# MONOIDS ARISING FROM POSITIVE MATRICES OVER COMMUTATIVE $C^*$-ALGEBRAS.

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**Abstract**

In this note we study the structure of the Cuntz monoid $S(C_0(X))$ associated to a locally compact space $X$. It is shown that for a wide class of spaces, namely the locally compact $\sigma$-compact ones, the Riesz decomposition on the monoid forces the (covering) dimension of the space to be zero. It is possible then to diagonalize matrices over $C_0(X)$ in a unique way with respect to an equivalence relation. As a consequence, we give a representation of $S(C_0(X))$ as a monoid of lower semi-continuous functions over $X$, from which order-cancellation on $S(C_0(X))$ follows.

# Introduction

In [3], Cuntz introduced an equivalence relation for positive elements of $A \otimes K$, where $A$ is a $C^*$-algebra and $K$ the algebra of compact operators on a separable infinite-dimensional Hilbert space. The set of all such equivalence classes was denoted $S(A)$ in [11], and it is an abelian monoid that can be endowed with a partial order, which turns out to extend the Murray-von Neumann subequivalence of projections to general positive elements. The study of the monoid $S(A)$ has proved to be useful in different instances. For example, it was used in [3] to prove the existence of dimension functions for stably finite simple unital $C^*$-algebras (see also [1]), and in [11] to study structural properties of $C^*$-algebras of the form $B \otimes D$, where $D$ is unital, simple and stably finite and $B$ is a $UHF$-algebra.

If $A$ is a $\sigma$-unital $C^*$-algebra with real rank zero and stable rank one, the author established in [9] the exact relation between $S(A)$ and $V(A)$. Our aim in this paper is to push forward that relation in case the $C^*$-algebra is moreover commutative, hence of the form $C_0(X)$ for some locally compact $\sigma$-compact zero-dimensional topological space

*Partially supported by MEC-DGICYT grant no.PB92-0586, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

$X$. It turns out then that the algebraic structure of the monoid $S(C_0(X))$ is stored topologically in $X$.

In [8, Corollary 5.8], it is shown that the totally disconnected compact Hausdorff spaces with the property that for every $n$, each normal element in $M_n(C(X))$ can be continuously diagonalized, are precisely the sub-Stonean spaces (see [7]). In order to achieve our goal, we also prove that it is possible to eliminate the sub-Stonean condition (for locally compact and $\sigma$-compact spaces), thus obtaining a weaker form of diagonalization, namely with respect to the equivalence relation mentioned before.

Here is a brief outline of the paper. Section 1 is devoted to summarizing the necessary basic notions. In Section 2 we relate the zero-dimensionality of a $\sigma$-compact Hausdorff space to a decomposition property of $\sigma$-compact open sets. In Section 3, we prove that the Riesz decomposition property on $S(C_0(X))$ is equivalent to $X$ having covering dimension zero. From this, a special diagonalization of matrices over $C_0(X)$ follows, and we subsequently represent $S(C_0(X))$ as a monoid of lower semicontinuous functions over $X$, which is cancellative.

# 1 Preliminaries

Let $A$ be a $C^*$-algebra and $a, b \in A$. Following [3], [11], we write $a \precsim b$ if there is a sequence $\{x_n\}$ in $A$ such that $x_n = r_n b s_n$ for some $r_n, s_n \in A$ and $x_n \to a$ in norm. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Then $\sim$ is an equivalence relation. Let $M_\infty(A) = \varinjlim M_n(A)$ with inclusions $M_n(A) \ni x \mapsto diag(x, 0) \in M_{n+1}(A)$. For $x, y \in M_\infty(A)_+$, say that $x \sim y$ if and only if $x \sim y$ in $M_n(A)$ for some $n$. By [11, Proposition 2.1], the relation $\precsim$, when restricted to projections, gives the usual Murray-von Neumann subequivalence. For the case of $\sim$, its restriction to projections gives the Murray-von Neumann equivalence when the algebra has stable rank one. Denote by $\langle x \rangle$ the $\sim$-equivalence class of $x$ and define $\langle x \rangle + \langle y \rangle = \langle x' + y' \rangle$ where $x', y' \in M_\infty(A)_+$ satisfy $x' \sim x$, $y' \sim y$ and $x'y' = 0$, or by $\langle x \rangle + \langle y \rangle = \langle \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \rangle$. We sometimes write $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ as $x \oplus y$. Note that if $xy = 0$, then $x + y \sim x \oplus y$. Define $S(A)$ to be the set of all $\sim$-equivalence classes in $M_\infty(A)_+$, and define a partial ordering on it by:

$$\langle x \rangle \leq \langle y \rangle \text{ if and only if } x \precsim y.$$

Thus $(S(A), \leq)$ is a partially ordered abelian monoid.

Let $X$ be a locally compact Hausdorff space, and let $f \in C_0(X)$. We denote by $\mathrm{Coz}(f)$ the cozero set of $f$, that is, $\mathrm{Coz}(f) = \{x \in X \mid |f(x)| > 0\}$.

**Lemma 1.1** *Let $A$ be a $C^*$-algebra and $x, y \in A_+$. Then $x \precsim y$ if and only if there exists $\{r_j\}$ in $A$ with $x = \lim_{j \to \infty} r_j y r_j^*$. If $g, h \in C(\mathrm{Spec}(x))_+$ and $\mathrm{Coz}(g) \subseteq \mathrm{Coz}(h)$, then $g(x) \precsim h(x)$. In particular, if $A$ is commutative, it follows that $f \precsim g$ for $f, g \in A$ if and only if $\mathrm{Coz}(f) \subseteq \mathrm{Coz}(g)$.*

**Proof.** The first conclusion is [11, Proposition 2.4], while the second is observed in [11, Section 1]. The last assertion follows from the two first conclusions. □

Let $M$ be a monoid. All monoids in this paper will be abelian, written additively. We say that $M$ is a **refinement monoid** provided that whenever $x_1, x_2, y_1, y_2 \in M$ satisfy $x_1 + x_2 = y_1 + y_2$, then $\sum_{j=1}^{2} z_{ij} = x_i$ and $\sum_{i=1}^{2} z_{ij} = y_j$ for some elements $z_{ij} \in M$. If $(M, \leq)$ is preordered, then we say that $M$ is a **Riesz monoid** provided that $M$ satisfies the **Riesz Decomposition Property**, that is, whenever $x, y_1, y_2 \in M$ satisfy $x \leq y_1 + y_2$, then there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leq y_i$ for all $i$. Finally, $M$ is said to satisfy the **Riesz Interpolation Property** provided that, whenever $x_1, x_2, y_1, y_2 \in M$ such that $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in M$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$.

If $M$ is the positive cone of a partially ordered abelian group, then these three properties are all equivalent, by [6, Proposition 2.1]. However, for general orderings, there is no relation among them.

A non-zero element $u \in M$ is called an **order-unit** provided that for any $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leq nu$. If $A$ is a unital $C^*$-algebra, then $\langle 1_A \rangle$ is trivially an order-unit for $S(A)$.

Let $M$ be a partially ordered abelian monoid. An **interval** over $M$ is a nonempty subset $I \subseteq M$ which is upward directed and order-hereditary (i.e.: if $x \leq y$ and $y \in I$, then $x \in I$). Denote by $\Lambda(M)$ the set of all intervals of $M$. We can endow $\Lambda(M)$ with a natural abelian monoid structure, namely if $I, J \in \Lambda(M)$

$$I + J = \{z \in M \mid z \leq x + y \text{ for some } x \in I, y \in J\}.$$

If $M$ is a Riesz monoid, then $I + J = \{x + y \mid x \in I, y \in J\}$.

We say that an interval $I$ is **countably generated** if there exists a countable cofinal subset $X \subseteq I$ (that is, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq I$ such that for all $x \in I$, there exists $n \in \mathbb{N}$ such that $x \leq x_n$). Notice that in this context, we can write $I = \{y \in M \mid y \leq x_n \text{ for some } n\}$. If given intervals $I$ and $J$, we have that $I \subseteq J$, then we say that $I$ is **bounded** by $J$.

# 2 Some general topology

The purpose of this section is to express the condition of zero-dimensionality in terms of decomposition of cozero sets of continuous functions. We will make use of the following topological property, similar to normality, which is a modification of the open reduction property introduced by Wehrung in [13, Definition 3.2].

**Definition 2.1** *Let $X$ be a topological space. We say that $X$ satisfies the $\sigma$-**compact open reduction property** if given open and $\sigma$-compact subsets $U$ and $V$ of $X$, there exist open and $\sigma$-compact subsets $U' \subseteq U$ and $V' \subseteq V$ such that $U' \cap V' = \emptyset$ and $U \cup V = U' \cap V'$.*

Note that the last condition may be written as $U \cup V = U' \sqcup V'$, where $\sqcup$ denotes disjoint union of sets.

Our goal is to show that for locally compact $\sigma$-compact Hausdorff spaces, this property coincides precisely with having zero covering dimension. To prove the desired equivalence, we need some preliminary notions that can be found in [5].

Assume now that $X$ is a completely regular space. We will consider two notions for dimension of $X$: the usual covering dimension, denoted by $\dim X$ (see [4, 1.6.7]) and the one considered in [5, Chapter 16], which we will denote by $\dim_1 X$. The definition of $\dim_1 X$ is the same as $\dim X$, except that only covers consisting of cozero sets are considered. It is shown in [5, Corollary 16.9] that $\dim X = \dim_1 X$ if $X$ is a normal space. If $\dim_1 X = 0$, then $X$ is called **strongly zero-dimensional** (see [12, 3.34]), while if $\dim X = 0$ we call $X$ **zero-dimensional**. As noted in [12, 3.39], not all zero-dimensional spaces are strongly zero-dimensional, so that the concepts of $\dim X$ and $\dim_1 X$ are not identical in general, but they coalesce for Lindelöf spaces, by [12, 3.35].

**Lemma 2.2** *Let $X$ be a completely regular space. Then $\dim_1 X = 0$ if and only if whenever $f, g \in C(X)$ satisfy $X = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$, then there exist clopen subsets $U \subseteq \mathrm{Coz}(f)$ and $V \subseteq \mathrm{Coz}(g)$ of $X$ such that $U \cap V = \emptyset$ and $U \cup V = X$.*

**Proof.** It is a rephrasing of the equivalence $(a) \Leftrightarrow (b)$ in [5, Theorem 16.17]. □

**Remark 2.3** If $X$ is a locally compact Hausdorff space, then the $\sigma$-compact open subsets of $X$ are precisely the cozero sets of functions in $C_0(X)_+$. In fact, this follows from an easy application of Urysohn's Lemma (for a proof, see [7]).

**Proposition 2.4** *Let $X$ be a locally compact Hausdorff space. If $X$ is strongly zero-dimensional, then $X$ satisfies the $\sigma$-compact open reduction property. If $X$ is $\sigma$-compact, then the converse is also true.*

**Proof.** First, we show that if $Y \subseteq X$ is a $\sigma$-compact open subset of $X$, then $Y$ is Lindelöf.

Take an open cover $\{U_\alpha\}_\alpha$ of $Y$, and write $Y = \cup_n K_n$, where $K_n$ are compact subsets of $X$. Each $K_n$ is covered by finitely many elements of $\{U_\alpha\}_\alpha$, say $K_n \subseteq \cup_{i=1}^{s(n)} U_{\alpha_{i,n}}$ for some $s(n)$, whence $Y \subseteq \cup_n \cup_i U_{\alpha_{i,n}}$.

Now assume that $X$ is strongly zero-dimensional. Let $Y \subseteq X$ be a $\sigma$-compact open subset of $X$. Clearly, $Y$ has a basis of clopen sets. Therefore, since $Y$ is Lindelöf it follows after the implication $(c) \Rightarrow (a)$ of [5, Theorem 16.17] that $Y$ is strongly zero-dimensional.

Let $U$ and $V$ be open and $\sigma$-compact subsets of $X$. By Remark 2.3, there exist $f, g \in C_0(X)_+$ such that $U = \mathrm{Coz}(f)$ and $V = \mathrm{Coz}(g)$. Let $Y = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$. Then $Y$ is open and $\sigma$-compact, so by the previous observation $Y$ is strongly zero-dimensional. Notice that $f_{|U}$ and $g_{|U}$ belong to $C(U)$ (because $\mathrm{Coz}_X(f), \mathrm{Coz}_X(g) \subseteq U$). Then, there exist relatively clopen sets $U' \subseteq \mathrm{Coz}(f)$ and $V' \subseteq \mathrm{Coz}(g)$ of $Y$ such that $U' \cap V' = \emptyset$ and $U' \cup V' = Y$. Now, if $Y = \cup_n K_n$, for some compact sets $K_n$, then

$U' = U' \cap Y = \cup_n(K_n \cap U')$. Notice that $U'$ is a relatively closed subset of $Y$, hence $U' = Y \cap T$, for a closed set $T$ of $X$. Then $K_n \cap U' = K_n \cap Y \cap T = K_n \cap T$, because $K_n \subseteq Y$. Observe that $K_n \cap T$ is closed in $K_n$, a compact space, so $K_n \cap T$ is compact. Thus $K_n \cap U'$ is compact for all $n$, whence $U'$ is $\sigma$-compact open. Similarly, $V'$ is $\sigma$-compact open. Therefore $X$ satisfies the $\sigma$-compact open reduction property. For the converse, assume that $X$ is $\sigma$-compact and that satisfies the $\sigma$-compact open reduction property. Then $X$ is Lindelöf and hence completely regular, by [5, 3.15].

Suppose that there exist $f, g \in C(X)$ such that $X = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$. Then $\mathrm{Coz}(f)$ and $\mathrm{Coz}(g)$ are open and $\sigma$-compact subsets of $X$. Therefore, there exist open and $\sigma$-compact sets $U' \subseteq \mathrm{Coz}(f)$ and $V' \subseteq \mathrm{Coz}(g)$ such that $U' \cap V' = \emptyset$ and $U' \cup V' = \mathrm{Coz}(f) \cup \mathrm{Coz}(g) = X$. In particular, $U'$ and $V'$ are also clopen sets, so that $X$ is strongly zero-dimensional by Lemma 2.2. □

**Corollary 2.5** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space. Then* $\dim X = 0$ *if and only if $X$ satisfies the $\sigma$-compact open reduction property.*

**Proof.** Since $X$ is $\sigma$-compact, it is Lindelöf and therefore normal (see, e.g. [5, Exercise 3D]). By [5, Corollary 16.9], we have that $\dim X = \dim_1 X$. Thus the equivalence follows from Proposition 2.4. □

# 3 The structure of $S(C_0(X))$

In this section we analyze the structure of the monoid $S(C_0(X))$ for a locally compact $\sigma$-compact Hausdorff space $X$. Since the Riesz decomposition property is the key that leads to a representation of $S(C_0(X))$, we first establish for which spaces this decomposition holds. It is possible then to diagonalize matrices over $C_0(X)$ in a unique way with respect to the equivalence relation $\sim$. By using techniques from [9] and [13], we finally give an explicit order-isomorphism between $S(C_0(X))$ and a monoid of lower semicontinuous functions over $X$.

Let $X$ be a compact space. From [10, Proposition 1.7] and [2, Proposition 1.1] we have that $sr(C(X)) = [\dim X/2] + 1$ and that $RR(C(X)) = \dim X$. In case the space is locally compact, then by definition $sr(C_0(X)) = sr(\widetilde{C_0(X)}) = sr(C(\alpha X))$ and $RR(C_0(X)) = RR(C(\alpha X))$, where $\alpha X$ is the one-point compactification of $X$. It follows that if $X$ is a zero-dimensional $\sigma$-compact Hausdorff space, then the real rank of $C_0(X)$ is zero and the stable rank is one.

When dealing with arbitrary matrices over a ring of continuous functions, it is often useful to evaluate at some point of the space where the functions are defined. It is easy to see that the usual matrix rank is an invariant for the order-relation $\precsim$ (that is, if $\langle a \rangle \leq \langle b \rangle$ in $S(C_0(X))$, then we have $rank(a(x)) \leq rank(b(x))$ for all $x \in X$).

**Theorem 3.1** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space. Then the following conditions are equivalent:*

1) $\dim X = 0$*;*

2) $RR(C_0(X)) = 0$;

3) $S(C_0(X))$ *is a Riesz monoid.*

**Proof.** 1) $\Leftrightarrow$ 2) is clear.

2) $\Rightarrow$ 3). Let $A = C_0(X)$. If $RR(A) = 0$, then $\dim X = 0$. Hence $sr(A) = 1$ and so $S(A)$ is a Riesz monoid by [9, Theorem 2.13].

3) $\Rightarrow$ 1). By Corollary 2.5, it suffices to show that $X$ satisfies the $\sigma$-compact open reduction property. Let $A = C_0(X)$ and let $f, g \in A$. Since $\mathrm{Coz}(f) = \mathrm{Coz}(|f|)$ and $\mathrm{Coz}(g) = \mathrm{Coz}(|g|)$, we can assume with no loss of generality that $f, g \geq 0$.

Notice that $\mathrm{Coz}(f+g) = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$. As $X$ is $\sigma$-compact, the $C^*$-algebra $C_0(X)$ is $\sigma$-unital, hence there exists a sequence $\{f_n\}$ of functions that form an approximate unit. Now observe that:

$$\begin{pmatrix} f_n & f_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} f_n & 0 \\ f_n & 0 \end{pmatrix} = \begin{pmatrix} f_n(f+g)f_n & 0 \\ 0 & 0 \end{pmatrix},$$

and this tends to $\begin{pmatrix} f+g & 0 \\ 0 & 0 \end{pmatrix}$ as $n \to \infty$. Therefore $f + g \precsim f \oplus g$. By the Riesz decomposition property, there exist $F, G \in M_n(A)$ for some $n$ such that $f + g \sim F \oplus G$ while $F \precsim f$ and $G \precsim g$. In particular, if $F = (f_{ij})$ for some $f_{ij} \in A$, then for each $k \in \mathbb{N}$ there exist functions $a_1^k, \ldots, a_n^k \in A$ such that:

$$\begin{pmatrix} a_1^k & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^k & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{a_1^k} & \ldots & \overline{a_n^k} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \to F,$$

hence $f_{ij} = \lim\limits_{k\to\infty} f a_i^k \overline{a_j^k}$ and so $\mathrm{Coz}(f_{ij}) \subseteq \mathrm{Coz}(f)$ for all $i$ and $j$.

Similarly, if $G = (g_{ij})$, then $\mathrm{Coz}(g_{ij}) \subseteq \mathrm{Coz}(g)$ for all $i$ and $j$. Further, we also have $\mathrm{Coz}(\sum\limits_{i,j} |f_{ij}|) = \cup_{i,j} \mathrm{Coz}|f_{i,j}| \subseteq \mathrm{Coz}(f)$, and $\mathrm{Coz}(\sum\limits_{k,l} |g_{k,l}|) \subseteq \mathrm{Coz}(g)$, because $F \precsim f$ and $G \precsim g$.

Now, using the fact that the matrix rank is an invariant for the relation $\precsim$, the condition $f + g \sim F \oplus G$ implies that

$$(\sum_{i,j} |f_{i,j}|)(\sum_{k,l} |g_{k,l}|) = 0,$$

as well as $\mathrm{Coz}(\sum\limits_{i,j} |f_{i,j}|) \sqcup \mathrm{Coz}(\sum\limits_{k,l} |g_{k,l}|) = \mathrm{Coz}(f+g) = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$. Hence, letting $U = \mathrm{Coz}(\sum\limits_{i,j} |f_{ij}|)$ and $V = \mathrm{Coz}(\sum\limits_{k,l} |g_{kl}|)$, we get that $U \sqcup V = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$ and that $U \subseteq \mathrm{Coz}(f)$ and $V \subseteq \mathrm{Coz}(g)$, whence $X$ satisfies the $\sigma$-compact open reduction property. $\square$

**Corollary 3.2** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space. If $S(C_0(X))$ is a Riesz monoid, then it satisfies the refinement and interpolation properties.*

**Proof.** Let $A = C_0(X)$. If $S(A)$ is a Riesz monoid, then $RR(A) = 0$ by Theorem 3.1, and so $sr(A) = 1$. Apply then [9, Theorem 2.13]. □

We will see now that it is possible to describe more precisely the structure of $S(C_0(X))$, diagonalizing matrices in a special way. First we state a general fact concerning $\sigma$-unital $C^*$-algebras with real rank zero and stable rank one. It follows from [9, Lemma 2.2] that if $A$ is a $\sigma$-unital $C^*$-algebra and $\{e_n\}$ is an increasing approximate unit, then the element $e = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \in A$ satisfies that the class $u := \langle e \rangle$ is an order-unit for $S(A)$.

**Lemma 3.3** *Let $A$ be a $\sigma$-unital $C^*$-algebra with real rank zero, stable rank one, and let $u = \langle e \rangle$ (for some $e \in A$) be an order-unit for $S(A)$. If $a \in M_\infty(A)_+$ and $\langle a \rangle \leq u$, then there exists $a' \in A$ such that $a \sim a'$. In particular, every element in $M_\infty(A)_+$ is $\sim$-equivalent to a diagonal matrix over $A$.*

**Proof.** Let $a \in M_\infty(A)_+$ and assume that $\langle a \rangle \leq u$. Take an (increasing) approximate unit $\{p_n\}$ of projections for the hereditary algebra generated by $a$. Let $r = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n$. Then $a \sim r$ by [9, Lemma 2.2]. Observe that $e \lesssim 1$ (in $\widetilde{A}$). Now, since $p_1 \lesssim a \lesssim e \lesssim 1$, we have that $p_1 \sim q_1 \leq 1$ for some projection $q_1$ and, in fact, $q_1 \in A$. Moreover, since $p_2 \lesssim 1$, there exists a projection $r_2 \in M_\infty(A)$ such that $(p_2 - p_1) \oplus p_1 \oplus r_2 \sim 1$. By cancellation on projections, it follows from the relation $(p_2 - p_1) \oplus p_1 \oplus r_2 \sim q_1 \oplus (1 - q_1)$ that $(p_2 - p_1) \oplus r_2 \sim (1 - q_1)$, whence there exists a projection $q_2' \leq 1 - q_1$ such that $(p_2 - p_1) \sim q_2'$. Notice that $q_2' \in A$, and define $q_2 = q_1 \oplus q_2' \in A$. Observe that $p_2 = (p_2 - p_1) \oplus p_1 \sim q_2' \oplus q_1 = q_2$.

By induction on $n$, we obtain an increasing sequence $\{q_n\}$ of projections in $A$ such that $p_n \sim q_n$. Let $a' = \sum_{n=1}^{\infty} \frac{1}{2^n} q_n \in A$. By [9, Proposition 2.3] $r \sim a'$ and so $a \sim r \sim a'$.

Now let $a \in M_\infty(A)_+$. There exists $n \in \mathbb{N}$ such that $\langle a \rangle \leq nu$. Using the Riesz decomposition property ([9, Theorem 2.13]), there exist $a_1, \ldots, a_n \in M_\infty(A)_+$ such that $\langle a \rangle = \langle a_1 \rangle + \ldots + \langle a_n \rangle$ and $\langle a_i \rangle \leq u$ for all $i$. By the argument in the preceding paragraph, there exist $a_i' \in A_+$ such that $a_i \sim a_i'$, so $a \sim \mathrm{diag}(a_1', \ldots, a_n')$. □

**Proposition 3.4** *Let $X$ be a zero-dimensional locally compact $\sigma$-compact Hausdorff space. Given continuous functions $f_1, \ldots, f_n \in C_0(X)_+$, there exist $g_1, \ldots, g_n \in C_0(X)_+$ such that $\mathrm{Coz}(g_i) \subseteq \mathrm{Coz}(g_{i-1})$ for $i = 2, \ldots, n$ and $\mathrm{Coz}(g_1) = \cup_{i=1}^{n} \mathrm{Coz}(f_i)$, which satisfy:*

$$f_1 \oplus \ldots \oplus f_n \sim g_1 \oplus \ldots \oplus g_n.$$

**Proof.** We proceed by induction on $n$.

$n=2$. Suppose we have $f,g \in C_0(X)_+$. Because of the zero- dimensionality of $X$ and Corollary 2.5, we get $\mathrm{Coz}(f) \cup \mathrm{Coz}(g) = \mathrm{Coz}(f') \sqcup \mathrm{Coz}(g')$ for some $f', g' \in C_0(X)$ such that $\mathrm{Coz}(f') \subseteq \mathrm{Coz}(f)$ and $\mathrm{Coz}(g') \subseteq \mathrm{Coz}(g)$. Therefore $f \sim ff' \oplus fg'$ and $g \sim gf' \oplus gg'$, hence

$$f \oplus g \sim ff' \oplus fg' \oplus gf' \oplus gg' \sim f' \oplus fg' \oplus gf' \oplus g' \sim$$

$$\sim f' \oplus g' \oplus fg' \oplus gf'.$$

Let $g_1 = f' \oplus g'$ and $g_2 = fg' \oplus gf'$. Then $\mathrm{Coz}(g_2) \subseteq \mathrm{Coz}(g_1) = \mathrm{Coz}(f) \cup \mathrm{Coz}(g)$.

$n > 2$. Using the $\sigma$-compact open reduction property, write $\mathrm{Coz}(f_1) \cup \ldots \cup \mathrm{Coz}(f_n) = \mathrm{Coz}(f_1') \sqcup \ldots \sqcup \mathrm{Coz}(f_n')$ for some $f_i' \in C_0(X)_+$, with $\mathrm{Coz}(f_i') \subseteq \mathrm{Coz}(f_i)$. Then, for $i = 1, \ldots, n$ we have

$$f_i \sim f_i f_1' \oplus \ldots \oplus f_i f_i' \oplus \ldots \oplus f_i f_n' \sim f_i f_1' \oplus \ldots \oplus f_i' \oplus \ldots \oplus f_i f_n'.$$

Therefore:

$$f_1 \oplus \ldots \oplus f_n \sim$$

$$\sim (f_1' \oplus f_1 f_2' \oplus \ldots \oplus f_1 f_n') \oplus \ldots \oplus (f_n f_1' \oplus \ldots \oplus f_n f_{n-1}' \oplus f_n') \sim$$

$$\sim (f_1' \oplus \ldots \oplus f_n') \oplus (f_1(f_2' \oplus \ldots \oplus f_n') \oplus f_2 f_1') \oplus \ldots \oplus (f_{n-1} f_n' \oplus f_n(f_1' \oplus \ldots \oplus f_{n-1}')).$$

By induction hypothesis, there exist functions $G_2, \ldots, G_n \in C_0(X)_+$ with $\mathrm{Coz}(G_i) \subseteq \mathrm{Coz}(G_{i-1})$ for $i = 3, \ldots, n$ and

$$\mathrm{Coz}(G_2) = \mathrm{Coz}(f_1(f_2' \oplus \ldots \oplus f_n') \oplus f_2 f_1') \cup \ldots \cup Coz(f_{n-1} f_n' \oplus f_n(f_1' \oplus \ldots \oplus f_{n-1}')).$$

Note that $\mathrm{Coz}(G_2) \subseteq \mathrm{Coz}(f_1' \oplus \ldots \oplus f_n')$. Set $G_1 = f_1' \oplus \ldots \oplus f_n' \in C_0(X)_+$. Thus:

$$f_1 \oplus \ldots \oplus f_n \sim G_1 \oplus \ldots \oplus G_n. \ \square$$

**Remark 3.5** The situation for projections is simpler, as follows. Let $X$ be a locally compact Hausdorff space. Let $p_1, \ldots, p_n \in C_0(X)$ be projections. Then there exist projections $q_1, \ldots, q_n \in C_0(X)$ such that $\mathrm{Coz}(q_i) \subseteq \mathrm{Coz}(q_{i-1})$ for $i = 2, \ldots, n$, and $\mathrm{Coz}(q_1) = \cup_{i=1}^n \mathrm{Coz}(p_i)$, which satisfy:

$$p_1 \oplus \ldots \oplus p_n \sim q_1 \oplus \ldots \oplus q_n.$$

**Proof.** Note that a function $f \in C_0(X)$ is equivalent to a projection $p$ if and only if $\mathrm{Coz}(f)$ is a compact open subset of $X$. Now arguments similar to the ones used in Proposition 3.4 carry over. $\square$

The proof of the following Lemma is straightforward and we omit it.

**Lemma 3.6** *Let $X$ be a zero-dimensional locally compact $\sigma$-compact Hausdorff space. Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be elements in $C_0(X)$. Suppose that $\mathrm{Coz}(f_n) \subseteq \ldots \subseteq \mathrm{Coz}(f_1)$ and that $\mathrm{Coz}(g_n) \subseteq \ldots \subseteq \mathrm{Coz}(g_1)$. If*

$$f_1 \oplus \ldots \oplus f_n \precsim g_1 \oplus \ldots \oplus g_n,$$

*then $f_i \precsim g_i$ for all $i$.* $\square$

**Theorem 3.7** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space with* $\dim X = 0$. *Then for all $n \in \mathbb{N}$, every element in $M_n(C_0(X))_+$ is equivalent, with respect to $\sim$, to a diagonal matrix with entries in $C_0(X)_+$ that have decreasing cozero sets. Further, this expression is unique up to $\sim$-equivalence classes.*

**Proof.** Apply Lemma 3.3, Proposition 3.4 and Lemma 3.6. □

This special form of the elements in the commutative case can be used to obtain a nice representation of the monoid, which yields an interesting consequence, namely cancellation on $S(C_0(X))$.

**Proposition 3.8** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space with* $\dim X = 0$. *Then*

$$V(C_0(X)) \cong C_0(X, \mathbb{Z}^+)$$

*as ordered monoids.*

**Proof.** Let $u \in S(C_0(X))$ be an order-unit. Let $[p] \in V(C_0(X))$. Then there exists $n \in \mathbb{N}$ such that $\langle p \rangle \leq nu$. By Riesz decomposition on $S(C_0(X))$ and by [9, Proposition 3.12], there exist projections $p_1, \ldots, p_n \in C_0(X)$ such that $p \sim p_1 \oplus \ldots \oplus p_n$. By Remark 3.5, we get that

$$p_1 \oplus \ldots \oplus p_n \sim q_1 \oplus \ldots \oplus q_n,$$

for some projections $q_1, \ldots, q_n \in C_0(X)$ such that $q_i \precsim q_{i-1}$ for $i = 2, \ldots n$. Therefore, there exist compact open subsets $U_i$ of $X$ such that $q_i = \chi_{U_i}$ for each $i$. Define a map:

$$\varphi : V(C_0(X)) \to C_0(X, \mathbb{Z}^+), \quad [p] \mapsto \chi_{U_1} + \ldots + \chi_{U_n}.$$

By Remark 3.5 and Lemma 3.6, $\varphi$ is a well-defined map which is also an ordered monoid morphism. It is easy to see that $\varphi$ is an order-isomorphism onto its image.

Let $f \in C_0(X, \mathbb{Z}^+)$. Then $f$ is locally constant, and since it is assumed to vanish at infinity, it only takes a finite number of values, so we can write $f = \sum_{i=1}^{k} n_i \chi_{U_i}$, where $n_i$ are positive integers and $U_i$ are compact open sets. Thus $p_i := \chi_{U_i}$ is a projection in $C_0(X)$. Let $[p] = n_1[p_1] + \ldots + n_k[p_k] \in V(C_0(X))$. Then $\varphi([p]) = f$, and therefore $\varphi$ is also surjective, hence an ordered monoid isomorphism. □

Let $X$ be a locally compact space. Denote by $LSC_{\sigma,b}(X, \mathbb{Z}^+)$ the abelian monoid of bounded lower semicontinuous functions over $X$ that are countable (pointwise) suprema of functions from $C_0(X, \mathbb{Z}^+)$. This monoid has a natural ordered structure given by the pointwise ordering.

Let $M$ be a monoid. Let $\Lambda_\sigma(M)$ be the monoid (under addition) of countably generated intervals over $M$, (partially) ordered under set inclusion. Fix $D \in \Lambda_\sigma(M)$, and let $\Lambda_{\sigma,D}(M)$ be the hereditary submonoid of $\Lambda_\sigma(M)$ generated by $D$. Note that the elements of $\Lambda_{\sigma,D}(M)$ are countably generated intervals over $M$ bounded by $nD$ for some $n$.

For a $C^*$-algebra $A$, denote $D(A) = \{[p] \in V(A) \mid p \text{ is a projection from } A\}$. In case the algebra is commutative, $\sigma$-unital and has real rank zero, that is, of the form $C_0(X)$ for some zero-dimensional locally compact $\sigma$-compact Hausdorff space $X$, then by Proposition 3.8 we can write

$$D = D(C_0(X)) = \{\chi_U \mid U \text{ is a compact open subset of } X\}.$$

The next result is similar to [13, Lemma 5.6], and for completeness we include a proof.

**Proposition 3.9** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space with* $\dim X = 0$*. Define the following maps:*

$$\Phi : \Lambda_{\sigma,D}(C_0(X, \mathbb{Z}^+)) \to LSC_{\sigma,b}(X, \mathbb{Z}^+), \quad I \mapsto \sup_{g \in I} g$$

$$\Psi : LSC_{\sigma,b}(X, \mathbb{Z}^+) \to \Lambda_{\sigma,D}(C_0(X, \mathbb{Z}^+)), \quad f \mapsto \{g \in C_0(X, \mathbb{Z}^+) \mid g \leq f\}.$$

*Then $\Phi$ and $\Psi$ are (order-preserving) mutually inverse maps and $\Phi$ is an ordered-monoid isomorphism.*

**Proof.** Let $I$ be a countably generated interval, bounded above by $nD$ for some $n$. Let $\{h_i\}$ be a countable cofinal sequence for $I$. Define $f = \sup_{h \in I} h$. Then $f = \sup_i h_i$ and $f \leq n$, so that $f \in LSC_{\sigma,b}(X, \mathbb{Z}^+)$. Note that $\{g \in C_0(X, \mathbb{Z}^+) \mid g \leq f\} \supseteq I$. If $g \leq f$ and $g \in C_0(X, \mathbb{Z}^+)$, then $g = \sum_{i=1}^{k} n_i \chi_{U_i}$ for some positive integers $n_i$ and compact open subsets $U_i$ of $X$; hence $U := \mathrm{Supp}(g) = \cup U_i$ is compact and open. Define $\mathcal{U}$ by

$$\mathcal{U} = \{V \subseteq U \mid V \text{ is open and there exists } b \in I \text{ such that } g_{|V} \leq b_{|V}\},$$

and notice that is an open covering for $U$: for all $x \in U$, there exists $b \in I$ such that $g(x) \leq b(x)$ (because both $g$ and $f$ have discrete ranges). Since both $g$ and $b$ are continuous and $\mathbb{Z}$-valued, they are locally constant and thus there exists an open neighbourhood $V'$ (in $X$) of $x$ such that $g(y) = g(x)$ and $b(y) = b(x)$ for all $y \in V'$. Thus $V = V' \cap U$ is an open neighbourhood of $x$ in $U$ such that $g(y) = g(x)$ and $b(y) = b(x)$ for all $y \in V$, so that $V \in \mathcal{U}$.

By compactness $U = \cup_{i=1}^m V_i$ for some $V_i \in \mathcal{U}$, and so there exist $b_i \in I$ such that $g_{|V_i} \leq b_{i|V_i}$. Let $b \in I$ such that $b \geq b_i$. Then $b \geq g$ and thus $g \in I$. This shows that $\{g \in C_0(X, \mathbb{Z}^+) \mid g \leq f\} = I$, and therefore $(\Psi \circ \Phi)(I) = I$.

Let $f \in LSC_{\sigma,b}(X, \mathbb{Z}^+)$, and let $x \in X$. Since $f$ is lower semicontinuous, the set $U = \{y \in X \mid f(y) \geq f(x)\} = \{y \in X \mid f(y) > f(x) - 1\}$ is an open subset of $X$ which contains $x$. Since $X$ is locally compact, there exists a compact neighbourhood $V'$ of $x$ contained in $U$, and as $\dim X = 0$, we can find a clopen neighbourhood $V$ of $x$ such that $V \subseteq V' \subseteq U$ (note that in particular $V$ is compact). Thus

$$g = f(x).\chi_V \in \{h \in C_0(X, \mathbb{Z}^+) \mid h \leq f\},$$

and so $f(x) = g(x) \leq (\Phi \circ \Psi)(f)(x)$. Thus $f \leq (\Phi \circ \Psi)(f)$, and since the other inequality is obvious, we get $(\Phi \circ \Psi)(f) = f$.

We have to check now that if $f \in LSC_{\sigma,b}(X, \mathbb{Z}^+)$, then

$$I = \{g \in C_0(X, \mathbb{Z}^+) \mid g \leq f\} \in \Lambda_{\sigma,D}(C_0(X, \mathbb{Z}^+)).$$

Clearly, $I$ is nonempty and order-hereditary. Since the pointwise supremum of two $\mathbb{Z}$-valued continuous functions vanishing at infinity is also continuous and vanishes at infinity, it follows that $I$ is upward directed. If $f \leq n$ for some $n$ and if $g \in I$ then, taking $U = \mathrm{Supp}(g)$, we have that $U$ is compact open in $X$ and $g \leq n\chi_U$, whence $I \subseteq nD$. Write $f = \sup_n g_n$, with $g_n \in C_0(X, \mathbb{Z}^+)$. Since $I$ is an interval and $f = \sup_{g \in I} g$, we may assume that $g_n$ is an increasing sequence. Take $g \in I$, let $U = \mathrm{Supp}(g)$, and define

$$\mathcal{U}' = \{V \subseteq U \mid V \text{ is open and there exists } n \in \mathbb{N} \text{ such that } g_{|V} \leq g_{n|V}\}.$$

Arguing as above, $\mathcal{U}'$ is an open covering for $U$. As $U$ is compact, we obtain that $g \leq g_k$ for some $k$, whence $I$ is countably generated by $\{g_n\}$. Finally, it is easy to check that both $\Phi$ and $\Psi$ are order-preserving. □

**Definition 3.10** *Let $(M, \leq)$ be a preordered monoid. We say that $M$ is* **order-cancellative** *if whenever $x, y, z \in M$ satisfy $x + z \leq y + z$, then $x \leq y$.*

**Theorem 3.11** *Let $X$ be a locally compact $\sigma$-compact Hausdorff space with* $\dim X = 0$. *Then $S(C_0(X))$ is order-isomorphic to $LSC_{\sigma,b}(X, \mathbb{Z}^+)$. In particular, $S(C_0(X))$ is order-cancellative.*

**Proof.** By [9, Theorem 2.13], $S(C_0(X))$ is order-isomorphic to $\Lambda_{\sigma,D}(V(C_0(X)))$, where $D = D(C_0(X))$. Now apply Propositions 3.8 and 3.9. □

# Acknowledgments

I am grateful to Pere Ara and Ken Goodearl for their useful suggestions.

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