorem. *For each left ideal L in a unital ring R we have*

$$\ker\mathrm{hull}\,(L) = \mathrm{cl}(L)\,.$$

Proof. We first observe that if $x \in \ker\mathrm{hull}\,(L)$, then $c(1-x)+y = 1$ for some c in R and y in L. Indeed, if $1 \notin R(1-x)+L$, then this set is a proper left ideal (containing both $1-x$ and L), hence contained in a maximal left ideal L_m of R. However, $L \subset L_m$ so $\ker\mathrm{hull}\,(L) \subset L_m$. Consequently,

$$1 = 1 - x + x \in L_m + \ker\mathrm{hull}\,(L) \subset L_m\,,$$

a contradiction.

To prove that this implies that $x \in \mathrm{cl}(L)$ consider an equation $ax+b = 1$ with a, b in R. Since $\ker\mathrm{hull}\,(L)$ is a left ideal this means that $1-b \in \ker\mathrm{hull}\,(L)$, so by our first observation we have $cb+y = 1$ for some c in R and y in L. Therefore

$$x - (xc)b = x - x(cb + y - y) = xy \in L\,,$$

whence $x \in \mathrm{cl}(L)$.

To prove the reverse inclusion $\mathrm{cl}(L) \subset \ker\mathrm{hull}\,(L)$, assume first that L is a maximal left ideal of R. We claim that this means that $\mathrm{cl}(L) = L$. Evidently $L \subset \mathrm{cl}(L)$, and if $x \notin L$, then $Rx + L = R$ by the maximality of L, so $ax + b = 1$ for some a in R and b in L. If now $x \in \mathrm{cl}(L)$, this would imply that $x + yb \in L$ for some y in R, whence $x \in L$, a contradiction. Therefore $x \notin \mathrm{cl}(L)$, so $\mathrm{cl}(L) \subset L$, as desired.

¿From condition (xii) for cl mentioned in 1.2 it now follows that $\ker\mathrm{hull}\,(L)$, being an intersection of maximal left ideals of R, is closed for any left ideal L. Using that $L \subset \ker\mathrm{hull}\,(L)$, this implies that

$$\mathrm{cl}(L) \subset \mathrm{cl}(\ker\mathrm{hull}\,(L)) = \ker\mathrm{hull}\,(L)\,.$$

□

2.3. Corollary. *Let I be a two-sided ideal in a unital ring R, and let* $\mathrm{prim}\,(R)$ *denote the primitive ideal space of R equipped with the Jacobson topology. Then*

$$\mathrm{cl}(I) = \pi^{-1}(\mathrm{rad}(R/I)) = \ker\mathrm{hull}\,(I)\,,$$

where $\pi\colon R \to R/I$ is the quotient morphism, and where the operations ker *and* hull *are now the ones associated with the topology on* $\mathrm{prim}\,(R)$*.*

Proof. From Theorem 2.2 and the formula (∗) in 2.1 we obtain the equality

$$\mathrm{cl}(I) = \pi^{-1}(\mathrm{rad}(R/I))\,.$$

That the ideal on the left equals $\ker\mathrm{hull}\,(I)$, computed in $\mathrm{prim}\,(R)$, is well known. Just recall that if L is a maximal left ideal containing I, and if

$$J = \{x \in R \mid xR \subset L\}$$

is the primitive ideal associated with L, then $\mathrm{cl}(I) \subset J$, since J is the largest ideal contained in L. □

2.4. Remark. It follows from Corollary 2.3 that for a two-sided ideal I of R we have $\mathrm{cl}(I) = I$ if and only if $\mathrm{rad}(R/I) = 0$, i.e. when I is a semi-primitive ideal of R. Since $\pi(\mathrm{rad}(R)) \subset \mathrm{rad}(R/I)$ by [**12**, Proposition 5.1], this can only happen if $\mathrm{rad}(R) \subset I$.

The fact that the operation hull ker defines the closure in the topological space $\mathrm{prim}\,(R)$ has an immediate consequence for cl, applied to two-sided ideals. In a slightly generalized form this gives the following result:

2.5. Proposition. *If L is a left and I a two-sided ideal in a unital ring R, then*

$$\mathrm{cl}(L \cap I) = \mathrm{cl}(L) \cap \mathrm{cl}(I)\,.$$

Proof. The inclusion $\subset$ is obvious. Consider therefore an element x not in $\mathrm{cl}(L \cap I)$. By Theorem 2.2 there is then a maximal left ideal $L_m \supset L \cap I$, such that $x \notin L_m$. Either $L \subset L_m$, whence $\mathrm{cl}(L) \subset L_m$, so $x \notin \mathrm{cl}(L)$; or else $L \not\subset L_m$. But in that case $R = L + L_m$ by maximality, whence

$$I \subset IL + IL_m \subset I \cap L + L_m \subset L_m\,,$$

so $\mathrm{cl}(I) \subset L_m$ and $x \notin \mathrm{cl}(I)$. In both cases we see that $x \notin \mathrm{cl}(L) \cap \mathrm{cl}(I)$, and therefore $\mathrm{cl}(L) \cap \mathrm{cl}(I) \subset \mathrm{cl}(L \cap I)$. □

2.6. Proposition. *Let E be a subset of a two-sided ideal I in a unital ring R. Then $x \in \mathrm{cl}(E) \cap I$ if and only if for every a in I we have*

$$\{x + I(1 - ax)\} \cap E \neq \emptyset .$$

Proof. By definition $x \in \mathrm{cl}(E)$ if $U_x(b) \cap E \neq \emptyset$, whenever $ax + b = 1$ for some a, b in R. Note now that if $x \in I$ then $1 - b \in I$. Therefore, if $x + yb \in E$ $(\subset I)$ for some y in R, then necessarily $y \in I$. Thus,

$$U_x(b) \cap E = \{x + I(1 - ax)\} \cap E .$$

It follows that $x \in \mathrm{cl}(E) \cap I$ if and only if for every a in R we have

$$(*) \qquad \{x + I(1 - ax)\} \cap E \neq \emptyset .$$

Denote by $\mathrm{cl}_0(E)$ the set of elements x in I that satisfy the condition in Proposition 2.6. Since $I \subset R$ this is evidently weaker than the condition in $(*)$, so

$$\mathrm{cl}(E) \cap I \subset \mathrm{cl}_0(E) .$$

On the other hand,

$$\mathrm{cl}_0(E) \subset \mathrm{cl}(\mathrm{cl}_0(E)) \cap I ,$$

so we have the desired equality if we can show that $\mathrm{cl}(\mathrm{cl}_0(E)) \cap I \subset \mathrm{cl}(E) \cap I$.

Toward this end consider x in $\mathrm{cl}(\mathrm{cl}_0(E)) \cap I$ and take any a in R. By $(*)$ there is a y in I such that

$$x_0 = x + y(1 - ax) \in \mathrm{cl}_0(E) .$$

Let $a_0 = axa$ in I. Since $x_0 \in \mathrm{cl}_0(E)$ there is a z in I such that $x_0 + z(1 - a_0x_0) \in E$. However, with $w = y + z(1 + ax - axay)$ we get

$$\begin{aligned} x + w(1 - ax) &= x + (y + z(1 + ax - axay))(1 - ax) \\ = x + y(1 - ax) + z(1 - axax - axay(1 - ax)) &= x_0 + z(1 - a_0x_0) \in E . \end{aligned}$$

This proves that $x \in \mathrm{cl}(E)$, as desired. □

2.7. Remark. Proposition 2.6 shows that if R is a non-unital ring contained as an ideal in a unital ring R_1, then the set $\mathrm{cl}(E) \cap R$ does not depend on R_1 for any subset E of R. Moreover, if R is a semi-primitive ideal in R_1, e.g. $R_1 = R \oplus \mathbb{Z}$, then $\mathrm{cl}(E) \subset R$ by Remark 2.4, so that the operation cl does not involve R_1 at all.

2.8. Proposition. *If e is an idempotent in a unital ring R with Jacobson radical* $\mathrm{rad}(R)$*, then*

$$\mathrm{cl}(Re) = Re + \mathrm{rad}(R) .$$

Proof. By conditions (iii) and (iv) for cl in 1.2 we have

$$L + \mathrm{rad}(R) \subset \mathrm{cl}(L) + \mathrm{cl}(L) = \mathrm{cl}(L)$$

for any left ideal L of R, in particular for $L = Re$.

To prove the other inclusion we denote by $\pi\colon R \to R/Re$ the quotient morphism between the left R–modules R and R/Re, but now we identify R/Re with the submodule $R(1-e)$ of R. Since the embedding $R(1-e) \subset R$ is a module morphism we have $\mathrm{rad}(R(1-e)) \subset \mathrm{rad}(R)$ by [**12**, Proposition 5.1], and thus by $(*)$ in 2.1

$$\ker \mathrm{hull}\,(Re) = \pi^{-1}(\mathrm{rad}(R(1-e))) \subset \pi^{-1}(\mathrm{rad}(R)) = Re + \mathrm{rad}(R) .$$

By Theorem 2.2 we have the desired inclusion. □

2.9. Corollary. *If e is idempotent in a semi-primitive ring R, then the left ideal Re is the intersection of the maximal left ideals that contains it.* □

2.10. Proposition. *If e and f are idempotents in a semi-primitive ring R, then*

$$\mathrm{cl}(eRf) = eRf\,.$$

Proof. Evidently $\mathrm{cl}(eRf) \subset \mathrm{cl}(Rf) = Rf$ by Proposition 2.8. On the other hand, using condition (vii) for cl in 1.2 we get

$$(1-e)\,\mathrm{cl}(eRf) \subset \mathrm{cl}((1-e)eRf) = \mathrm{cl}(0) = 0\,,$$

since R is semi-primitive; so $\mathrm{cl}(eRf) \subset eR$. As $Rf \cap eR = eRf$, the desired equality follows. □

2.11. Proposition. *Let e be an idempotent in a unital, semi-primitive ring R, and let E be a subset of eRe. Then* $\mathrm{cl}(E)$*, computed in R, is equal to* $\mathrm{cl}(E)$*, computed in eRe.*

Proof. Denote by $\mathrm{cl}_e(E)$ the closure of E in eRe, and take x in $\mathrm{cl}_e(E)$. If now $ax + b = 1$ for some a, b in R, then $eaex + ebe = e$, and therefore $x + yebe \in E$ for some y in eRe. However, $b = 1 - ax$, so $eb(1 - e) = 0$, i.e. $ebe = eb$, and thus $x + yeb \in E$, proving that $a \in \mathrm{cl}(E)$.

Conversely, if $x \in \mathrm{cl}(E)$ and $ax+b = e$ for some a, b in eRe, then $ax+(b+1-e) = 1$, so $x + y(b + 1 - e) \in E$ for some y in R. Since $x \in eRe$ by Proposition 2.10, it follows that $x + eyeb \in E$, whence $x \in \mathrm{cl}_e(E)$, as desired. □

2.12. Theorem. *If e is an idempotent in a unital, semi-primitive ring R, then* $\mathrm{cl}(e)$ *consists of all central idempotents in eRe.*

Proof. In view of Proposition 2.11 we may replace R by eRe (which is again semi-primitive) and therefore assume that $e = 1$.

If now p is a central idempotent in R and $ap + b = 1$ for some a, b in R, then $1 - p = b(1 - p) = (1 - p)b$, so

$$p + (1-p)b = p + 1 - p = 1\,,$$

proving that $p \in \mathrm{cl}(1)$.

Conversely, if p is an element in $\mathrm{cl}(1)$ consider a maximal left ideal L of R. If $p \notin L$, then $Rp + L = R$, so $ap + b = 1$ for some a in R and b in L. Since $p \in \mathrm{cl}(1)$ we have $p + yb = 1$ for some y in R, whence $1 - p \in L$. Thus, either $p \in L$ or $1 - p \in L$ for any maximal left ideal L of R. In particular, $p(1 - p) \in L$ for all L, whence $p = p^2$, since R is semi-primitive.

If $x \in pR(1 - p)$ then $q = p + x$ is an idempotent in R, and setting $u = 1 + x$ we have an invertible element in R (with $u^{-1} = 1 - x$) such that $u^{-1}pu = q$. Since $y \to u^{-1}yu$ is an automorphism of R it follows from condition (xi) for cl in 1.2 that $q \in \mathrm{cl}(1)$, so either $q \in L$ or $1 - q \in L$ for any maximal left ideal L. Fixing L this leads to the following four possibilities:

(i) $p \in L$ and $p + x \in L$.
(ii) $p \in L$ and $1 - p - x \in L$.
(iii) $1 - p \in L$ and $p + x \in L$.
(iv) $1 - p \in L$ and $1 - p - x \in L$.

Of these, (ii) and (iii) are clearly impossible, since by addition they imply that $u^{-1} \in L$ or $u \in L$, respectively. On the other hand, both (i) and (iv) imply by subtraction that $x \in L$, which is therefore always the case. Since R is semi-primitive it follows that $x = 0$, and since x was arbitrary, $pR(1-p) = 0$, so p is central, as desired. □

2.13. Examples. The last three results have deliberately been stated for semi-primitive rings, since the presence of a radical seems to complicate the results beyond recognition. To see what may happen, take any vector space N over a field $\mathbb{F}$ and put $A = \mathbb{F} \oplus N$. Writing 1 for the vector $(1, 0)$ we obtain a trivial algebra structure on A by defining $1 \cdot 1 = 1$, $1 \cdot n = n$ and $n \cdot m = 0$ for all n, m in N. Evidently N is the radical of A. Now put $R = \mathbb{M}_2(A)$, which is a unital algebra with $\mathrm{rad}(R) = \mathbb{M}_2(N)$, and denote by e_{ij} the usual matrix units. Then for the right principal ideal $e_{11}R$ in R we have that $\mathrm{cl}(e_{11}R)$ is a proper subset of $e_{11}R + \mathrm{rad}(R)$ containing $e_{11}R \cup \mathrm{rad}(R)$. In fact, for n in N computation shows that $e_{12} + ne_{22} \in \mathrm{cl}(e_{11}R)$ if and only if $n = 0$. In particular, $\mathrm{cl}(e_{11}R)$ is not a linear subspace. Similarly we find that

$$\mathrm{cl}(e_{11}) = \mathrm{rad}(R) \cup \{e_{11} + (\mathrm{rad}(R))(1 - e_{11})\} .$$

In the same vein, consider the ring R of upper triangular $2 \times 2-$matrices over a field $\mathbb{F}$. Then e_{11} is in $\mathrm{cl}(1)$, but not in $\mathrm{cr}(1)$ (with cr as in 1.5). Analogously, e_{22} is in $\mathrm{cr}(1)$, but not in $\mathrm{cl}(1)$. Note that e_{11} and e_{22} are not central idempotents in R. The radical of R is $\mathbb{F}e_{12}$, so that $R/\mathrm{rad}(R) = \mathbb{F} \oplus \mathbb{F}$. By computation we find that

$$\mathrm{cl}(1) = \{\, e_{11} + \mathrm{rad}(R) \,\} \cup \{\, 1 + \mathrm{rad}(R) \,\} \cup \mathrm{rad}(R) .$$

As noted in Remark 1.9.e any surjective ring homomorphism $\pi \colon R \to S$ between unital rings R and S is "continuous" and "open" with respect to cl. On the other hand, for our ring above the quotient morphism $\pi \colon R \to R/\mathrm{rad}(R)$ is not a "closed" map, because $\pi(\mathrm{cl}(1))$ is not "closed" in $\mathbb{F} \oplus \mathbb{F}$. Indeed, $\pi(\mathrm{cl}(1)) = \{\, (1, 0), (1, 1), (0, 0) \,\}$, missing the central idempotent $(0, 1)$ in $\mathbb{F} \oplus \mathbb{F}$.

3. Persistent Regularity

3.1. Definitions. An element x in a (unital) ring R is *von Neumann regular* if $x = xyx$ for some y in R. We shall refer to y as a *partial inverse* for x and note that, replacing if necessary y with yxy, we may assume that also y is regular, with x as its partial inverse. The set of von Neumann regular elements will be denoted by R^r.

An element x such that $p = 1 - xy$ and $q = 1 - yx$ satisfy $pRq = qRp = 0$ (in symbols $p \perp q$) for some y in R is said to be *quasi-invertible* with y as its *quasi-inverse*. Necessarily $x \in R^r$ and one may choose y to be a partial inverse for x. The set of quasi-invertible elements in R is denoted by R_q^{-1}. These elements are maximal in the partial ordering on R^r in [**6**, Proposition 2.5]; and in good cases, notably if R is an exchange ring or a QB-ring, R_q^{-1} are precisely the maximal elements in R^r, cf. Corollary 5.11 and Proposition 8.2 in [**6**].

Following the spirit if not precisely the letter of [**10**, §7] we say that an element x in R is *persistently regular*, if $x \in U \subset R^r$ for some subset U in R with $\mathrm{inl}(U) = U$. Thus, by Proposition 1.7 there must exist some b in R with $Rx + Rb = R$, such that $U_x(b) \subset R^r$. We denote by R^{pr} the set of persistently regular elements, and note from Proposition 1.7 that

$$R^{pr} = \mathrm{inl}(R^r).$$

It is clear that by choosing inl, depending on cl, for the definition of persistent regularity we have an asymmetric concept favouring left multiplication over right. The next result shows that the definition is in fact symmetric, and that persistent regularity has important structural properties. In particular, if we define inr in analogy with inl, i.e. $\mathrm{inr}(E) = R\backslash(\mathrm{cr}(R\backslash E))$, with cr as in 1.5, then $R^{pr} = \mathrm{inr}(R^r)$.

3.2. Theorem. *Let x be a von Neumann regular element in a unital ring R and choose any partial inverse y for x. The following conditions are then equivalent:*

- **(i)** $x \in R^{pr}$.
- **(ii)** $x + (1 - xy)R(1 - yx) \subset R^r$.
- **(iii)** $(1 - xy)R(1 - yx) \subset R^r$.
- **(iv)** $x + R(1 - yx) \subset R^r$.
- **(v)** $x + (1 - xy)R \subset R^r$.

Proof. For ease of notation set $p = 1 - xy$ and $q = 1 - yx$.
(i) $\implies$ (ii) By assumption we can find a, b in R such that $ax + b = 1$ and $U_x(b) \subset R^r$. Since $(1 - b)q = axq = 0$ we have for each t in R that

$$x + ptq = x + ptbq = x + ptb - ptbyx = (1 - ptby)x + ptb\,.$$

Set $u = 1 - ptby$ and note that $u \in R^{-1}$ with $u^{-1} = 1 + ptby$. Moreover, $up = p$. Thus,

$$x + ptq = u(x + ptb) \in uR^r \subset R^r\,.$$

(ii) $\implies$ (iii) For each t in R there is by assumption a partial inverse w for the element $x + ptq$. Consequently $ptq \in R^r$, since

$$ptqwptq = p(x + ptq)w(x + ptq)q = p(x + ptq)q = ptq\,.$$

(iii) $\implies$ (ii) For each t in R there is by assumption a partial inverse w for the element ptq. Consequently $x + ptq \in R^r$, since

$$(x + ptq)(y + qwp)(x + ptq) = xyx + ptqwptq = x + ptq\,.$$

(ii) $\iff$ (iv) For each t in R we have

$$x + tq = x + xytq + ptq\,.$$

Setting $v = 1 + ytq$ we see that $v \in R^{-1}$ with $qv = q$. Thus

$$x + tq = (x + ptq)v\,,$$

which shows that $x + Rq$ and $x + pRq$ are simultaneously in R^r.
(ii) $\iff$ (v) By an argument symmetric to (ii) $\iff$ (iv).
(iv) $\implies$ (i) By assumption $x + Rq = U_x(q) \subset R^r$, where $yx + q = 1$, whence $x \in R^{pr}$ by definition. □

3.3. Corollary. $R_q^{-1} \subset R^{pr}$.

Proof. Immediate from condition (ii), since $R_q^{-1} \subset R^r$. □

3.4. Corollary. *If R is semi-prime, every maximal element in R^{pr} (with respect to the ordering $\prec$ in R^r) lies in R_q^{-1}.*

Proof. We already know from [**6**, Proposition 2.8] that each element in R_q^{-1} is maximal. But by [**6**, Lemma 2.7] we have $x \prec z$ in R^r if and only if there exist idempotents p and q in R, such that

$$z - x \in pRq \qquad \text{and} \qquad (1-p)z = x = z(1-q).$$

If now $x \in R^{pr}$, we see from condition (iii) in Theorem 3.2 that x can be extended to a strictly larger element in R^r (actually in R^{pr}), unless $(1-xy)R(1-yx) = 0$ for some partial inverse y for x. Since R is semi-prime this implies that also $(1-yx)R(1-xy) = 0$, so $1-yx \perp 1-xy$ and $x \in R_q^{-1}$. □

3.5. Remarks. It is worth mentioning that three of the conditions in Theorem 3.2 can be formally strengthened to:

(ii') $\quad x + (1-xy)R(1-yx) \subset R^{pr}$.
(iv') $\quad x + R(1-yx) \subset R^{pr}$.
(v') $\quad x + (1-xy)R \subset R^{pr}$.

To see this, note that Remark 1.9.c, applied with $E = R^r$, shows that

$$(*) \qquad R^{-1}R^{pr} = R^{pr}R^{-1} = R^{pr}.$$

Now observe that condition (iv) is equivalent to condition (iv') by Proposition 1.7, since $x + R(1-yx)$ has the form $U_x(1-yx)$. However, in the arguments for (ii) $\Longleftrightarrow$ (iv) and (ii) $\Longleftrightarrow$ (v) we showed that for each t in R there are invertible elements v and u (depending on t) such that

$$x + tq = (x + ptq)v \quad \text{and} \quad x + pt = u(x + ptq),$$

(again with $p = 1 - xy$ and $q = 1 - yx$). Since $x + tq \in R^{pr}$, it follows from $(*)$ that also $x + ptq$ and $x + pt$ belong to R^{pr}, as desired.

Upon writing an element z in the form $zyx + zq$ it follows that we have the equation $x + Rq = \{z \in R \mid zy = xy\}$. Consequently we can combine conditions (iv') and (v') to the seemingly stronger condition:

(vi) $\quad \{z \in R \mid zy = xy\} \cup \{z \in R \mid yz = yx\} \subset R^{pr}$.

The results in Theorem 3.2 concern an unspecified partial inverse for a von Neumann regular element. The next result determines what freedom we actually have.

3.6. Proposition. *If x and y are regular elements in a unital ring R with $xyx = x$ and $yxy = y$, then also $xy'x = x$, provided that*

$$(*) \qquad y' = y + s(1 - xy) + (1 - yx)t$$

for some s, t in R. Conversely, if $xy'x = x$, then necessarily y' has the form in $(*)$. Moreover, the element y' in $(*)$ will satisfy $y'xy' = y'$, if and only if

$$(**) \qquad (1 - yx)(s + t - txs)(1 - xy) = 0\,,$$

a condition satisfied e.g. for $s = 0$ and $t = t'xy$.

Proof. Evidently every element y' of the form $(*)$ will satisfy $xy'x = x$. Assume now that $xy'x = x$ and rewrite

$$\begin{aligned} y' &= yxy' + (1 - yx)y' = yxy'xy + yxy'(1 - xy) + (1 - yx)y' \\ &= y + (yxy')(1 - xy) + (1 - yx)y'\,, \end{aligned}$$

which shows that y' satisfies $(*)$. The formula in $(**)$ is a simple computation. □

3.7. Remark. If $p = 1 - xy$ and $p' = 1 - xy'$ denote the defect idempotents for x corresponding to the two partial inverses y and y' mentioned above, then

$$p' = 1 - x(y + s(1 - xy) + (1 - yx)t) = (1 - xsp)p = up\,.$$

Here $u = 1 - xsp \in R^{-1}$ (with $u^{-1} = 1 + xsp$) and $ux = x$. Similarly, if $q = 1 - yx$ and $q' = 1 - y'x$, then $q' = qv$ for some v in R^{-1} with $xv = x$. Consequently,

$$x + p'Rq' = x + upRqv = u(x + pRq)v\,.$$

This means that although the conditions (ii)-(v) in Theorem 3.2 are formulated for a partial inverse y for x that satisfies $yxy = y$ (and this is used in the arguments), they are, in fact, valid for any choice of a partial inverse y'.

For the convenience of the reader we include the following well-known result.

3.8. Lemma. *Let I be a two-sided ideal of a ring R contained in R^r and denote by $\pi\colon R \to R/I$ the quotient morphism. If $u \in R$ such that $\pi(u) = (R/I)^r$, then $u + I \subset R^r$.*

Proof. By [**8**, Lemma 1] each element y i R, such that $y - yvy$ is regular for some v in R, is itself regular. Indeed, if $(y - yvy)a(y - yvy) = y - yvy$, then

$$(*) \qquad y = yvy + (y - yvy)a(y - yvy) = y(v + (1 - vy)a(1 - yv))y\,.$$

In our case there is by assumption an element v such that $\pi(u) = \pi(u)\pi(v)\pi(u)$. This means that $u - uvu \in I$. Thus, if $y = u + x$ for some x in I, we still have $y - yvy \in I$. Since $I \subset R^r$ it follows that $y \in R^r$. □

3.9. Theorem (cf. [**8**])**.** *For each ring R there is a largest ideal $I_{\rm reg}(R)$ contained in R^r, and the quotient $R/I_{\rm reg}(R)$ has no non-zero ideals contained in $(R/I_{\rm reg}(R))^r$. We always have $I_{\rm reg}(R)+R^r \subset R^r$, and if R is additively generated by its invertibles ($R = R^{-1} + R^{-1} + \cdots$), then*

$$I_{\rm reg}(R) = \{x \in R \mid x + R^r \subset R^r\}\,.$$

Proof. It follows from Lemma 3.8 that the sum of two ideals contained in R^r is again in R^r, so that we can define $I_{\rm reg}(R)$ as the sum of all ideals of R contained in R^r. Applying Lemma 3.8 with $I = I_{\rm reg}(R)$ we see that $R/I_{\rm reg}(R)$ can have no non-zero ideals contained in its regular part, since the counter-images in R would again be contained in R^r, contradicting the maximality of $I_{\rm reg}(R)$. The lemma also shows that $I_{\rm reg}(R) + R^r \subset R^r$.

For the last assertion, put $J = \{x \in R \mid x + R^r \subset R^r\}$. Evidently $I_{\rm reg}(R) \subset J \subset R^r$, $J = -J$ and $J + J \subset J$. If $u \in R^{-1}$, then

$$uJ + R^r = u(J + u^{-1}R^r) = u(J + R^r) \subset uR^r = R^r\,,$$

so $R^{-1}J = J$. Similarly, $JR^{-1} = J$. By assumption each element a in R has a representation $a = \sum a_i$, with a_i in R^{-1}, whence $aJ \subset J$ and $Ja \subset J$. Consequently J is an ideal of R contained in R^r. By maximality, $J = I_{\rm reg}(R)$. □

3.10. Corollary. *For each ring R we have*

$$R^{pr} + I_{\rm reg}(R) \subset R^{pr}\,.$$

Proof. Combine Theorem 3.9 and condition (x) from Lemma 1.8. □

3.11. Proposition. *If L is a left ideal in a unital ring R, then* $\mathrm{inl}(X) = X$, *when X is one of the four subsets $R^{-1} + L$, $R_\ell^{-1} + L$, $R_r^{-1} + L$ and $R_q^{-1} + L$.*

Proof. By Remark 1.9.d it suffices to prove the result when $L = 0$.

For $X = R^{-1}$ and $X = R_\ell^{-1}$ the result follows from condition (vi) in Lemma 1.8.

If $x \in R_r^{-1}$ and $xy = 1$, then $(x + a(1 - yx)y = 1$ for every a in R. Thus, $U_x(1 - yx) \subset R_r^{-1}$, so $x \in \mathrm{inl}(R_r^{-1})$.

If $x \in R_q^{-1}$ with quasi-inverse y, then $x + R(1 - yx) \subset R_q^{-1}$ by by [**6**, Theorem 2.3]. Thus, $U_x(1 - yx) \subset R_q^{-1}$, so $R_q^{-1} = \mathrm{inl}(R_q^{-1})$. □

3.12. Remark. It follows from Corollaries 3.3 and 3.10 and Proposition 3.11 that if R is a unital ring, then

$$\mathrm{inl}(R_q^{-1} + I_{\rm reg}(R)) = R_q^{-1} + I_{\rm reg}(R) \subset R^{pr}\,.$$

It is tempting to conjecture that the inclusion in the last statement is actually an equality. At least this will be the case for many of the accessible examples, like C^*-algebras, cf. [**10**, Theorem 7.7], and also the algebra $\mathbb{B}(\mathbb{F})$ of countably infinite, but row- and column-finite matrices over a field $\mathbb{F}$, cf. [**6**, Example 8.8.A]. However, the ring R of upper triangular $2 \times 2-$matrices over a field $\mathbb{F}$, considered in Example 2.13, provides a counterexample (albeit not a semi-primitive example), since $R^{pr} \neq R_q^{-1} + I_{\text{reg}}(R)$. In fact, $I_{\text{reg}}(R) = 0$ here, and $R_q^{-1} (= R^{-1})$ equals the set of matrices where both diagonal terms are non-zero. However, R^r is the set of matrices where just one of the diagonal terms is non-zero, together with the zero element. It follows that $R^r \setminus R^{pr} = \{0\}$.

Returning to a general unital ring R we observe that if $x + t \in R_q^{-1} + I_{\text{reg}}(R)$ and y is a quasi-inverse for x, then $x + t - (x+t)y(x+t) \in I_{\text{reg}}(R)$. From formula $(*)$ in the proof of Lemma 3.7 it follows that we can find a partial inverse for $x+t$ of the form $y + s$ for some s in $I_{\text{reg}}(R)$.

If z is any other partial inverse for $x + t$, then by Proposition 3.6

$$z = y + s + ap + bq\,,$$

for some a, b in R, where $p = 1 - (x+t)(y+s)$ and $q = 1 - (y+s)(x+t)$. By computation this shows that for some r in $I_{\text{reg}}(R)$ we have

$$z = y + a(1 - xy) + (1 - yx)b + r\,,$$

and this element belongs to $R_q^{-1} + I_{\text{reg}}(R)$ by [**6**, Theorem 2.3].

The conclusion is that any partial inverse for an element in $R_q^{-1} + I_{\text{reg}}(R)$ again belongs to $R_q^{-1} + I_{\text{reg}}(R)$.

4. C*-Algebras

The category of C^*-algebras, i.e. norm closed $^*-$subalgebras of $\mathbb{B}(\mathfrak{H})$, where $\mathfrak{H}$ is a Hilbert space, is one of the most important sources of examples of non-commutative rings. On the surface these are very well-behaved rings, and the spectral theorem makes many of the ordinary matrix algebra techniques available; yet when $\mathfrak{H}$ is infinite-dimensional the variety of C^*-subalgebras of $\mathbb{B}(\mathfrak{H})$ is tremendous, providing us with a wide range of phenomena. The fact that the operation cl has so many pleasant properties in the category of C^*-algebras should therefore be taken as a sign of important future developments.

Note that by Remark 2.7 any property of cl that does not specifically mention a unit is valid also for non-unital C^*-algebras.

4.1. Theorem. *Let G denote the set of invertible elements in the positive part A_+^1 of the unit ball of a unital C^*-algebra A. If E is a subset of A such that $EG \subset E^=$ (where $E^=$ denotes the norm closure of E), then also $EA_+^1 \subset E^=$, and in that case* $\text{cl}(E) \subset E^=$.

Proof. If $EG \subset E^=$ and $a \in A_+^1$, then $\delta 1 + (1-\delta)a \in G$ for every $\delta > 0$. Since $E(\delta 1 + (1-\delta)a) \subset E^=$ by assumption, we see that $Ea \subset E^=$ by continuity.

Next, consider a subset E such that $EA_+^1 \subset E^=$ and take x in $\text{cl}(E)$. For each $\varepsilon > 0$ define a continuous function g_ε on $\mathbb{R}$ by

$$g_\varepsilon(t) = \begin{cases} 0 & \text{for } t \leq \varepsilon, \\ \varepsilon^{-1} - t^{-1} & \text{for } \varepsilon \leq t \leq 2\varepsilon, \\ t^{-1} & \text{for } t \geq 2\varepsilon. \end{cases}$$

Set $f_\varepsilon(t) = tg_\varepsilon(t)$, and note that $f_\varepsilon f_{2\varepsilon} = f_{2\varepsilon}$, since $f_\varepsilon(t) = 1$ for $t \geq 2\varepsilon$. Now define $a = g_\varepsilon(x^*x)x^*$ and $b = 1 - f_\varepsilon(x^*x)$. Then $ax + b = 1$, so by assumption $x + yb \in E$ for some y in A. But then

$$xf_{2\varepsilon}(x^*x) = (x + yb)f_{2\varepsilon}(x^*x) \in E^= .$$

By spectral theory

$$\|x - xf_{2\varepsilon}(x^*x)\|^2 = \|(1 - f_{2\varepsilon}(x^*x))x^*x(1 - f_{2\varepsilon}(x^*x))\| \leq \sup |(1 - f_{2\varepsilon}(t))^2 t| \leq 4\varepsilon ,$$

whence $x \in E^=$. □

4.2. Corollary. *If R is a closed right ideal of A, then* $\mathrm{cl}(R) = R$. □

4.3. Corollary. *If B is a hereditary C^*−subalgebra of A, then* $\mathrm{cl}(B) = B$.

Proof. By definition $B = L \cap L^*$ for some closed left ideal L in A, cf. [**15**, Theorem 1.5.2]. As $\ker \mathrm{hull}\,(L) = L$ for every closed left ideal in A by [**15**, Lemma 3.13.5] we have $\mathrm{cl}(L) = L$ by Theorem 2.2, and thus, by Corollary 4.2,

$$B \subset \mathrm{cl}(B) \subset \mathrm{cl}(L) \cap \mathrm{cl}(L^*) = L \cap L^* = B .$$

□

4.4. Corollary. *Let E be a bounded subset of A contained in the closed ball $\mathbf{B}_r(A)$ with center 0 and radius r. Then also* $\mathrm{cl}(E) \subset \mathbf{B}_r(A)$.

Proof. Since $\mathbf{B}_r(A)A^1_+ \subset \mathbf{B}_r(A)$ it follows from Theorem 4.1 that $\mathrm{cl}(\mathbf{B}_r(A)) = \mathbf{B}_r(A)$. Consequently $\mathrm{cl}(E) \subset \mathrm{cl}(\mathbf{B}_r(A)) = \mathbf{B}_r(A)$ by the monotonicity property (ii) for cl in 1.2. □

4.5. Examples. Just to show that the operation cl is not completely predictable, consider the open unit ball $\mathbf{B}_1(A)^\circ$ in a C^*−algebra A. Here we actually have $\mathrm{cl}(\mathbf{B}_1(A)^\circ) = \mathbf{B}_1(A)^\circ$. To see this note first that $\mathrm{cl}(\mathbf{B}_1(A)^\circ) \subset \mathbf{B}_1(A)$ by Corollary 4.4, so we only have to show that no element x in A with $\|x\| = 1$ can belong to $\mathrm{cl}(\mathbf{B}_1(A)^\circ)$. Toward this end consider the trivial equation $x^*x + (1 - x^*x) = 1$. If now $x \in \mathrm{cl}(\mathbf{B}_1(A)^\circ)$ we could find a y in A such that $\|x + y(1 - x^*x)\| < 1$. However, if we let $x = u|x|$ be the polar decomposition of x (say, in the universal representation for A) and choose a state φ of A such that $\varphi(|x|) = 1$ (which is possible since $\|\,|x|\,\| = 1$), then since $1 - |x| \geq 0$ it follows from the Cauchy-Schwarz inequality that $\varphi(z|x|) = \varphi(z)$ for any z in the weak closure of A. (φ is "definite" on $|x|$ in Kadison's terminology.) Consequently we obtain the contradiction

$$\|x + y(1 - x^*x)\| \geq \varphi(u^*(x + y(1 - |x|^2)))$$
$$= \varphi(|x| + u^*y(1 - |x|^2)) = 1 + \varphi(u^*y) - \varphi(u^*y) = 1 .$$

Another interesting case occurs when we consider the unit sphere $\mathbf{S}(A)$ in a unital C^*−algebra A. Then

$$\mathrm{cl}(\mathbf{S}(A)) = \mathbf{B}_1(A) \setminus (A_\ell^{-1} \cap \mathbf{B}_1(A)^\circ) .$$

To prove this note first that $\mathrm{cl}(\mathbf{S}(A)) \subset \mathbf{B}_1(A)$ by Corollary 4.4. If now $x \in \mathbf{B}_1(A)$ and $ax + b = 1$ we shall try to find a y in A such that $\|x + yb\| = 1$. If $b \neq 0$ this can be achieved simply with a scalar y. But if $b = 0$, which can occur only if x is left invertible, then $\|x + yb\| = 1$ means that $\|x\| = 1$. The elements in $\mathbf{B}_1(A)$ excluded from $\mathrm{cl}(\mathbf{S}(A))$ are therefore precisely those in $A_\ell^{-1} \cap \mathbf{B}_1(A)^\circ$.

4.6. Proposition. *If A is a unital C^*-algebra, then*

$$\mathrm{cl}(A_q^{-1}) = (A_q^{-1})^= .$$

Proof. Since $A_q^{-1}A^{-1} = A_q^{-1}$, it follows from Theorem 4.1 that $\mathrm{cl}(A_q^{-1}) \subset (A_q^{-1})^=$.

Assume now that $x \in (A_q^{-1})^=$ and consider an equation $ax + b = 1$. Then

$$\begin{aligned} 1 &= (ax+b)^*(ax+b) \leq 2(x^*a^*ax + b^*b) \\ &\leq 2\|a\|^2 x^*x + 2b^*b \leq 2\|a\|^2\, \|x\|\, |x| + 2b^*b\,. \end{aligned}$$

With $\gamma = \max(2, 2\|a\|^2\|x\|)$ this means that $|x| + b^*b \geq \gamma^{-1}1$, so $c = |x| + b^*b \in A^{-1}$.

Let $x = u|x|$ be the polar decomposition of x in some faithful representation of A on a Hilbert space. Choose for each n a continuous function f_n on $\mathbb{R}_+$ vanishing in a neighbourhood of 0 and such that $f_n(t) = t$ for $t \geq \frac{1}{n}$. Since $x \in (A_q^{-1})^=$, there is by [**9**, Corollary 2.3] an extreme partial isometry u_n in A_q^{-1} such that

$$uf_n(|x|) = u_n f_n(|x|)\,.$$

This implies that

$$\begin{aligned} &\|x - u_n|x|\| \\ &\leq \|u(|x| - f_n(|x|))\| + \|uf_n(|x|) - u_n f_n(|x|)\| + \|u_n(f_n(|x|) - |x|)\| \\ &\leq \tfrac{1}{n} + 0 + \tfrac{1}{n} = \tfrac{2}{n}\,. \end{aligned}$$

By the estimate above,

$$\begin{aligned} &\|u_n - (x + u_n b^*b)c^{-1}\| \leq \|u_n - (u_n|x| + u_n b^*b)c^{-1}\| + \tfrac{2}{n}\|c^{-1}\| \\ &= \|u_n - u_n(|x| + b^*b)c^{-1}\| + \tfrac{2}{n}\|c^{-1}\| = \tfrac{2}{n}\|c^{-1}\|\,. \end{aligned}$$

For an element y in A_q^{-1} the number $m_q(y)$ denotes the length of the gap around zero in the spectrum of $|y|$, cf. [**9**, Definition 1.4]. However, $m_q(y)$ also measures the distance from y to $A \setminus A_q^{-1}$ by [**9**, Proposition 1.5]. Since $m_q(u_n) = 1$ it follows that $m_q((x + u_n b^*b)c^{-1}) > 0$ for $n > 2\|c^{-1}\|$. Thus $(x + u_n b^*b)c^{-1} \in A_q^{-1}$, and therefore also $x + u_n b^*b \in A_q^{-1}$; proving that $x \in \mathrm{cl}(A_q^{-1})$. □

4.7. Remark. We apologize to the readers for having inadvertently used the assumption $(A_q^{-1})^= = A$ instead of the one stated (viz. $x \in (A_q^{-1})^=$) in the proof of [**6**, Proposition 9.1]. The present Proposition 4.6 is meant to rectify this error.

Under the stronger asssumption that A_q^{-1} is dense in A, it is shown in [**9**, Theorem 3.3] that whenever $ax + yb = 1$ for some a, y in A (equivalently, when $x^*x + b^*b \in A^{-1}$), then $x + ub \in A_q^{-1}$ for some extreme partial isometry u in A, i.e. an element for which $(1 - uu^*)A(1 - u^*u) = 0$, so that u^* is a quasi-inverse for u. Inspection of the argument reveals that it suffices to know that x *and* b both belong to $(A_q^{-1})^=$. If just $x \in (A_q^{-1})^=$, then by Proposition 4.6 we only get $x + ub^*b \in A_q^{-1}$.

4.8. Remark. The key ingredient in the proof of Proposition 4.6, viz. [**9**, Corollary 2.3], is also available for the more elementary subsets A^{-1} and A_ℓ^{-1}. In the case of A^{-1} this is [**16**, Theorem 5], and for A_ℓ^{-1} the argument is given in [**17**, Corollary 7.2]. It follows that we can assert that

$$\mathrm{cl}(A^{-1}) = (A^{-1})^{=} \qquad \text{and} \qquad \mathrm{cl}(A_\ell^{-1}) = (A_\ell^{-1})^{=}\,.$$

For the subset A_r^{-1} of right invertible elements a direct argument is at hand. Note first that $\mathrm{cl}(A_r^{-1}) \subset (A_r^{-1})^{=}$ by Theorem 4.1. Then take x in $(A_r^{-1})^{=}$ and consider an equation $ax + b = 1$. This implies, as in the proof of Proposition 4.6, that $x^*x + b^*b \in A^{-1}$. Since A^{-1} is open we can find y in A_ℓ^{-1}, approximating x^*, so that $yx + b^*b \in A^{-1}$. Now choose z in A_r^{-1} with $zy = 1$ and observe that

$$x + (zb^*)b = z(yx + b^*b) \in zA^{-1} \subset A_r^{-1}\,;$$

which proves that $x \in \mathrm{cl}(A_r^{-1})$. Thus, also in this case we have

$$\mathrm{cl}(A_r^{-1}) = (A_r^{-1})^{=}\,.$$

4.9. Proposition. *The set of persistently regular elements in a unital C^*-algebra A coincides with the (topological) interior of A^r, i.e.*

$$\mathrm{inl}(A^r) = (A^r)^\circ\,.$$

Proof. Since $A^rA^{-1} = A^r$ also $(A \setminus A^r)A^{-1} = A \setminus A^r$, so $\mathrm{cl}(A \setminus A^r) \subset (A \setminus A^r)^{=}$ by Theorem 4.1. Thus, $(A^r)^\circ \subset \mathrm{inl}(A^r) = A^{pr}$.

To prove the reverse inclusion take x in A^{pr} and choose a partial inverse y for x. Since 0 is persistently regular if and only if $A = A^r$ $(= A^{pr})$ we may assume that $x \neq 0$, whence also $y \neq 0$. Then $1 + ay \in A^{-1}$ for each a in A with $\|a\| < \|y\|^{-1}$. Consequently,

$$\begin{aligned} x + a &= x + ayx + a(1 - yx) = (1 + ay)x + a(1 - yx) \\ &= (1 + ay)(x + (1 + ay)^{-1}a(1 - yx)) \subset R^{-1}R^r = R^r \end{aligned}$$

by condition (iv) in Theorem 3.2. □

4.10. Remark. Being a persistently regular element in a C^*-algebra A is a very strong notion, cf. [**10**, §7], where it is labeled *persistently closed range.* The reason, presumably, is that the maximal regular ideal of A coincides with the socle of A, i.e. the ideal $\mathrm{Soc}(A)$ generated by the minimal projections in A, isomorphic to a direct sum of algebras $\mathbb{B}_f(\mathfrak{H})$ of finite rank operators on Hilbert spaces, so that finite-dimensionality is just below the surface. In fact, it is shown in [**10**. Theorem 7.7] that

$$A^{pr} = A_q^{-1} + \mathrm{Soc}(A)\,.$$

Moreover, for each element in A^{pr} its component in $\mathrm{Soc}(A)$ can be chosen with arbitrarily small norm, so that the two sets A^{pr} and A_q^{-1} are simultaneously dense in A. Since $A^{pr}A^{-1} = A^{pr}$ it follows from Theorem 4.1 that $\mathrm{cl}(A^{pr}) \subset (A^{pr})^{=}$. For a unital C^*-algebra A we therefore have the result:

$$\{\,\mathrm{cl}(A^{pr}) = A\,\} \qquad \Longrightarrow \qquad \{\,\mathrm{cl}(A_q^{-1}) = A\,\}\,.$$

Unfortunately, this is not necessarily true for all rings. In fact, [**4**, Example 3.2] provides an example of a stably finite, von Neumann regular ring R (so $R = R^r = R^{pr}$) for which $\mathrm{cl}(R_q^{-1}) \neq R$, cf. [**6**, Example 8.8.e].

4.11. Theorem. *For each C^*−algebra A we have* $\mathrm{cl}(A_{sa}) = A_{sa}$.

Proof. As in the proof of Theorem 4.1 we define for each $\varepsilon > 0$ the continuous function g_ε on $\mathbb{R}$ for which

$$g_\varepsilon(t) = \begin{cases} 0 & \text{for } t \le \varepsilon, \\ \varepsilon^{-1} - t^{-1} & \text{for } \varepsilon \le t \le 2\varepsilon, \\ t^{-1} & \text{for } t \ge 2\varepsilon. \end{cases}$$

We then define $f_\varepsilon(t) = tg_\varepsilon(t)$ and note that $0 \le f_\varepsilon \le 1$ and $f_\varepsilon f_{2\varepsilon} = f_{2\varepsilon}$.

Consider x in $\mathrm{cl}(A_{sa})$ and for ease of notation put $h_\varepsilon = f_\varepsilon(x^*x)$. For the first step we let $a = g_\varepsilon(x^*x)x^*$ and $b = 1 - h_\varepsilon$. Then $ax + b = 1$, so $x + yb \in A_{sa}$ for some y in A by assumption. Since $(1 - f_\varepsilon)f_{2\varepsilon} = 0$ this implies that

$$h_{2\varepsilon}xh_{2\varepsilon} = h_{2\varepsilon}(x + yb)h_{2\varepsilon} \in A_{sa}.$$

For the second step in the proof we let $a = (1 - i(1 - h_{\varepsilon/2})x)g_\varepsilon(x^*x)x^*$ and put $b = 1 - h_\varepsilon + i(1 - h_{\varepsilon/2})xh_\varepsilon$. Again we have $ax + b = 1$ so $x + yb \in A_{sa}$ for some y in A, i.e.

$$(*) \qquad x + y(1 - h_\varepsilon) + iy(1 - h_{\varepsilon/2})xh_\varepsilon \in A_{sa}.$$

Multiplying this expression left and right with $h_{2\varepsilon}$, and recalling that $h_{2\varepsilon}xh_{2\varepsilon} \in A_{sa}$ we learn that

$$(**) \qquad ih_{2\varepsilon}y(1 - h_{\varepsilon/2})xh_{2\varepsilon} \in A_{sa},$$

since $(1 - h_\varepsilon)h_{2\varepsilon} = 0$. Multiplying the expression $(*)$ with $1 - h_{\varepsilon/2}$ left and right leads to the equation

$$(1 - h_{\varepsilon/2})(x + y)(1 - h_{\varepsilon/2}) \in A_{sa},$$

from which we deduce that

$$(***) \qquad (1 - h_{\varepsilon/2})(y - y^*)(1 - h_{\varepsilon/2}) = (1 - h_{\varepsilon/2})(x^* - x)(1 - h_{\varepsilon/2})\,.$$

Writing out $(*)$ in the form $z - z^* = 0$ and multiplying left with $h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})$ and right with $h_{2\varepsilon}$ yields the equation

$$\begin{aligned} &h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})xh_{2\varepsilon} + ih_{2\varepsilon}x^*(1 - h_{\varepsilon/2})y(1 - h_{\varepsilon/2})xh_{2\varepsilon} \\ &- h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})x^*h_{2\varepsilon} - h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})y^*h_{2\varepsilon} = 0\,. \end{aligned}$$

In this equation the first summand is positive, the third summand is small, since $\|(1 - h_{\varepsilon/2})x^*\| \le \varepsilon$ by spectral theory, and the fourth summand is skew-adjoint by $(**)$. The self-adjoint part of the second summand equals

$$h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})i(x^* - x)(1 - h_{\varepsilon/2})xh_{2\varepsilon}$$

by $(***)$, and thus has norm at most $2\|x\|^2\varepsilon$. Taken together this implies that

$$\|h_{2\varepsilon}x^*(1 - h_{\varepsilon/2})xh_{2\varepsilon}\| \le 2\|x\|^2\varepsilon + \|x\|\varepsilon\,.$$

¿From spectral theory we know that $xh_\varepsilon \to x$ as $\varepsilon \to 0$, and it follows from above that $(1 - h_\varepsilon)x \to 0$, whence $h_\varepsilon xh_\varepsilon \to x$. Since $h_\varepsilon xh_\varepsilon \in A_{sa}$ from the first part of the proof, we conclude that $x \in A_{sa}$. □

4.12. Corollary. *If A is a C^*-algebra, then* $\mathrm{cl}(A_+) = A_+$.

Proof. If $x \in \mathrm{cl}(A_+)$ we know from Theorem 4.11 that $x = x^*$. As in the first part of the proof of Theorem 4.11 we take $a = g_\varepsilon(x^2)x$ and $b = 1 - f_\varepsilon(x^2)$, so that $ax + b = 1$. By assumption $x + yb \in A_+$ for some y in A. As before this implies that

$$h_{2\varepsilon} x h_{2\varepsilon} = h_{2\varepsilon}(x + yb)h_{2\varepsilon} \in A_+ \,,$$

and since $h_\varepsilon x = x h_\varepsilon \to x$ we conclude that $x \in A_+$. □

4.13. Proposition. *If A is a unital C^*-algebra, then* $\mathrm{cl}(A^{-1}_{sa}) = (A^{-1}_{sa})^=$.

Proof. If $x \in \mathrm{cl}(A^{-1}_{sa})$, then $x = x^*$ by Theorem 4.11. Moreover, since $aA^{-1}_{sa}a = A^{-1}_{sa}$ for every a in A^{-1}_+ it follows, as in Theorem 4.1, that $aA^{-1}_{sa}a \subset (A^{-1}_{sa})^=$ for every a in A_+.

Now, with notations as in Theorem 4.11, put $h_\varepsilon = f_\varepsilon(x^2)$, so that we have the equation

$$g_\varepsilon(x^2)x^2 + 1 - h_\varepsilon = 1 \,.$$

Since $x \in \mathrm{cl}(A^{-1}_{sa})$ there is a y in A such that $x + y(1 - h_\varepsilon) \in A^{-1}_{sa}$. But then

$$h_{2\varepsilon} x h_{2\varepsilon} = h_{2\varepsilon}(x + y(1 - h_\varepsilon))h_{2\varepsilon} \in (A^{-1}_{sa})^=$$

by the argument above; and since $\|x - h_\varepsilon x h_\varepsilon\| \to 0$, it follows that $x \in (A^{-1}_{sa})^=$.

To prove the reverse implication we first observe that if $x \in A_{sa}$ and $y \in A_+$, such that $x^2 + y^2 \geq \varepsilon 1$ for some $\varepsilon > 0$, then the matrix $\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ is invertible. Otherwise, representing A on a sufficiently large Hilbert space $\mathfrak{H}$, e.g. in the universal (or just the reduced atomic) representation, there would be a non-zero vector (ξ, η) in $\mathfrak{H} \oplus \mathfrak{H}$, such that

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x\xi + y\eta \\ y\xi - x\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

This implies that $0 \leq (y\xi|\xi) = (x\eta|\xi) = (\eta|x\xi) = -(\eta|y\eta) \leq 0$, whence $y\xi = y\eta = 0$, since $y \geq 0$. But then also $x\xi = x\eta = 0$, in contradiction with $\|x\xi\|^2 + \|y\xi\|^2 = ((x^2 + y^2)\xi|\xi) \geq \varepsilon\|\xi\|^2$.

If now $x \in (A^{-1}_{sa})^=$ and $ax + b = 1$ for some a, b in A, then as in the proof of Proposition 4.6 we see that $\gamma 1 \leq |x| + b^*b$ for some $\gamma > 0$. Setting $y = b^*b$ the same argument gives $\gamma^2 1 \leq (|x| + y)^2 \leq 2x^2 + 2y^2$, so that $x^2 + y^2 \geq \varepsilon 1$ for some $\varepsilon = \frac{1}{2}\gamma^2 > 0$. By the argument above the matrix $\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ is invertible, so if we choose z in A^{-1}_{sa} sufficiently close to x (which we can), then also the matrix $\begin{pmatrix} x & y \\ y & -z \end{pmatrix}$ is invertible. However, this matrix is diagonalizable with

$$\begin{pmatrix} 1 & yz^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ y & -z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{-1}y & 1 \end{pmatrix} = \begin{pmatrix} x + yz^{-1}y & 0 \\ 0 & -z \end{pmatrix} \in (\mathbb{M}_2(A))^{-1}_{sa} <, ,$$

from which we conclude that $x + yz^{-1}y \in A^{-1}_{sa}$. Since $x + yz^{-1}y = x + (b^*bz^{-1}b^*)b$ it follows that $x \in \mathrm{cl}(A^{-1}_{sa})$, as desired. □

4.14. Remark. Proposition 4.13 shows that a C^*-algebra A has real rank zero if and only if $\mathrm{cl}(A^{-1}_{sa}) = A_{sa}$. Since a C^*-algebra has real rank zero precisely when it is an exchange ring, cf. [**3**, Theorem 7.2], this prompts the question whether the operation cl can be used to characterize exchange rings in general.

References

[1] Frank W. Anderson & Kent R. Fuller, *"Rings and Categories of Modules"*, Graduate Texts in Mathematics, **13**, Springer-Verlag, Berlin, 1974.

[2] Pere Ara, *Extensions of exchange rings*, Journal of Algebra **197** (1997), 409–423.

[3] Pere Ara, Ken R. Goodearl, Kevin C. O'Meara & Enric Pardo, *Separative cancellation for projective modules over exchange rings*, Israel Journal of Mathematics **105** (1998), 105–137.

[4] Pere Ara, Kevin C. O'Meara & Dmtry V. Tyukavkin, *Cancellation of projective modules over regular rings with comparability*, Journal of Pure and Applied Algebra **107** (1996), 19–38.

[5] Pere Ara & Francesc Perera, *Multipliers of von Neumann regular rings*, Communications in Algebra, to appear.

[6] Pere Ara, Gert K. Pedersen & Francesc Perera, *An infinite analogue of rings with stable rank one*, Journal of Algebra **230** (2000), 608–655.

[7] Hyman Bass, *"Introduction to some Methods of Algebraic $K-$Theory"*, Regional Conference Series in Mathematics, **21**, American mathematical Society, Providence R.I. 1974.

[8] Bayley Brown & Neal H. McCoy, *The maximal regular ideal of a ring*, Proceedings of the American Mathematical Society **1** (1950), 165–171.

[9] Lawrence G. Brown & Gert K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, Journal für die reine und angewandte Mathematik **469** (1995), 113–147.

[10] Lawrence G. Brown & Gert K. Pedersen, *Extremal $K-$theory and index for C^*-algebras*, K-Theory **20** (2000), 201–241.

[11] Lawrence G. Brown & Gert K. Pedersen, *Ideal theory in C^*-algebras pertaining to ranks and dimension*, Preprint 2000.

[12] Charles W. Curtis & Irving Reiner, *"Methods of Representation Theory,* vol **I**", Interscience Series in Pure and Applied Mathematics, J. Wiley & Sons, New York, 1981.

[13] Ken R. Goodearl, *"Von Neumann Regular Rings"*, Pitman, London, 1979, second edition Krieger, Malabar, FL, 1991.

[14] Richard H. Herman & Leonid N. Vaserstein, *The stable range of C^*-algebras*, Inventiones Mathematicæ **77** (1984), 553–555.

[15] Gert K. Pedersen, *"C^*-Algebras and their Automorphism Groups"*, LMS Monographs **14**, Academic Press, London, 1979.

[16] Gert K. Pedersen, *Unitary extensions and polar decompositions in a C^*-algebra*, Journal of Operator Theory **17** (1987), 357–364.

[17] Gert K. Pedersen, *The $\lambda-$function in operator algebras*, Journal of Operator Theory **26** (1991), 345–381.

[18] Francesc Perera, *Ideal structure of multiplier algebras of simple C^*-algebras with real rank zero*, Preprint.

[19] Francesc Perera, *Extremal richness of multiplier and corona algebras of simple C^*-algebras with real rank zero*, Journal of Operator Theory, to appear.

[20] Francesc Perera, *Lifting units modulo exchange ideals and C^*-algebras with real rank zero*, Journal für die reine und angewandte Mathematik **522** (2000), 51–62.

[21] Marc A. Rieffel, *Dimension and stable rank in the $K-$theory of C^*-algebras*, Proceedings of the London Mathematical Society **46** (1983), 301–333.

[22] Leonid N. Vaserstein, *Bass' first stable range condition*, Journal of Pure and Applied Algebra **34** (1984), 319–330.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193, Bellaterra (Barcelona), Spain

E-mail address: para@mat.uab.es, perera@mat.uab.es

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

E-mail address: gkped@math.ku.dk