MULTIPLIERS OF VON NEUMANN REGULAR RINGS

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Abstract. We analyse the structure of the multiplier ring $\mathcal{M}(R)$ of a (nonunital) Von Neumann regular ring R. We show that $\mathcal{M}(R)$ is not regular in general, but every principal right ideal is generated by two idempotents. This, together with Riesz Decomposition on idempotents of $\mathcal{M}(R)$, furnishes a description of the monoid $V(\mathcal{M}(R))$ of Murray-Von Neumann equivalence classes of idempotents which is used to examine efficiently the lattice of ideals of $\mathcal{M}(R)$. The techniques developed here will allow as well other applications to the category of projective modules over regular rings.

Introduction

The ring of multipliers of an associative algebra R was introduced by Hochschild in [17] as a helpful tool for studying, given two algebras A and C, the possible extensions of A by C. After Busby's work ([10]) on the extension theory for C^*-algebras, the multiplier algebra $\mathcal{M}(A)$ of a C^*-algebra A has been an object intensively studied and used in many circumstances: see, e.g. [18], [2], [11], [9], [27], [33], [22]. In the general setting of semiprime rings, the multiplier ring $\mathcal{M}(R)$ can be described as the (unique) solution to the universal problem of adjoining a unit to a (nonunital) ring R, as follows: There exist a unital ring $\mathcal{M}(R)$ and an injective ring homomorphism $\varphi : R \to \mathcal{M}(R)$ such that (1) $\varphi(R) \lhd \mathcal{M}(R)$, and (2) If S is a ring and $\varphi_1 : R \to S$ is an injective ring homomorphism such that $\varphi_1(R) \lhd S$, then there exists a unique homomorphism $\overline{\varphi} : S \to \mathcal{M}(R)$ such that $\overline{\varphi}\varphi_1 = \varphi$. Moreover, if S is unital then $\overline{\varphi}(1_S) = 1_{\mathcal{M}(R)}$, and $\overline{\varphi}$ is injective if and only if $\varphi_1(R)$ is an essential ideal of S (see also [4]).

An important technical feature of multiplier rings of semiprime rings is that they can be equipped with a topology under which they are complete topological rings. (For C^*-algebras, this was already done in [10].) This topology allows to define a so-called approximate unit for R as a net that converges to the unit of $\mathcal{M}(R)$, so that rings with local units are particular cases of rings having approximate units. Thus, an important part of Section 1 is devoted to defining this topology and establishing its basic properties. This tool can be used successfully to show that $\mathcal{M}(R)$ is not regular whenever R is a nonunital prime ring with a countable unit (hence, in particular, there are nonunital regular rings R such that $\mathcal{M}(R)$ is not regular).

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The description of the ideal theory of the multiplier algebras for C^*-algebras has been a common objective pursued by several authors (to cite a few examples, see [20], [21], [31], [32], [25]), with special emphasis on the class of C^*-algebras with real rank zero, which can be algebraically described as those C^*-algebras that are exchange rings ([6, Theorem 7.2]). The abundance of idempotents in these algebras suggests that to achieve a good description of multiplier rings of semiprime rings, the efforts should be directed first to the class of regular rings. Also, many of the methods introduced in the C^*-algebraic case make an intensive use of purely analytic tools, hence one of the key points of Section 2 is to provide an algebraic framework that allows to derive results that apply to a wide class of regular rings. More precisely, we concentrate on the non-stable K-theory of $\mathcal{M}(R)$, that is, $V(\mathcal{M}(R))$, whose order-ideal lattice is isomorphic to the ideal lattice of $\mathcal{M}(R)$ (see also [32] and [25]). For proving this, Riesz decomposition in $V(\mathcal{M}(R))$ is established (parallel to [32, Theorem 1.1]), as well as the important fact that each finitely generated right ideal of $\mathcal{M}(R)$ is generated by idempotents, even though $\mathcal{M}(R)$ itself need not be regular. This latter result will also use some of the technical facts proved in Section 1. Next, we give a representation of $V(\mathcal{M}(R))$ that allows to systematize the study of the ideal lattice of $\mathcal{M}(R)$ (Theorem 2.11), and that involves both $V(R)$ and a semigroup of affine and lower semicontinuous functions defined on its state space. This representation is directed to the class of simple, σ-unital regular rings R with stable rank one and such that $V(R)$ is strictly unperforated. It is remarkable that no examples are known of simple regular rings whose $V(R)$'s are not strictly unperforated. Also, this result will be used in a subsequent paper ([5]), in order to study the category of non-finitely generated projective modules over a unital regular ring satisfying various comparability conditions.

For a ring R lying in the abovementioned class, the analysis of the ideal lattice of $\mathcal{M}(R)$ is taken up in Section 3, in analogy with [25]. We benefit considerably from the monoid-theoretical approach carried out in the previous section and hence some of the proofs use monoid techniques. Thus, we characterize those regular rings R for which $\mathcal{M}(R)/R$ is simple: this occurs exactly when R is elementary or has continuous scale. If the scale of R is not finite, then a rich ideal structure in $\mathcal{M}(R)/R$ opens up. For example, if R has n infinite extremal pseudo-rank functions, then there is a quotient of $\mathcal{M}(R)/R$ that has exactly 2^n ideals, modulo each of which a purely infinite ring results.

We now fix some notations. Given any ring R, we denote by $M_\infty(R) = \varinjlim M_n(R)$, under the maps $M_n(R) \to M_{n+1}(R)$ defined by $x \mapsto \operatorname{diag}(x, 0)$. Notice that $M_\infty(R)$ can also be described as the ring of countably infinite matrices over R with only finitely many nonzero entries, and it is sometimes denoted by $FM(R)$. If $e, f \in M_\infty(R)$ are idempotents, we write $e \leq f$ provided that $e = ef = fe$. Also, we say that e and f are **equivalent**, and we write $e \sim f$, if there exist elements $x, y \in M_\infty(R)$ such that $e = xy$ and $yx = f$. Finally, we write $e \lesssim f$ (respectively, $e \prec f$) provided that there exists an idempotent $e' \in M_\infty(R)$ such that $e \sim e' \leq f$ (respectively, $e \sim e' < f$). We then define $V(R) = \{[e] \mid e \in M_\infty(R)\}$, where $[e]$ denotes the $\sim$-equivalence class of e. Note that

$V(R)$ is naturally an abelian monoid, with operation defined by $[e]+[f]=[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}]$. In case R is unital $V(R)$ can also be described as the abelian monoid of isomorphism classes of finitely generated projective (right) R-modules, and its universal enveloping group is then $K_0(R)$. If $x, y \in V(R)$, we write $x \leq y$ if and only if $x+z=y$, for some $z \in V(R)$. This defines a pre-ordering on $V(R)$ (sometimes referred as to the **algebraic preordering**). As usual, we write $x < y$ if $x \leq y$ and $x \neq y$. If M is an algebraically ordered abelian monoid and $S \subseteq M$ is a submonoid, we say that S is an **order-ideal** provided that whenever $x \leq y$ and $y \in S$, then $x \in S$. Given a monoid M and an order-ideal S, we define an equivalence relation in M by setting $x \sim y$ if and only if there exist $z, w \in S$ such that $x+z=y+w$. We denote by M/S the quotient of M modulo this equivalence relation, and by $\overline{x}$ the equivalence class of $x \in M$. Then M/S is an abelian monoid, with an operation defined by $\overline{x}+\overline{y}=\overline{x+y}$.

A monoid M is said to be **conical** provided that the set M^* of nonzero elements in M is closed under addition. Finally, we say that M is a **Riesz monoid** if the following Riesz decomposition property holds: whenever $x, y_1, y_2 \in M$ satisfy $x \leq y_1+y_2$, then there exist $x_1, x_2 \in M$ such that $x=x_1+x_2$ and $x_i \leq y_i$ for $i=1,2$. It is a well-known fact that if R is either a von Neumann regular ring or a C^*-algebra with real rank zero, then $V(R)$ is a conical Riesz monoid (see [12, Theorem 2.8], [32, Theorem 1.1]).

1. The strict topology

Although most of the results in this section hold with more generality, we will restrict to the class of semiprime rings in order to guarantee enough 'nondegeneracy'.

Let R be a semiprime ring, possibly without unit. A **double centralizer** on R is a pair (f, g), where $f : R \to R$ is a right module homomorphism, $g : R \to R$ is a left module homomorphism, and they satisfy the balanced condition $xf(y) = g(x)y$, for all $x, y \in R$. We denote by $\mathcal{M}(R)$ the set of all double centralizers on R. Note that $\mathcal{M}(R)$ is a unital ring, with componentwise sum, and with a product defined by the rule: $(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_2 g_1)$. The unit of $\mathcal{M}(R)$ is $1 = (\mathrm{id}, \mathrm{id})$, where id is the identity homomorphism from R to R. We call the ring $\mathcal{M}(R)$ the **multiplier ring** of R.

There is an injective ring homomorphism $\varphi : R \to \mathcal{M}(R)$, given by $\varphi(x) = (f_x, g_x)$, where $f_x(y) = xy$ and $g_x(y) = yx$, for $x, y \in R$. The image of R under φ is a two-sided essential ideal of $\mathcal{M}(R)$. As mentioned in the Introduction, this construction solves the universal problem of adjoining a unit to R. In what follows, we will identify R with its image $\varphi(R)$ in $\mathcal{M}(R)$ without further comment.

We now present a classical example (see [8], and also [16]). Recall that if D is a division ring and ${}_DV$, W_D are vector spaces over D, then the pair (V, W) is called a **pair of dual spaces** provided there exists a D-valued nondegenerate bilinear form $\langle , \rangle : V \times W \to D$. A D-linear map $f : V \to V$ is called **adjointable** if there exists a D-linear map $f^* : W \to W$ such that $\langle (v)f, w \rangle = \langle v, f^*(w) \rangle$, for all $v \in V$ and $w \in W$.

Denote by $\mathfrak{L}_W(V)$ the set of all adjointable endomorphisms of V, and by $R := \mathfrak{F}_W(V)$ the subset of $\mathfrak{L}_W(V)$ consisting of the elements with finite rank. Then R is an ideal of $\mathfrak{L}_W(V)$, and by [8, Theorem 4.3.7(vi), Theorem 4.3.8(iv)], we have that $R = \mathrm{Soc}(R)$ and it is simple. Then it follows from [4, Proposition 2] that $\mathcal{M}(R) \cong \mathfrak{L}_W(V)$. In case we have a field K, and if $V = K^{(\omega)}$, the K-vector space which is a direct sum of countably many copies of K, then (V, V) with the product $\langle (v_i)_{i\in\mathbb{N}}, (w_i)_{i\in\mathbb{N}}\rangle = \sum_{i=1}^{\infty} v_i w_i$ becomes a dual pair. Notice that $R := \mathfrak{F}_V(V) = M_\infty(K)$, and it follows from the previous considerations that $\mathcal{M}(R) \cong \mathbb{B}(K)$, the ring of countably infinite matrices over K such that each row and each column have finitely many nonzero entries. It is well known that $\mathbb{B}(K)$ is not a regular ring, even though R is a regular ring. A related example, which will be useful for [5], states that the multiplier ring of $M_\infty(R)$, where R is any unital semiprime ring, is exactly the ring of row- and column-finite countably infinite matrices over R, denoted by $FRCM(R)$. Although this is possibly well known, we give the short argument for completeness.

Proposition 1.1. *Let R be a unital semiprime ring. Then the multiplier ring of $M_\infty(R)$ is $FRCM(R)$.*

Proof. Denote by $FCM(R)$ (respectively $FRM(R)$) the ring of all column-finite (respectively, row-finite) countably infinite matrices over R. If $A \in FRCM(R)$, define $T_A \in \mathcal{M}(M_\infty(R))$ by $T_A = (L_A, R_A)$, where L_A (resp. R_A) is left (resp. right) multiplication by A. It is clear that this defines an embedding $FRCM(R) \subseteq \mathcal{M}(M_\infty(R))$. Now let $T \in \mathcal{M}(M_\infty(R))$, and denote by $\{e_{ij} \mid i, j \geq 1\}$ the usual system of matrix units for $M_\infty(R)$. Set $e_i := e_{ii}$. Since $Te_i = (Te_i)e_i$, the element $Te_i \in M_\infty(R)$ has all its columns, except possibly the i-th, equal to zero. Let A be the matrix in $FCM(R)$ which agrees in the i-th column with Te_i for all i. Similarly, let B be the matrix in $FRM(R)$ which agrees in the j-th row with e_jT for all j. By using the relation $e_j(Te_i) = (e_jT)e_i$, we get $A = B$, so $A \in FRCM(R)$. Now observe that $TM = AM$ and that $MT = MA$ for all $M \in M_\infty(R)$, whence it follows that $T = A$. $\square$

Our next task will be to introduce a topology on $\mathcal{M}(R)$ which, in analogy with the case of C^*-algebras, will be called the **strict topology** on $\mathcal{M}(R)$ induced by R. It will turn out that $\mathcal{M}(R)$ is a complete topological ring and that in cases of interest, $\mathcal{M}(R)$ is the completion of R under this topology. This topology will also become a useful tool for establishing structural properties of $\mathcal{M}(R)$.

Define a basis of open neighbourhoods of 0 in $\mathcal{M}(R)$ as follows: for any $a_1, \dots, a_n \in R$, set

$$(\dagger) \quad U(a_1, \dots, a_n) = \{x \in \mathcal{M}(R) \mid xa_i = a_ix = 0 \text{ for } i = 1, \dots, n\}.$$

If $x \in \mathcal{M}(R)$ is any other point, then the basis of open neighbourhoods of x is defined simply by taking $x + U$, where U is a set of the form $(\dagger)$. (Observe that this topology is the discrete topology in case the ring R is unital.) In terms of convergence, we say that a net $(x_\lambda)_{\lambda\in\Lambda}$ converges **strictly** to an element $x \in \mathcal{M}(R)$, if and only if for all

$a_1, \dots, a_n \in R$, there exists $\lambda_0 \in \Lambda$ such that $(x_\lambda - x)a_i = a_i(x_\lambda - x) = 0$ for $\lambda \geq \lambda_0$ and for $i = 1, \dots, n$. Equivalently, (x_λ) converges strictly to x if and only if for any $a \in R$, there exists λ_0 such that $(x_\lambda - x)a = a(x_\lambda - x) = 0$ for $\lambda \geq \lambda_0$.

Remark 1.2. The previous discussion can be done for semiprime rings R equipped with a metric $d : R \times R \to \mathbb{R}^+$. In this context, we define the strict topology on $\mathcal{M}(R)$ by giving a sub-basis of open neighbourhoods of 0 as follows: for $\epsilon > 0$ and $a_1, \dots, a_n \in R$, take $U(\epsilon; a_1, \dots, a_n) = \{x \in \mathcal{M}(R) \mid d(xa_i, 0), d(a_i x, 0) < \epsilon\}$. This generality applies both to regular rings (where $d(x, y) = 1 - \delta_{x,y}$), and to C^*-algebras (where $d(x, y) = \|x - y\|$). Since the more immediate applications will be directed to regular rings, we will not work in that level of generality.

It is clear that the sum is strictly continuous. For the product, we have the following:

Lemma 1.3. *Let R be a semiprime ring, and let $(x_\lambda)_\Lambda$, $(y_\mu)_{\mu\in\Gamma}$ be two nets in $\mathcal{M}(R)$ that converge, respectively, to x and y, in the strict topology induced by R. Then $(x_\lambda y_\mu)_{(\lambda,\mu)\in\Lambda\times\Gamma}$ converges strictly to xy.*

Proof. Let $a \in R$. Since x_λ converges to x, there exists λ_0 such that $a(x_\lambda - x) = 0$ and $(x_\lambda - x)ya = 0$ whenever $\lambda \geq \lambda_0$. Similarly, there exists μ_0 such that $ax(y_\mu - y) = 0$ and $(y_\mu - y)a = 0$ if $\mu \geq \mu_0$. Therefore, if $(\lambda, \mu) \geq (\lambda_0, \mu_0)$ we have that

$$x_\lambda y_\mu - xy = x_\lambda(y_\mu - y) + (x_\lambda - x)y,$$

hence

$$a(x_\lambda y_\mu - xy) = ax_\lambda(y_\mu - y) + a(x_\lambda - x)y = ax(y_\mu - y) = 0,$$

as well as

$$(x_\lambda y_\mu - xy)a = x_\lambda(y_\mu - y)a + (x_\lambda - x)ya = 0. \quad \square$$

Definition 1.4. *Let R be a semiprime ring. We say that R has an* **approximate unit** *if there exists a net $(a_i)_{i\in I}$ in R that converges to $1 \in \mathcal{M}(R)$ in the strict topology induced R. If R has an approximate unit consisting of idempotents, then we say that R has* **local units**.

Notice that our definition of a ring with local units is equivalent to other definitions already existing in the literature (see, for example, [3] and [1]).

If R is a semiprime ring with an approximate unit $(a_i)_{i\in I}$, then we say that $(a_i)_{i\in I}$ is **increasing** provided that, if $i < j$ for $i, j \in I$, then $a_i = a_i a_j = a_j a_i$.

Lemma 1.5. *Let R be a semiprime ring. Then R has an approximate unit if and only if R has an increasing approximate unit.*

Proof. The condition of sufficiency is clear. Suppose that $(a_i)_{i\in I}$ is an approximate unit for R. Let $\Lambda = \{a \in R \mid a = a_i \text{ for some } i \in I\}$. Define a partial order in Λ as follows: if $\lambda, \mu \in \Lambda$, set $\lambda \leq \mu$ if either $\lambda = \mu$ or if $\lambda = \lambda\mu = \mu\lambda$ in case $\lambda \neq \mu$. Note that Λ is an upward directed set. For, if $\lambda, \mu \in \Lambda$, then there exist $i, j \in I$ such that $\lambda = a_i$ and $\mu = a_j$. Since $a_l \to 1$ strictly, there exists $k \in I$ such that $a_i a_k = a_k a_i = a_i$

and $a_j a_k = a_k a_j = a_j$. Set $\nu = a_k$, and then it is clear that $\lambda, \mu \leq \nu$. Define $a_\lambda = \lambda$ if $\lambda \in \Lambda$. Then the net $(a_\lambda)_{\lambda\in\Lambda}$ is an increasing approximate unit for R. Indeed, it suffices to check that it converges strictly to 1. Let $x \in R$. There exists $i \in I$ such that $a_i x = x a_i = x$. Let $\lambda_0 = a_i \in \Lambda$. Then, if $\lambda \geq \lambda_0$ we have that $\lambda x = (\lambda\lambda_0)x = \lambda_0 x = x$, and analogously $x\lambda = x$. $\square$

If R has an approximate unit $(a_i)_{i\in I}$ with I countable, then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n a_{n+1} = a_n = a_{n+1}a_n$ for all $n \in \mathbb{N}$ and that converges strictly to 1 (Lemma 1.5). In this situation we say that R is **σ-unital** (see [19, Definition 1.2.1]) and that the approximate unit $(a_n)_{n\in\mathbb{N}}$ is a **σ-unit** for R.

Proposition 1.6. *Let R be a semiprime ring. Then $\mathcal{M}(R)$ is complete in the strict topology induced by R, that is, every strict Cauchy net in $\mathcal{M}(R)$ converges to an element of $\mathcal{M}(R)$. Moreover, R has an approximate unit if and only if $\mathcal{M}(R)$ is the completion of R in the strict topology.*

Proof. Let $(x_\lambda)_\lambda$ be a strict Cauchy net in $\mathcal{M}(R)$. Then, for every $a \in R$, there exists λ_0 such that $(x_\lambda - x_\mu)a = a(x_\lambda - x_\mu) = 0$ for all λ, $\mu \geq \lambda_0$. Hence $\lim\limits_{\lambda} x_\lambda a$ and $\lim\limits_{\lambda} a x_\lambda$ exist and belong to R. Define maps $f, g : R \to R$ by the rules $f(a) = \lim\limits_{\lambda} x_\lambda a$ and $g(a) = \lim\limits_{\lambda} a x_\lambda$. Given $a, b \in R$, there exists λ_0 such that $x_\lambda a = x_{\lambda_0} a$ and $b x_\lambda = b x_{\lambda_0}$ for all $\lambda \geq \lambda_0$. Then $bf(a) = b(x_{\lambda_0} a) = g(b)a$. Thus (f, g) is a double centralizer. Moreover, we have that $\lim\limits_{\lambda} x_\lambda = (f, g)$. To see this, take $a \in R$. Then, if λ is large enough, we have $x_\lambda a = (f, g)a$ and $a x_\lambda = a(f, g)$.

Assume now that R has an approximate unit $(a_i)_{i\in I}$. Then, if $x \in \mathcal{M}(R)$, we have that $a_i x \to 1 \cdot x = x$ in the strict topology, by Lemma 1.3, hence $\mathcal{M}(R)$ is the strict completion of R. The converse is obvious. $\square$

Let R be a semiprime ring, and let $\{p_i\}_{i\in I}$ be a set of orthogonal idempotents in $\mathcal{M}(R)$. If $J \subset I$ is finite, define $p_J = \sum\limits_{i\in J} p_i$. Then $(p_J)_J$ is a net consisting of idempotents, which is a Cauchy net if and only if, for all $a \in R$ there exists a finite subset J_0 of I such that $p_i a = a p_i = 0$ for all $i \in I \setminus J_0$. In this case, and since $\mathcal{M}(R)$ is strictly complete by Proposition 1.6, we write $\sum\limits_{i\in I} p_i \in \mathcal{M}(R)$ to denote the limit of $(p_J)_J$. By Lemma 1.3, we have that $\sum\limits_{i\in I} p_i$ is an idempotent of $\mathcal{M}(R)$. The following Lemma will be needed later.

Lemma 1.7. *Let R be a semiprime ring. Let $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ be two sets of orthogonal idempotents of $\mathcal{M}(R)$ such that $e := \sum\limits_{i\in I} e_i$ and $f := \sum\limits_{i\in I} f_i$ exist. For every $i \in I$, let $x_i \in \mathcal{M}(R)$ be an element satisfying $x_i \in e_i\mathcal{M}(R)f_i$. Then $x := \sum\limits_{i\in I} x_i$ exists and $x \in e\mathcal{M}(R)f$.*

Proof. Let $a \in R$. Then there exist two finite subsets J_1, J_2 of I such that $e_i a = ae_i = 0$ for all $i \in I \setminus J_1$, and $f_i a = af_i = 0$ for all $i \in I \setminus J_2$. Let $J_0 = J_1 \cup J_2 \subset I$, and notice that $x_i a = x_i f_i a = 0$ and $ax_i = ae_i x_i = 0$ for all $i \in I \setminus J_0$. Thus the net $(\sum_{i\in J} x_i)_{\{J\subset I; |J|<\infty\}}$ is Cauchy, hence strictly convergent. Let $x = \sum_{i\in I} x_i$. Then it is clear that $x \in e\mathcal{M}(R)f$. $\square$

Let R be a semiprime ring. We say that R has a **countable unit** provided that R is σ-unital and R has local units. Notice that this means exactly that $R = \cup_n p_n R p_n$, where (p_n) is an increasing sequence of idempotents in R. We will call such a sequence (p_n) a **countable unit for** R. Observe also that in case R is regular, then R has a countable unit if and only if R is σ-unital. This follows immediately from the fact that, for regular rings, the set of idempotents is an upward directed set (see [23, Lemma 1.1]). The following result extends to a wider class of rings the known fact that, for a field K, the ring $\mathbb{B}(K)$ is not a regular ring.

Proposition 1.8. *Let R be a nonunital prime ring with a countable unit. Then $\mathcal{M}(R)$ is not a regular ring.*

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be a countable unit for R. Write $e_n = p_n - p_{n-1}$, with $p_0 = 0$. Since R is nonunital, we may assume that $e_n \neq 0$ for all $n \in \mathbb{N}$. By an easy inductive argument (that uses the primeness of R), there exists a sequence (x_n) in R such that $x_n \in e_n R e_{n+1}$ and $x_1 x_2 \dots x_n \neq 0$ for all $n \geq 1$.

Let $x = \sum_{n=1}^{\infty} x_n \in \mathcal{M}(R)$, which exists by Lemma 1.7. We now claim that the element $1 + x$ is not von Neumann regular in $\mathcal{M}(R)$. Assume that there exists $y \in \mathcal{M}(R)$ such that

$$1 + x = (1 + x)y(1 + x). \tag{1}$$

Set $e_0 = x_0 = 0$. First, we will prove that $e_m y e_n = 0$ for all $m > n$ and that $e_n y e_n = e_n$ for all n. We proceed by induction on n. This is clear if $n = 0$. Suppose that there exists $k \geq 0$ such that $e_m y e_k = 0$ for all $m > k$ and $e_n y e_n = e_n$ for all $n \leq k$. Multiplying relation (1) on the right by e_{k+1} we get

$$(1 + x)y(e_{k+1} + x_k) = e_{k+1} + x_k. \tag{2}$$

Note that there exists $l > k + 1$ such that $ye_{k+1} \in p_l R$. By the induction hypothesis (using that $e_m y e_k = 0$ for all $m > k$), we also have that $yx_k \in p_k R$. Hence, multiplying relation (2) on the left by e_t with $k + 1 \leq t$, we get that $e_t y e_{k+1} = 0$ if $t \geq l$, that $e_t y e_{k+1} + x_t y e_{k+1} = 0$ if $k + 1 < t < l$, and that $e_{k+1} y e_{k+1} + x_{k+1} y e_{k+1} = e_{k+1}$ if $t = k + 1$. Using the first two relations, it turns out after a recurrent process (starting with $t = l - 1$) that $e_t y e_{k+1} = 0$ for all $k + 1 < t$. Thus $x_{k+1} y e_{k+1} = 0$ and therefore we have that $e_{k+1} y e_{k+1} = e_{k+1}$.

Now we will prove that for all $n \geq 1$ and all $0 \leq i < n$ we have

$$e_i y e_n = (-1)^{n-i} x_i x_{i+1} \dots x_{n-1}.$$

Again, we proceed by induction on n. The equality is clear if $n = 1$. Suppose that it is valid for some $n \geq 1$. Then, using that $e_m y e_n = 0$ if $m > n$ and the fact that $1 + x = \sum_{k=1}^{\infty}(e_k + x_k)$ we get

$$(1+x)yx_n = (1+x)(\sum_{i=1}^{n}(-1)^{n-i}x_i x_{i+1}\dots x_n)$$

$$= \sum_{i=1}^{n}(-1)^{n-i}x_i x_{i+1}\dots x_n + \sum_{i=1}^{n}(-1)^{n-i}x_{i-1}x_i x_{i+1}\dots x_n = x_n.$$

Using this and relation (2), we conclude that

$$e_{n+1} + x_n = (1+x)y(e_{n+1} + x_n) = (1+x)ye_{n+1} + x_n,$$

and therefore $e_{n+1} = (1+x)ye_{n+1}$. Multiplying this equation on the left by e_t with $1 \leq t \leq n$ we get that $0 = e_t y e_{n+1} + x_t(e_{t+1} y e_{n+1})$. Since $e_{n+1} y e_{n+1} = e_{n+1}$, the previous relations show (recurrently) that $e_t y e_{n+1} = (-1)^{n+1-t}x_t x_{t+1}\dots x_n$, and this completes the induction.

Finally, note that there exists $k > 1$ such that $e_1 y e_k = 0$ and thus we have $0 = e_1 y e_k = (-1)^{k-1}x_1 x_2 \dots x_{k-1} \neq 0$, which is a contradiction with our choice of the sequence (x_n). □

We remark that R being prime is an assumption needed in the previous result. For example, let $\{K_n\}$ be a sequence of fields, and consider the ring $R = \bigoplus_{n=1}^{\infty} K_n$, which is semiprime, nonunital and has a countable unit. Then $\mathcal{M}(R) = \prod_{n=1}^{\infty} K_n$, which is regular.

We close this section by showing two useful properties for multiplier rings of semiprime rings. Although these results are known for C^*-algebras, we could not locate references in the literature for semiprime rings, hence we provide proofs for the reader's convenience.

Lemma 1.9. *Let R be a semiprime ring with approximate unit, and let $p \in \mathcal{M}(R)$ be an idempotent. Then pRp is a semiprime ring and $\mathcal{M}(pRp) = p\mathcal{M}(R)p$.*

Proof. It is clear that pRp is a semiprime ring. We denote by $\mathcal{M}_p$ the set of all pairs (f, g) such that $f : pR \to pR$ and $g : Rp \to Rp$ are (respectively) right and left module maps, and $xf(y) = g(x)y$ for all $x \in Rp$ and $y \in pR$. Then it is easy to check that $\mathcal{M}_p$ is a unital ring under the natural sum and multiplication rules. Define a map $\Theta : \mathcal{M}_p \to \mathcal{M}(pRp)$ by $\Theta(f, g) = (f_{|pRp}, g_{|pRp})$. Clearly, Θ is a unital ring homomorphism. Assume that $\Theta(f, g) = 0$. Then $f_{|pRp} = g_{|pRp} = 0$. Using the balance condition on f and g, we get that:

$$pRf(pR) = pRpf(pR) = g(pRp)pR = 0,$$

whence $f(pR)^2 = f(pRf(pR)) = 0$. Thus $f(pR) = 0$, since R is semiprime, and therefore $f = 0$. Similarly $g = 0$. This shows that Θ is injective.

We claim that Θ is moreover surjective. Let $(f,g) \in \mathcal{M}(pRp)$. Then $f,g : pRp \to pRp$ are, respectively, right and left pRp-module maps, and $xf(y) = g(x)y$ for all $x,y \in pRp$. We want to extend these maps to a pair $(f_0, g_0) \in \mathcal{M}_p$.

Let $(a_i)_{i\in I}$ be an approximate unit of R. Then the net $(pa_ip)_i$ converges in the strict topology (induced by pRp) to $p \in \mathcal{M}(pRp)$. We claim that the nets $(f(pa_ip))_i$ and $(g(pa_ip))_i$ converge in the strict topology induced by pRp to elements $w,t \in \mathcal{M}(pRp)$, respectively. Indeed, it is enough to check that both nets are Cauchy nets and use the fact that $\mathcal{M}(pRp)$ is strictly complete (by Lemma 1.6). Let $x \in R$. Since $(pa_ip)_i$ is a Cauchy net, there exists $i_0 \in I$ such that $(pa_ip)(pxp) = (pa_jp)(pxp)$ and $g(pxp)(pa_ip) = g(pxp)(pa_jp)$ whenever $i,j \geq i_0$. Therefore, if $i,j \geq i_0$, we have that

$$f(pa_ip)(pxp) = f((pa_ip)(pxp)) = f(pa_jp)(pxp),$$

and also

$$(pxp)f(pa_ip) = g(pxp)(pa_ip) = g(pxp)(pa_jp) = (pxp)f(pa_jp).$$

It follows that $(f(pa_ip))_i$ is a Cauchy net. A similar argument shows that $(g(pa_ip))_i$ is a Cauchy net, establishing the claim.

If $x \in R$, define $f_0(px) = tpx$, and similarly set $g_0(xp) = xpw$. It is clear that both f_0 and g_0 are (respectively) right and left R-module maps. We have to check, finally, that $(f_0,g_0) \in \mathcal{M}_p$ and that $\Theta(f_0,g_0) = (f,g)$.

Note that $(f_0,g_0) \in \mathcal{M}_p$ if, and only if, $yptpx = ypwpx$ for any $x,y \in R$. If $j \in I$ and $z \in R$, we have that $g(pzp)pa_jp = pzpf(pa_jp)$, and computing strict limits it follows that $g(pzp) = pzpw$. Thus, if $i \in I$ and $x,y \in R$ we have that $ypg(pa_ip)px = ypa_ipwpx$, whence computing strict limits a second time it follows that $yptpx = ypwpx$. Thus $(f_0,g_0) \in \mathcal{M}_p$.

It is easy to check that $f_{0|pRp} = f$ and that $g_{0|pRp} = g$. Therefore, Θ is surjective and the claim is proved.

Now it is clear that $p\mathcal{M}(R)p \subseteq \mathcal{M}(pRp)$. Conversely, let $(f,g) \in \mathcal{M}(pRp)$ be a double centralizer. Let $(f_0,g_0) \in \mathcal{M}_p$ be the preimage of (f,g) under the map Θ, and define maps $f',g' : R \to R$ by $f'(x) = f_0(px)$ and $g'(x) = g_0(xp)$, for $x \in R$. Then it is clear that f' and g' are (respectively) right and left R-module maps, and in fact, if $x,y \in R$, then $xf'(y) = xf_0(py) = xpf_0(py) = g_0(xp)py = g'(x)y$. Therefore $(f',g') \in \mathcal{M}(R)$, and clearly $p(f',g')p = (f,g)$. □

That the hypotheses imposed on Lemma 1.9 are not superfluous follows from the next example:

Example 1.10. *There exist a prime ring R and an idempotent $p \in \mathcal{M}(R)$ such that $\mathcal{M}(pRp) \neq p\mathcal{M}(R)p$.*

Proof. Let

$$R = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} & 2^2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2^3\mathbb{Z} \\ 2^2\mathbb{Z} & 2^3\mathbb{Z} & 2^4\mathbb{Z} \end{pmatrix},$$

with the ring operations induced from $M_3(\mathbb{Z})$. Since for any $x \in M_3(\mathbb{Z})$, we have that $2^4x \in R$ and $M_3(\mathbb{Z})$ is a prime ring, it follows easily that R is also prime.

Let $p = e_{11} + e_{22} \in \mathcal{M}(R)$. Then

$$pRp = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} & 0 \\ 2\mathbb{Z} & 2\mathbb{Z} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $x \in \mathcal{M}(pRp)$ correspond to the multiplier (e_{21}, e_{21}), and suppose x agrees with multiplication by some $y \in p\mathcal{M}(R)p$. Then $y(2e_{11}) = x(2e_{11}) = 2e_{21}$, and therefore $y(2^3e_{13}) = y(2e_{11})(2^2e_{13}) = 2^3e_{23}$. But $y(2^3e_{13}) = 2[y(2^2e_{13})]$, so we would have $2^3e_{23} \in 2R$, a contradiction. $\square$

The previous example contrasts with the fact that if R is any semiprime ring and $p^2 = p \in Q := Q_s(R)$, then $pQp = Q_s(pQp \cap R)$ (see, for example, [8, Proposition 2.3.14]).

Lemma 1.11. *Let R be a semiprime ring. Then $M_n(\mathcal{M}(R)) = \mathcal{M}(M_n(R))$ for all $n \geq 1$.*

Proof. It is clear that $M_n(\mathcal{M}(R)) \subseteq \mathcal{M}(M_n(R))$. For $1 \leq i, j \leq n$, let $e_{ij} \in M_n(\mathcal{M}(R))$ be the matrix whose (i, j)-th entry is $1 \in \mathcal{M}(R)$, and all the other entries are zero. Denote $e_i = e_{ii}$.

Let $x \in \mathcal{M}(M_n(R))$. Note that for $1 \leq j \leq n$, the elements from $xe_jM_n(R)e_j$ are matrices (over R) that have (possibly nonzero) entries in the j-th column, and zeros elsewhere. Denote by $f_{ij} : R \to R$ the maps defined by the equation $f_{ij}(a)e_{ij} = e_ix(ae_j)$, where $a \in R$. Observe that f_{ij} are right R-module maps. In fact, they are clearly additive, whereas if $a, r \in R$, we have that $f_{ij}(ar)e_{ij} = e_ix(are_j) = e_i(xa)e_jre_j = f_{ij}(a)e_{ij}re_j = f_{ij}(a)re_{ij}$.

A similar computation, multiplying on the left by the element x, provides left R-module maps $g_{ij} : R \to R$, defined by $g_{ij}(a)e_{ij} = (e_ia)xe_j$, where $a \in R$.

It is easy to check that the pairs (f_{ij}, g_{ij}) are double centralizers and that $x = ((f_{ij}, g_{ij}))_{i,j}$. Thus $\mathcal{M}(M_n(R)) \subseteq M_n(\mathcal{M}(R))$, as desired. $\square$

In case the ring R has an approximate unit, the previous result can be derived from Proposition 1.6 by computing strict completions (this is, in fact, the argument used for C^*-algebras). Briefly, if $\overline{S}^\beta$ denotes the completion of a semiprime ring S in the strict topology, then we have that $\mathcal{M}(M_n(R)) = \overline{M_n(R)}^\beta \cong M_n(\overline{R}^\beta) = M_n(\mathcal{M}(R))$.

2. Multiplier rings of regular rings

The purpose of the current section is to establish the exact relation existing between the monoids $V(\mathcal{M}(R))$ and $V(R)$, for a σ-unital regular ring with stable rank one. We first relate $V(\mathcal{M}(R))$ to a certain monoid of intervals in $V(R)$, and then we shall specialize to the simple case, assuming also strict unperforation on $V(R)$. In this case the representation obtained for $V(\mathcal{M}(R))$ will allow to systematize the analysis of the ideal lattice of $\mathcal{M}(R)$, as we will see in the last section.

Two important ingredients in what follows will be the facts that $V(\mathcal{M}(R))$ is a Riesz monoid and that every principal right ideal of $\mathcal{M}(R)$ is generated by two idempotents, whenever R is σ-unital and regular.

Lemma 2.1. *Let R be a σ-unital regular ring, and let $p \in \mathcal{M}(R)$ be an idempotent. Then pRp is a regular ring and there exists an increasing sequence (f_n) of idempotents in pRp which is a σ-unit for pRp. Moreover, if $q \in R$ is an idempotent, then $q \lesssim p$ if and only if $q \lesssim f_n$ for some $n \in \mathbb{N}$.*

Proof. Let $x \in pRp$. Then $x \in R$, hence there exists $y \in R$ such that $x = xyx$. Now, since $x = pxp$ we have $x = (pxp)(pyp)(pxp)$, showing that pRp is regular.

Recall that by [23, Lemma 1.1], the set $E(S)$ of idempotents of a regular ring S is upward directed, and so $S = \cup_{e\in E(S)} eSe$.

Let (e_n) be a σ-unit for R, and write $R = \cup_{n=1}^{\infty} e_nRe_n$. Write also

$$pRp = \cup_{e\in E(pRp)} e(pRp)e.$$

Note that $pe_1p \in f_1(pRp)f_1$, for some idempotent $f_1 \in pRp$, and also $pe_2p \in f_2'(pRp)f_2'$ for some idempotent $f_2' \in pRp$. Using the upward directedness of the idempotents of pRp, we find an idempotent $f_2 \in pRp$ such that $f_1, f_2' \leq f_2$, whence $pe_2p \in f_2(pRp)f_2$. Continuing in this way, we obtain a sequence $f_1 \leq f_2 \leq f_3 \leq \dots$ of idempotents in pRp such that $pe_np \in f_n(pRp)f_n$ for all $n \in \mathbb{N}$. Clearly, $\cup_n f_nRf_n \subseteq pRp$. For the converse, if $x \in pRp$, then there exists $n \in \mathbb{N}$ such that $x = e_nxe_n$, and also $x = pxp$. Thus $x = (pe_np)x(pe_np) = f_n(pe_npxpe_np)f_n \in f_nRf_n$.

Note now that since $f_n \in pRp$ for all $n \in \mathbb{N}$, we have $f_n = f_np = pf_n$, hence $f_n \leq p$. Therefore, if $q \in R$ is an idempotent and $q \lesssim f_n$ for some $n \in \mathbb{N}$, then it is clear that $q \lesssim p$. Conversely, if $q \lesssim p$, then $q \sim q' \leq p$ for some idempotent $q' \in pRp$. Thus there exists $n \in \mathbb{N}$ such that $q' \in f_nRf_n$, whence $q' \leq f_n$ and so $q \lesssim f_n$. □

Remark 2.2. *Let R be a regular ring, and let $p \in \mathcal{M}(R)$ be an idempotent. Then the strict topologies on $\mathcal{M}(pRp)$ induced by pRp and by R coincide.*

Proof. First note that $\mathcal{M}(pRp) = p\mathcal{M}(R)p$ by Lemma 1.9. Suppose that (x_i) is a net in $p\mathcal{M}(R)p$ that converges strictly to $x \in p\mathcal{M}(R)p$ in the topology induced by pRp. Let $a \in R$. Using the regularity, we have that $pa = (pa)b(pa)$, for some $b \in R$. Hence $x_ia = x_ipa = x_i(pabp)a = x(pabp)a = xa$, if i is large enough. The argument works similarly for ax_i. □

The proof of the following lemma is straightforward, and we omit it.

Lemma 2.3. *Let R be a regular ring, and let $p, q \in \mathcal{M}(R)$ be orthogonal idempotents such that pRp and qRq have respective σ-units (consisting of idempotents) (e_n) and (f_n). Then $(e_n + f_n)$ is a σ-unit for $(p + q)R(p + q)$.* □

With these ingredients, we now establish Riesz decomposition in $V(\mathcal{M}(R))$ for σ-unital regular rings. The proof is analogous to the one known for C^*-algebras, and established by Zhang in [32, Theorem 1.1], so we just indicate which modifications should be adopted in the regular, σ-unital case.

Theorem 2.4. *Let R be a σ-unital regular ring. Then $V(\mathcal{M}(R))$ is a Riesz monoid.*

Proof. Let $[r]$, $[p]$, $[q] \in V(\mathcal{M}(R))$ be elements such that $[r] \leq [p]+[q]$. Since all matrix rings over R are regular (see [12, Lemma 1.6]) and σ-unital, and using Lemma 1.11, we may assume without loss of generality that r, p and q are idempotents in $\mathcal{M}(R)$, that $pq = qp = 0$ and that $r \leq p + q$. Also, since $(p+q)\mathcal{M}(R)(p+q) = \mathcal{M}((p+q)R(p+q))$ by Lemma 1.9, we may assume that $p + q = 1$.

Fix σ-units (p_n) for pRp, (q_n) for qRq, (r_n) for rRr and (r'_n) for $(1-r)R(1-r)$. Then, by Lemma 2.3, the idempotents $e_n := p_n + q_n$ and $f_n := r_n + r'_n$ form σ-units for R. Now we use that $R = \cup_n e_nRe_n = \cup_n f_nRf_n$, and that every subsequence of a σ-unit is also a σ-unit, whence we may assume by changing notation that:

$$e_1 \leq f_1 \leq e_2 \leq f_2 \leq \dots .$$

Now the proof ends as in [32, Proof of Theorem 1.1], using whenever necessary that $V(R)$ is a Riesz monoid (see [12, Theorem 2.8]). $\square$

The lack of regularity of $\mathcal{M}(R)$ (shown in Proposition 1.8) can be replaced by the existence of enough idempotents, as proved in the next result. This will be useful for studying the ideal lattice of $\mathcal{M}(R)$.

Theorem 2.5. *Let R be a σ-unital regular ring, and let $x \in \mathcal{M}(R)$. Then there exist idempotents $p_i \in x\mathcal{M}(R)$ and elements $x_i \in \mathcal{M}(R)$, for $i = 1, 2$, such that $x = p_1x_1 + p_2x_2$. Consequently, the ideals of $\mathcal{M}(R)$ are generated by idempotents.*

Proof. Let (e_n) be a σ-unit for R consisting of idempotents. Notice first that it follows from Lemma 1.3 that $\lim_n xe_n = \lim_n e_nx = x$.

Set $e_0 = 0$. We deduce immediately that $x = \sum_{n=0}^{\infty} x(e_n - e_{n-1})$ and that $x = \sum_{n=0}^{\infty}(e_n - e_{n-1})x$, in the strict topology. Consider $xe_1 \in R$. There exists $n \geq 2$ such that $xe_1 \in e_nRe_n$, and thus $xe_1 = e_nxe_1e_n = e_nxe_1$. Analogously, there exists $m \geq 2$ such that $e_1x = e_1xe_m$. Renumbering if necessary we may assume that $n = m = 2$, and hence we have $xe_1 = e_1xe_1 + (e_2 - e_1)xe_1$ and $e_1x = e_1xe_1 + e_1x(e_2 - e_1)$.

Using induction we will prove that, for $n \geq 2$ (and, again, up to renumbering the σ-unit):

$$\begin{aligned}
(1) \qquad & x(e_n - e_{n-1}) = (e_{n-1} - e_{n-2})x(e_n - e_{n-1}) + \\
& (e_n - e_{n-1})x(e_n - e_{n-1}) + (e_{n+1} - e_n)x(e_n - e_{n-1}), \\
(2) \qquad & (e_n - e_{n-1})x = (e_n - e_{n-1})x(e_{n-1} - e_{n-2}) + \\
& (e_n - e_{n-1})x(e_n - e_{n-1}) + (e_n - e_{n-1})x(e_{n+1} - e_n).
\end{aligned}$$

Proceeding in this way we can represent the element x as a tridiagonal matrix with countably many rows and columns: setting $a_n = (e_n - e_{n-1})x(e_{n+1} - e_n)$, $b_n = (e_n - e_{n-1})x(e_n - e_{n-1})$ and $c_n = (e_{n+1} - e_n)x(e_n - e_{n-1})$, for $n \geq 1$, we then have:

$$x = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ c_1 & b_2 & a_2 & 0 & \cdots \\ 0 & c_2 & b_3 & a_3 & \cdots \\ 0 & 0 & c_3 & b_4 & \cdots \\ 0 & 0 & 0 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us prove now that this decomposition holds. Suppose that $n = 2$, and consider the elements $x(e_2 - e_1)$ and $(e_2 - e_1)x$. As before, there exists $n \geq 3$ such that $x(e_2 - e_1), (e_2 - e_1)x \in e_n R e_n$. Renumbering, we may assume that $n = 3$ and thus we get

$$x(e_2 - e_1) = e_3 x(e_2 - e_1) = (e_3 - e_2)x(e_2 - e_1) + \\ (e_2 - e_1)x(e_2 - e_1) + e_1 x(e_2 - e_1),$$

and also

$$(e_2 - e_1)x = (e_2 - e_1)x(e_3 - e_2) + (e_2 - e_1)x(e_2 - e_1) + (e_2 - e_1)xe_1.$$

Suppose that for some $n > 2$, the equalities (1) and (2) hold. Consider in this case the elements $x(e_{n+1} - e_n)$ and $(e_{n+1} - e_n)x$. We may assume that both elements belong to $e_{n+2}Re_{n+2}$. Therefore

$$x(e_{n+1} - e_n) = \sum_{k=1}^{n+2} (e_k - e_{k-1})x(e_{n+1} - e_n),$$

$$(e_{n+1} - e_n)x = \sum_{k=1}^{n+2} (e_{n+1} - e_n)x(e_k - e_{k-1}).$$

If $k = 1$, then $e_1 x(e_{n+1} - e_n) = e_1 x e_2 (e_{n+1} - e_n) = 0$, since $n > 2$. On the other hand, if $1 < k < n$, then by the induction hypothesis $(e_k - e_{k-1})x(e_{n+1} - e_n) = (e_k - e_{k-1})x(e_{k+1} - e_{k-2})(e_{n+1} - e_n) = (e_k - e_{k-1})x(e_{k+1}(1 - e_n))e_{n+1} = 0$. This proves that $x(e_{n+1} - e_n) = \sum\limits_{k=n}^{n+2} (e_k - e_{k-1})x(e_{n+1} - e_n)$, and establishes equality (1). Equality (2) follows in a similar manner, and hence the induction is complete.

For $i = 1, 2$, let $x_i \in \mathcal{M}(R)$ be the elements defined as follows (which exist by Lemma 1.7):

$$x_1 = \sum_{n=1}^{\infty} (x(e_{4n-3} - e_{4n-4}) + x(e_{4n-2} - e_{4n-3})) = \sum_{n=1}^{\infty} x(e_{4n-2} - e_{4n-4}),$$

$$x_2 = \sum_{n=1}^{\infty} (x(e_{4n-1} - e_{4n-2}) + x(e_{4n} - e_{4n-1})) = \sum_{n=1}^{\infty} x(e_{4n} - e_{4n-2}).$$

Then $x = x_1 + x_2$. Let $y_n = x(e_{4n-2} - e_{4n-4}) \in R$. Since R is regular, there exist elements $z_n \in R$ such that $y_n = y_n z_n y_n$. The relation (1) applied to y_n yields

$$y_n = (e_{4n-1} - e_{4n-4})x(e_{4n-2} - e_{4n-3}) +$$
$$(e_{4n-2} - e_{4n-5})x(e_{4n-3} - e_{4n-4}) = (e_{4n-1} - e_{4n-5})y_n,$$

so we may assume that $z_n \in (e_{4n-1} - e_{4n-5})R(e_{4n-1} - e_{4n-5})$. Note that the elements $f_n := y_n z_n = x z_n$ are orthogonal idempotents. Let $p_1 = \sum\limits_{n=1}^{\infty} f_n \in \mathcal{M}(R)$. Then $p_1^2 = p_1$ and if $z = \sum\limits_{n=1}^{\infty} z_n \in \mathcal{M}(R)$, then we have $p_1 = \sum\limits_n y_n z_n = \sum\limits_n x z_n = x \sum\limits_n z_n = xz$. Note also that $x_1 = \sum\limits_n y_n = \sum\limits_n f_n y_n = p_1 x_1$.

Similarly, we construct an idempotent $p_2 \in \mathcal{M}(R)$ such that $p_2 \in x\mathcal{M}(R)$ and $x_2 = p_2 x_2$. Then we have $x = x_1 + x_2 = p_1 x_1 + p_2 x_2$. This proves that the ideal generated by $x \in \mathcal{M}(R)$ is generated by the idempotents p_1 and p_2, whence the result follows. $\square$

Remark 2.6. In [15, p.115], it was mentioned that if K is a field, then any one-sided ideal in $FRCM(K)$ (and also in some of its subalgebras) is generated by idempotents (see also [29] for related results). Combining Theorem 2.5 and Proposition 1.8 we obtain lots of examples of nonregular rings such that all their one-sided ideals are generated by idempotents.

The previous facts enable us to faithfully relate the ideal structure of $\mathcal{M}(R)$ to the order-ideal structure of the monoid $V(\mathcal{M}(R))$. This has already been done for σ-unital C^*-algebras with real rank zero ([32]); for an exchange ring R (and, in particular, for a regular ring), there is also a close relation between the ideal lattice of R and the order-ideal lattice of $V(R)$ ([6]). The essential ingredients in the proof of the next result (which we outline below) are Theorems 2.4 and 2.5, and we refer the reader to [32, Theorem 2.3], [24, Teorema 4.1.7] and [25, Theorem 2.1] for further details.

Theorem 2.7. *Let R be a σ-unital regular ring. Then the lattice of ideals of $\mathcal{M}(R)$ is isomorphic to the lattice of order-ideals of $V(\mathcal{M}(R))$.*

Proof. Denote by $L(\mathcal{M}(R))$ the lattice of ideals of $\mathcal{M}(R)$ and by $L(V(\mathcal{M}(R)))$ the lattice of order-ideals of $V(\mathcal{M}(R))$. Define $\phi : L(\mathcal{M}(R)) \to L(V(\mathcal{M}(R)))$ by $\phi(I) = V(I)$. We claim that ϕ is a lattice isomorphism.

First note that by Riesz decomposition in $V(\mathcal{M}(R))$, it follows that $V(I)$ is the submonoid of $V(\mathcal{M}(R))$ generated by $\{[e] \in V(R) \mid e \text{ is an idempotent in } I\}$ for any ideal I of $\mathcal{M}(R)$. It is clear that $V(I \cap J) = V(I) \cap V(J)$, and also that $V(I) + V(J) \subseteq V(I+J)$, whenever $I, J \in L(\mathcal{M}(R))$. If $[e] \in V(I+J)$, where e is an idempotent in $I+J$, then it follows after a standard argument that there exist idempotents $e_1, \dots, e_n \in I$ and $f_1, \dots, f_m \in J$ such that $[e] \leq \sum\limits_{i=1}^{n} [e_i] + \sum\limits_{j=1}^{m} [f_j]$. Therefore $[e] \in V(I) + V(J)$, using Riesz decomposition on $V(\mathcal{M}(R))$. This shows that ϕ is a lattice homomorphism.

Since the ideals of $\mathcal{M}(R)$ are generated by idempotents, it is clear that $I = J$ whenever $V(I) = V(J)$, hence ϕ is injective. Let S be an order-ideal of $V(\mathcal{M}(R))$, and let $I(S)$ be the ideal of $\mathcal{M}(R)$ generated by $\{e \in \mathcal{M}(R) \mid e \text{ is an idempotent and } [e] \in S\}$. Then $V(I(S)) = S$. For, if $e \in I(S)$ is an idempotent, then again a standard argument shows that there exist idempotents $e_1, \dots, e_n \in \mathcal{M}(R)$ with $[e_i] \in S$ for all i, such that $[e] \leq \sum_{i=1}^{n} [e_i]$. Since $[e_i] \in S$ for all i, we conclude that $[e] \in S$. If, conversely, $[e] \in S$, then we may write $[e] = \sum_{i=1}^{n} [e_i]$, for some idempotents $e_i \in \mathcal{M}(R)$, using again Riesz decomposition on $V(\mathcal{M}(R))$. It follows that $e_i \in I(S)$ for all i, whence $[e_i] \in V(I(S))$ for all i, so that $[e] \in V(I(S))$. Therefore ϕ is surjective, hence a lattice-isomorphism, as claimed. ☐

Now, in analogy with [25, Theorem 2.4] (see also [13, Theorem 1.10]), we can represent the monoid $V(\mathcal{M}(R))$ in terms of a monoid of intervals in $V(R)$. Recall that an **interval** in a monoid M is any nonempty, hereditary, upward directed subset I of M. An interval I in a monoid M is said to be **countably generated** provided that I has a countable cofinal subset. For any ring R, define $D(R) = \{[e] \in V(R) \mid e \in R\}$. In case R has a countable unit (e_n), then $D(R)$ may be described as the countably generated interval which has $([e_n])$ as a countable cofinal subset.

Assume now that R is a σ-unital regular ring with stable rank one. If $e \in M_n(\mathcal{M}(R))$ is an idempotent, then $eM_\infty(R)e$ is a σ-unital regular subring of $M_n(R)$. Define:

$$\theta(e) = \{[p] \in V(R) \mid p \text{ is an idempotent in } eM_\infty(R)e\}.$$

Then $\theta(e)$ is an interval in $V(R)$, and indeed $\theta(e) = \{[p] \in V(R) \mid p \lesssim e\} = \{[p] \in V(R) \mid p \lesssim e_k \text{ for some } k\}$, where $e_1 \leq e_2 \leq \dots$ is a σ-unit for $eM_n(R)e$ (see Lemma 2.1). Clearly, $\theta(e) \subseteq nD(R)$.

Theorem 2.8. *Let R be a σ-unital regular ring with stable rank one. Then there is a normalized monoid isomorphism between the monoid $(V(\mathcal{M}(R)), [1_{\mathcal{M}(R)}])$ and the monoid $(W_\sigma^D(V(R)), D)$, whose elements are those countably generated intervals I in $V(R)$ for which there exist $n \in \mathbb{N}$ and a countably generated interval J in $V(R)$ such that $I + J = nD$, where $D = D(R)$.*

Proof. Using the Riesz decomposition property on $V(\mathcal{M}(R))$ it follows that if $e, f \in M_\infty(\mathcal{M}(R))$ are idempotents, then $\theta(e \oplus f) = \theta(e) + \theta(f)$. From this, if $e \in M_n(\mathcal{M}(R))$ is an idempotent, then taking $g = 1_{M_n(\mathcal{M}(R))} - e$ we see that $\theta(e) + \theta(g) = nD(R)$.

Define a map $\gamma : V(\mathcal{M}(R)) \to W_\sigma^D(V(R))$ by $\gamma([e]) = \theta(e)$. The method used in the proof of [13, Proposition 1.7] shows that γ is an injective monoid homomorphism. That γ is surjective follows from a similar argument to the one used in [13, Proposition 1.8]. Finally, it is clear that D is an order-unit for $W_\sigma^D(V(R))$ and that $\gamma([1_{\mathcal{M}(R)}]) = D$, hence γ is normalized. ☐

Let R be a simple ring. We say that R is **elementary** provided that R has minimal idempotents. (Here, a nonzero idempotent $e \in R$ is **minimal** if eR is a minimal right ideal.) If R is simple with an approximate unit, then it can be shown that R is elementary if and only if there exist a division ring D and a dual pair ${}_DV, W_D$ such that $R \cong \mathfrak{F}_W(V)$. This is equivalent to saying that R is regular and $V(R) \cong \mathbb{Z}^+$ (see [26, Proposició 3.1.4]).

Remark 2.9. Notice that if R is a simple σ-unital (nonunital) elementary ring, then $\mathcal{M}(R)/R$ is simple. To see this, note that R can be identified with $\mathfrak{F}_V(W)$, for a certain dual pair ${}_DV, W_D$ of countably dimensional vector spaces over a division ring D. If I is an ideal of $\mathcal{M}(R)$ such that $R \subsetneqq I$, then there exists an idempotent $p \in I$ with $\mathrm{rank}(p) = \infty$. Since R is nonunital, we see that $\mathrm{rank}(1_{\mathcal{M}(R)}) = \infty$. Hence, we conclude that $p \sim 1_{\mathcal{M}(R)}$, and so $I = \mathcal{M}(R)$ (see also [26, Teorema 3.1.5].

Definition 2.10. *Let M be a cancellative monoid. We say that M is* **strictly unperforated** *if whenever $nx < ny$ for some $n \in \mathbb{N}$ and $x, y \in M$, it follows that $x < y$.*

Recall that a **state** on a monoid M with order-unit $u \in M$ is a monoid homomorphism $s : M \to \mathbb{R}^+$ such that $s(u) = 1$. The set of states on (M, u) is denoted by $S_u = St(M, u)$. It is clear that $St(M, u) = St(G(M), u)$, where $G(M)$ is the Grothendieck group of M. Let $\mathrm{Aff}(S_u)$ denote the ordered Banach space of affine continuous (real-valued) functions defined on the compact convex set S_u. Denote by $\phi_u : M \to \mathrm{Aff}(S_u)$ the natural representation map (given by evaluation). We define $\mathrm{LAff}_\sigma(S_u)^{++}$ as the semigroup whose elements are those affine and lower semicontinuous functions with values on $\mathbb{R}^{++} \cup \{+\infty\}$, and that can be expressed as countable (pointwise) suprema of an increasing sequence of (strictly positive) elements from $\mathrm{Aff}(S_u)$. Recall also that an interval I in a monoid M is called **soft** provided that $I \neq 0$ and for each $x \in I$, there exist $y \in I$ and $n \in \mathbb{N}$ such that $(n+1)x \leq ny$ (see [14]).

Let R be a σ-unital simple regular ring with stable rank one. Fix $u \in V(R)^*$, and set $d = \sup \phi_u(D(R))$. We define:

$$W_\sigma^d(S_u) = \{f \in \mathrm{LAff}_\sigma(S_u)^{++} \mid f + g = nd \text{ for some } g \in \mathrm{LAff}_\sigma(S_u)^{++} \text{ and } n \in \mathbb{N}\}.$$

Now consider the set $V(R) \sqcup W_\sigma^d(S_u)$, where $\sqcup$ denotes disjoint union of sets. This set can be endowed with a monoid structure, by extending the natural operations in $V(R)$ and $W_\sigma^d(S_u)$, and by setting $x + f = \phi_u(x) + f$, whenever $x \in V(R)$ and $f \in W_\sigma^d(S_u)$.

Theorem 2.11. *Let R be a simple σ-unital (nonunital) regular ring with stable rank one. Assume that R is nonelementary and that $V(R)$ is strictly unperforated. Fix a nonzero element $u \in V(R)$. Let $D = D(R)$ and $d = \sup \phi_u(D)$. Then there is a normalized monoid isomorphism*

$$\varphi : V(\mathcal{M}(R)) \to V(R) \sqcup W_\sigma^d(S_u),$$

such that $\varphi([p]) = [p]$ if $p \in R$, and $\varphi([p]) = \sup\{\phi_u([q]) \mid [q] \in V(R) \text{ and } q \lesssim p\}$ if $p \in \mathcal{M}(R) \setminus R$.

Proof. By Theorem 2.8 there is a normalized monoid isomorphism between the monoid $(V(\mathcal{M}(R)), [1_{\mathcal{M}(R)}])$ and $(W_\sigma^D(V(R)), D(R))$. Note now that $V(R)$ is a conical refinement monoid, which is simple and cancellative since R is regular, simple and has stable rank one. Further, the interval $D(R)$ is countably generated, because R has a countable unit. We claim that, as long as R is nonunital, $D(R)$ is a soft interval. Suppose that $D(R)$ is not soft. Then there exists a nonzero element $x \in D(R)$ such that $(n+1)x \not\leq ny$ for all $n \in \mathbb{N}$ and all $y \in D(R)$. Set $x = [e]$ for some idempotent $e \in R$. Since e is not a unit for R, there exists $a \in R$ such that $a - ea \neq 0$, whence $(1-e)R \neq 0$. Let $B = (1-e)R(1-e)$, which is a nonzero regular ring, (otherwise $(1-e)R = (1-e)R(1-e)R = BR = 0$). Therefore there exists a nonzero idempotent $f \in B$, whence $e + f$ is a nonzero idempotent of R. Now use the simplicity of R to find $k \in \mathbb{N}$ such that $[e] \leq k[f]$, and then $(k+1)[e] = k[e] + [e] \leq k[e+f]$, a contradiction.

Thus we can apply [25, Theorem 3.8], which establishes a (normalized) monoid isomorphism between $(W_\sigma^D(V(R)), D(R))$ and $(V(R) \sqcup W_\sigma^d(S_u), d)$. Finally, the composition of this isomorphism with the one given at the beggining of the proof provides an isomorphism between $(V(\mathcal{M}(R)), [1_{\mathcal{M}(R)}])$ and $(V(R) \sqcup W_\sigma^d(S_u), d)$, as desired. $\square$

We close this section proving a result that implies simplicity of the ring $\mathcal{M}(R)/R$ for a simple, σ-unital, purely infinite regular ring R.

Definition 2.12. *Let M be a monoid. We say that M is* **purely infinite** *if whenever $x \in M$ is nonzero, then there exists a nonzero element $y \in M$ such that $x + y = x$. If R is any ring, and $e \in R$ is a nonzero idempotent, we say that e is* **infinite** *provided that there exists $f \in R$ such that $f < e$ and $e \sim f$. Finally, a ring is said to be* **purely infinite** *if any nonzero right ideal I of R contains an infinite idempotent.*

It is easy to see that R is purely infinite if and only if any nonzero principal right ideal contains an infinite idempotent, as well as that the condition of pure infiniteness is indeed symmetric. Observe that if R is purely infinite, then every nonzero idempotent is infinite. For, if $e \in R$ is a nonzero idempotent, then there is an infinite idempotent $f \in R$ such that $fR \subseteq eR$. Thus $f = ef$ and so $g = fe$ is an idempotent with $f \sim g \leq e$. Now g is infinite, so there is an idempotent h such that $g \sim h < g$. Thus $e \sim h \oplus (e - g) < e$. Notice also that if R is a simple purely infinite ring containing nonzero idempotents e and f, then it follows from the simplicity of R that $e \prec f$. If R is a regular ring, then R is purely infinite if and only if $V(R)$ is a purely infinite monoid.

For C^*-algebras, it is known that if A is a σ-unital stable C^*-algebra (see, for example, [30] for definitions), then $\mathcal{M}(A)/A$ is simple if and only if A is elementary or A is purely infinite simple (see [21, Theorem 3.8] and [28, Theorem 3.2]). Also, it was proved in [33, Theorem 1.2 (i)]) that if A is σ-unital, purely infinite and simple, then A is either unital or stable, whence it follows that if A is nonunital, then $\mathcal{M}(A)/A$ is simple.

A monoid M is said to be **separative** provided that whenever $a+a = a+b = b+b$, for $a, b \in M$, it follows that $a = b$. We say that a ring R is **separative** if the corresponding monoid $V(R)$ is. In the presence of separativity, simple regular rings are shown to have a very extreme behaviour: either they have stable rank one or they are purely infinite (see [6]). Also, there are no examples known of non separative regular rings, so this seems to suggest that only these two cases are to be considered. We are indebted to Enric Pardo for the following observation:

Proposition 2.13. *Let R be a σ-unital simple regular ring. Assume that R is purely infinite. Then $e \sim f$ for evey pair of idempotents $e, f \in \mathcal{M}(R) \setminus R$. Consequently, if R is nonunital, $V(\mathcal{M}(R)) \cong V(R) \sqcup \{\infty\}$, where $x + \infty = \infty$, for any $x \in V(R)$.*

Proof. Let $e, f \in \mathcal{M}(R) \setminus R$ be idempotents. Take countable units (e_n) and (f_n) for eRe and fRf respectively (see Lemma 2.1). Since R is purely infinite simple, we have that $e_1 \prec f_2 \prec e_3 \prec f_4 \prec \dots$

Without loss of generality, we may assume that $e_i \neq e_{i+2}$ and $f_i \neq f_{i+2}$ for all i. Let $h_1 \in R$ be an idempotent such that $h_1 \sim e_1$ and $h_1 < f_2$. Then we have that $f_2 - h_1 \prec e_3 - e_1 \neq 0$. Let $g' \in R$ be an idempotent such that $g' < e_3 - e_1$ and $g' \sim f_2 - h_1$, and let $g_2 := e_1 \oplus g'$. Then $g_2 \in R$ and $e_1 < g_2 < e_3$ with $g_2 \sim f_2$. Continuing in this way we define idempotents g_{2j}, $h_{2j+1} \in R$ such that $e_{2j-1} < g_{2j} < e_{2j+1}$, and $g_{2j} \sim f_{2j}$ for all j, and $f_{2j} < h_{2j+1} < f_{2j+2}$, and $h_{2j+1} \sim e_{2j+1}$ for all j. Let us summarize this situation as follows:

$$\begin{array}{ccccccccc} h_1 & < & f_2 & < & h_3 & < & f_4 & < & \dots \\ \wr & & \wr & & \wr & & \wr & & \\ e_1 & < & g_2 & < & e_3 & < & g_4 & < & \dots \end{array}$$

Let $g_0 = h_0 = 0$, $g_i = e_i$ for i odd, and $h_i = f_i$ for i even. Then (g_n) and (h_n) are increasing approximate units consisting of idempotents for eRe and fRf respectively, that satisfy $g_i \sim h_i$ for all i. Note also that $h_i < h_{i+1}$ and $g_i < g_{i+1}$ for all i. Now observe that $g_i \oplus (g_{i+1} - g_i) = g_{i+1} \sim h_{i+1} = h_i \oplus (h_{i+1} - h_i)$, and $h_i \sim g_i$. Taking into account that $V(R)^*$ is a group (see [6, Proposition 2.4]), it follows that $h_{i+1} - h_i \sim g_{i+1} - g_i$ for all i.

Then there exist elements x_i, $y_i \in R$ such that $x_i y_i = g_{i+1} - g_i$ and $y_i x_i = h_{i+1} - h_i$ for all i. Set $x = \sum_i x_i$ and $y = \sum_i y_i$. Then $xy = e$ and $yx = f$, and therefore $e \sim f$.

Now, if $e \in \mathcal{M}(R) \setminus R$ is an idempotent, it follows from the above that $e \sim 1_{\mathcal{M}(R)}$. If, on the other hand, $e \sim 1_{\mathcal{M}(R)}$, for an idempotent $e \in R$, then $1_{\mathcal{M}(R)} = xy = xey$ for some $x, y \in \mathcal{M}(R)$, which would imply $1_{\mathcal{M}(R)} \in R$. $\square$

3. Ideals of multiplier rings

Combined with [32, Theorem 2.3] (see also [25, Theorem 2.1]), Theorem 2.11 provides an effective method to study the ideal structure of multiplier rings of regular rings in the class we are considering. For simple C^*-algebras with real rank zero, an extensive account of what type of results one may expect using this technique can be found in [25]. In fact, the arguments used in [25] derive from the representation of the monoid of equivalence classes of projections of $M_\infty(\mathcal{M}(A))$ (for a C^*-algebra A) as a monoid involving a semigroup of functions over the quasitrace space of A. In order to make these results available for regular rings, we first need to establish the relation between the state space S_u and the space of pseudo-rank functions over R, in a parallel way to what is done for unital regular rings (see [12, Proposition 17.12]).

Definition 3.1. [7, Section 2] *Let R be a regular ring. A* **pseudo-rank function** *over R is a map $N : R \to \mathbb{R}^+$ such that $N(xy) \leq N(x), N(y)$ for $x, y \in R$, and such that $N(e+f) = N(e) + N(f)$ whenever $e, f \in R$ are orthogonal idempotents. If* $\sup N(R) = 1$, *we say that N is* **normalized**.

We denote by $\mathbb{P}(R)$ the set of normalized pseudo-rank functions, which is a convex subspace of $\mathbb{R}^R$, not compact in general. Let R be a regular ring, and let $x \in R$ be a nonzero element. Denote by $\mathbb{P}(R)_x$ the set of unnormalized pseudo-rank functions N such that $N(x) = 1$. Notice that $\mathbb{P}(R)_x$ is a convex subset of $\mathbb{R}^R$. We will prove now that if R is simple, $e \in R$ is a nonzero idempotent and if $u = [e] \in V(R)$, then the spaces S_u and $\mathbb{P}(R)_e$ are affinely homeomorphic. The argument we use is based on the unital case.

Proposition 3.2. *Let R be a simple regular ring and let $e \in R$ be a nonzero idempotent. Set $u = [e] \in V(R)$. Then there exists an affine homeomorphism $\alpha : \mathbb{P}(R)_e \to S_u$ such that $\alpha(N)([f]) = N(f)$, for every $N \in \mathbb{P}(R)_e$ and every idempotent $f \in R$.*

Proof. First, we prove that $\mathbb{P}(R)_e$ is affinely homeomorphic to $\mathbb{P}(eRe)$. If $N \in \mathbb{P}(R)_e$, then it is easily verified that its restriction to eRe gives a pseudo-rank function on eRe. Denote by $r : \mathbb{P}(R)_e \to \mathbb{P}(eRe)$ the restriction map, which is affine and continuous. Now, let $N \in \mathbb{P}(eRe)$, and let $x \in R$. There exists an idempotent $f \in R$ such that $xR = fR$. Since R is simple, there exists $n \in \mathbb{N}$ such that $[f] \leq n[e]$ in $V(R)$, and by Riesz decomposition there exist idempotents $f_1, \dots, f_n \in eRe$ such that $[f] = \sum_i [f_i]$, and $[f_i] \leq [e]$ for all i. Define $\overline{N}(x) = \sum_i N(f_i)$. By [7, Lemma 2.1(a)], this map is well-defined, and it is an unnormalized pseudo-rank function on R such that $\overline{N}(e) = 1$. Hence, the assignment $N \mapsto \overline{N}$ defines a map $\overline{r} : \mathbb{P}(eRe) \to \mathbb{P}(R)_e$. It is not difficult to check that r and $\overline{r}$ are mutually inverse, and so r is an affine continuous isomorphism. To see that $\overline{r}$ is continuous, observe that whenever $N_i \to N$ in $\mathbb{P}(eRe)$, we have that $N_i(exe) \to N(exe)$ for all $x \in R$. Now, let $x \in R$. Using the regularity and simplicity of R as before, we find $n \in \mathbb{N}$ and idempotents $f_1, \dots, f_n \in eRe$ such that $\overline{r}(N_i)(x) = \overline{N_i}(x) = \sum_{j=1}^{n} N_i(f_j)$ and $\overline{r}(N)(x) = \overline{N}(x) = \sum_{j=1}^{n} N(f_j)$. Since $N_i(f_j) \to N(f_j)$ for all j, we conclude that $\overline{r}(N_i)(x) \to \overline{r}(N)(x)$ for all $x \in R$, and so $\overline{r}(N_i) \to \overline{r}(N)$. Therefore r is a homeomorphism.

By [12, Proposition 17.12], there is an affine homeomorphism θ_e between $\mathbb{P}(eRe)$ and $St(V(eRe), [e])$. On the other hand, there is an isomorphism $\psi : V(R) \to V(eRe)$ that maps $[e]$ to $[e]$, cf. [6, Lemma 1.5(c)]. Therefore the map $\beta : St(V(eRe), [e]) \to S_u$ defined by $\beta(s)([f]) = s\psi([f])$, is an affine homeomorphism. Set $\alpha := \beta \circ \theta_e \circ r : \mathbb{P}(R)_e \to S_u$. Then α is an affine homeomorphism. Now, let $N \in \mathbb{P}(R)_e$ and let $f \in R$ be an idempotent. By simplicity, there exists $n \in \mathbb{N}$ such that $[f] \leq n[e]$, and again by

Riesz decomposition there exist idempotents $f_1, \dots, f_n \in eRe$ such that $[f] = \sum_{i=1}^{n}[f_i]$. Let $f_1', \dots, f_n' \in R$ be orthogonal idempotents such that $f = \sum_{i=1}^{n} f_i'$ and $f_i' \sim f_i$ for all i. Then

$$\alpha(N)([f]) = \beta((\theta_e r)(N))[f] = (\theta_e r)(N)(\psi[f]) = \sum_{i=1}^{n}(\theta_e r)(N)[f_i] = \sum_{i=1}^{n} N(f_i') = N(f),$$

as desired. □

Remark 3.3. The proof of Proposition 3.2 gives the following more general result: let R be a regular ring and let $e \in R$ be an idempotent with $R = ReR$. Then $u := [e] \in V(R)$ is an order-unit, and the spaces $\mathbb{P}(R)_e$ and S_u are affinely homeomorphic. In this case, $\mathbb{P}(R)_e$ is a compact convex set.

For a compact convex set K, we denote by $\partial_{\mathfrak{e}} K$ the set of all its extreme points.

Definition 3.4. *Let M be a monoid with order unit u, and let D be a generating interval. We say that (M, D) has* **continuous scale** *if the affine function $d := \sup \phi_u(D)$ is a continuous function from S_u to $\mathbb{R}$. We say that (M, D) has* **finite scale** *if the restriction of $d = \sup \phi_u(D)$ to $\partial_{\mathfrak{e}} S_u$ is finite. If R is a simple regular ring, and if $u \in V(R)$ is a nonzero element, then we say that R has* **continuous scale** *(resp.* **finite scale***) provided that $(V(R), D(R))$ has continuous scale (resp. finite scale).*

Now we obtain a description of the rings in our class whose multiplier rings have only the trivial ideals.

Proposition 3.5. *Let R be a simple σ-unital (nonunital) regular ring. Suppose that R has stable rank one and that $V(R)$ is strictly unperforated. Then $\mathcal{M}(R)/R$ is simple if and only if R is elementary or R has continuous scale.*

Proof. If R is elementary, then $\mathcal{M}(R)/R$ is simple by Remark 2.9. Hence, we may assume that R is nonelementary. Now the proof follows along the lines of [25, Proposition 4.1, Corollary 4.4]. □

The argument used in [25, Proposition 4.1] shows that in fact $\mathcal{M}(R)$ contains a unique ideal $L(R)$ which properly contains R and such that is contained in every ideal that properly contains R. Moreover, if $u \in V(R)^*$, then $V(L(R)) \cong V(R) \sqcup \mathrm{Aff}(S_u)^{++}$. For C^*-algebras, the existence of this ideal was noted first by Lin in [20] for simple AF algebras, and later in [21] for simple and separable C^*-algebras.

Theorem 3.6. *Let R be a σ-unital simple regular ring with stable rank one. Assume that R is nonelementary, that $V(R)$ is strictly unperforated and that the state space S_u is metrizable, where $u \in V(R)^*$. Then R has finite scale if and only if the monoid $V(\mathcal{M}(R))/V(L(R))$ is cancellative.*

Proof. This is proved using arguments similar to the ones in [25, Theorem 4.8]. □

Proposition 3.7. *Let R be a σ-unital (nonunital) simple regular ring with stable rank one. Assume that $V(R)$ is strictly unperforated and that R is nonelementary. Fix a nonzero idempotent $e \in R$, and suppose that $\mathbb{P}(R)_e$ is metrizable. Then there exists a unique ideal $I_{fin}(R)$ of $\mathcal{M}(R)$ properly containing R, which is maximal with respect to the property that $V(I_{fin}(R))/V(L(R))$ is cancellative.*

Proof. Let $u = [e] \in V(R)$, and set $I_{fin} := V(R) \sqcup \{f \in W_\sigma^d(S_u) \mid f_{|\partial_{\mathfrak{c}} S_u}$ is finite$\}$. Define $I_{fin}(R)$ to be the unique ideal of $\mathcal{M}(R)$ such that $\varphi(V(I_{fin}(R))) = I_{fin}$, where φ is the monoid isomorphism in Theorem 2.11. The rest of the proof follows now the lines of [25, Proposition 6.1]. $\square$

Definition 3.8. *Let R be a regular ring. We say that a pseudo-rank function N over R is* **infinite** *provided that $\sup_i N(u_i) = \infty$, for some (hence any) approximate unit (u_i) for R.*

Lemma 3.9. *Let R be a σ-unital regular ring. Let I be an ideal of $\mathcal{M}(R)$. If the monoid $V(\mathcal{M}(R))/V(I)$ is purely infinite, then $\mathcal{M}(R)/I$ is a purely infinite ring.*

Proof. Denote by $\pi : \mathcal{M}(R) \to \mathcal{M}(R)/I$ the natural quotient map and let $x \in \mathcal{M}(R)/I$ be a nonzero element. Take $y \in \mathcal{M}(R) \setminus I$ such that $\pi(y) = x$. By Theorem 2.5, there exist idempotents $p_1, p_2 \in \mathcal{M}(R)$ and elements $y_1, y_2, z, w \in \mathcal{M}(R)$ such that $y = p_1y_1 + p_2y_2$ and $p_1 = yz$, $p_2 = yw$. Since $y \notin I$, we may assume that $p_1 \notin I$. Thus, setting $f = p_1$, we have that $\pi(f) \neq 0$, as well as $\pi(f)(\mathcal{M}(R)/I) \subseteq x(\mathcal{M}(R)/I)$. Denote by $\psi : V(\mathcal{M}(R)) \to V(\mathcal{M}(R))/V(I)$ the natural quotient map. If $V(\mathcal{M}(R))/V(I)$ is a purely infinite monoid, there exists an idempotent $q \in M_\infty(\mathcal{M}(R))$ such that $[q] \notin V(I)$ and $\psi[f] = \psi[f] + \psi[q]$. Hence we can find idempotents $g, h \in M_\infty(I)$ such that $[f] + [g] = [f] + [q] + [h]$. Finally we get that $\pi(f) \sim \pi(f) \oplus \pi(q)$ in $M_\infty(\mathcal{M}(R)/I)$, and $\pi(q) \neq 0$ since $q \notin M_\infty(I)$. This shows that $\pi(f)$ is an infinite idempotent, whence $x(\mathcal{M}(R)/I)$ contains infinite idempotents, whence $\mathcal{M}(R)/I$ is a purely infinite ring. $\square$

Theorem 3.10. *Let R be a σ-unital (nonunital) simple regular ring with stable rank one. Assume that R is nonelementary and that $V(R)$ is strictly unperforated. Let $e \in R$ be a nonzero idempotent, and suppose that $\mathbb{P}(R)_e$ is metrizable. Let $\mathfrak{c}$ be the cardinality of the set of infinite extremal pseudo-rank functions in $\mathbb{P}(R)_e$.*

(*a*) *If $\mathfrak{c} = n$, then there exist precisely 2^n ideals between $I_{fin}(R)$ and $\mathcal{M}(R)$.*
(*b*) *If $\mathfrak{c}$ is infinite, then $\mathcal{M}(R)$ has at least $\mathfrak{c}$ different maximal ideals that properly contain $I_{fin}(R)$.*
(*c*) *If $\mathfrak{c}$ is infinite and $\partial_{\mathfrak{c}}\mathbb{P}(R)_e$ is compact Hausdorff, then $\mathcal{M}(R)/I_{fin}(R)$ has exactly $\mathfrak{c}$ minimal ideals.*

In each case, the quotient of $\mathcal{M}(R)$ by any of these ideals is a purely infinite ring.

Proof. The first three assertions of the theorem follow using similar arguments to the ones in [25, Theorem 6.3, Theorem 6.6, Proposition 6.7]. For the last part, let I be an ideal of $\mathcal{M}(R)$ such that $I_{fin}(R) \subseteq I$. Proceeding as in [25, Proposition 6.5], we get that $V(\mathcal{M}(R))/V(I)$ is a purely infinite monoid, and by Lemma 3.9, this implies that $\mathcal{M}(R)/I$ is a purely infinite ring. □

Remark 3.11. The ideal lattice of $\mathcal{M}(R)$ is, in general, very intricate. Indeed, under the same hypotheses as in Theorem 3.10, it can be shown that if $\partial_{\mathfrak{e}}\mathbb{P}(R)_e$ is a compact Hausdorff space that contains a nonisolated infinite pseudo-rank function, then there exist uncountably many (proper) different ideals between $L(R)$ and $\mathcal{M}(R)$ that form a chain with respect to inclusion (see [25, Theorem 6.8]). Similar to [13], [25] and [26], all these pathologies can be realized with nonunital ultramatricial F-algebras, for any field F.

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