THE STRUCTURE OF POSITIVE ELEMENTS FOR C^*-ALGEBRAS WITH REAL RANK ZERO.

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Abstract

In this paper we give a representation theorem for the Cuntz monoid $S(A)$ of a σ-unital C^*-algebra A with real rank zero and stable rank one, which allows to prove several Riesz decomposition properties on the monoid. As a consequence, it is proved that the comparability conditions (FCQ), stable (FCQ) and (FCQ+) are equivalent for simple C^*-algebras with real rank zero. It is also shown that the Grothendieck group $K_0^*(A)$ of $S(A)$ is a Riesz group, and lattice-ordered under some additional assumptions on A.

1980 Mathematics Subject Classification (revised 1985): Primary 46L80; Secondary 06F15.

Introduction

One important question in the study of simple C^*-algebras was posed in [4, 3.1.1] (FCQ): *Let A be a simple C^*-algebra. If p and q are nonzero projections in A such that $\tau(p) < \tau(q)$ for every quasi-trace τ, is then $p \prec q$?* If A satisfies this property, A is said to have *strict comparability.* There are important cases for which a positive answer to the FCQ is known. Such examples include the class of AF C^*-algebras ([4, Section 5]) and the irrational rotation algebras ([16], [17]). However, Villadsen has constructed recently an example of a simple C^*-algebra with perforated K_0 that doesn't satisfy the FCQ ([20]). This example doesn't have real rank zero, so it remains possible that the FCQ has a positive answer within the class of simple C^*-algebras with real rank zero and stable rank one. Moreover, Villadsen's example has stable rank one, and hence the monoid $V(A)$ of Murray-von Neumann equivalence classes of projections is cancellative.

Comparability questions such as the FCQ are intimately related with the concept of order in $V(A)$. It is natural, then, to investigate extensions of the Murray-von Neumann

*Partially supported by MEC-DGICYT grant no.PB92-0586, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

subequivalence of projections to all positive elements in $M_\infty(A)$. For this purpose, a notion of order $\lesssim$ in $M_\infty(A)_+$ is needed and a monoid $S(A)$ analogous to $V(A)$ has to be built. Such a relation was introduced by Cuntz in [9] and has been studied by several authors, see e.g. [19]. It is then possible to give a version of the (FCQ) for positive elements (see [4]) (FCQ+): *Let A be a simple C^*-algebra. If $a, b \in M_\infty(A)_+$ satisfy $d(a) < d(b)$ for all lower semicontinuous dimension functions d, is then $a \lesssim b$?* An affirmative answer to this question is given in [19] for C^*-algebras of the form $B \otimes D$, where B is a UHF-algebra and D is unital, simple and stably finite. Again Villadsen's construction gives an example of a simple C^*-algebra with stable rank one that doesn't satisfy the FCQ+.

In the case of C^*-algebras with real rank zero, it is expected that the structure of $S(A)$ is completely determined by $V(A)$. One of the main objectives of this paper is to establish the exact relation between $S(A)$ and $V(A)$ in the case of a σ-unital C^*-algebra with real rank zero and stable rank one. This enables us to obtain Riesz decomposition and Riesz interpolation with respect to the natural order in $S(A)$. It also allows to show that A has strict comparability if and only if the (FCQ+) holds, when A is simple and has real rank zero.

Another interesting aspect of $S(A)$ is its Grothendieck group, denoted by $K_0^*(A)$. This group was introduced in [9] to prove the existence of dimension functions in the stably finite and simple case, while in the purely infinite case $K_0^*(A) = \{0\}$. In [7, Section III], it is shown that $K_0^*(A)$ has a certain duality with quasi-traces, and as a consequence in some cases it can be represented faithfully onto the group of differences of affine and lower semicontinuous dimension functions defined over the quasi-trace space.

The paper is organized as follows. In Section 1 the necessary basic notions are summarized. In Section 2 we consider σ-unital C^*-algebras with real rank zero and stable rank one. It is proved that $S(A)$ is a Riesz and refinement monoid, and satisfies the Riesz interpolation property. On the way to establish these facts, a representation of $S(A)$ as a certain monoid of intervals over $V(A)$ is given. A monoid-theoretical point of view of the lower semicontinuous functions, strict comparability for positive elements (under some additional assumptions on A) and some characterizations of the real rank zero condition in terms of the monoid $S(A)$ are presented as applications in Section 3. Some of the techniques of this Section give a generalization of a Theorem of [19]. Finally, the structure of the group $K_0^*(A)$ is analysed in Section 4. It is shown that this group is a Riesz group and if $K_0(A)$ is a lattice-ordered group, then so is $K_0^*(A)$.

1 Preliminaries

Let A be a C^*-algebra and $a, b \in A$. Following [9], [19], we write $a \lesssim b$ if there is a sequence $\{x_n\}$ in A such that $x_n = r_n b s_n$ for some $r_n, s_n \in A$ and $x_n \to a$ in norm. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Then $\sim$ is an equivalence relation. Let $M_\infty(A) = \varinjlim M_n(A)$ with inclusions $M_n(A) \ni x \mapsto diag(x, 0) \in M_{n+1}(A)$. For $x, y \in M_\infty(A)_+$, say that $x \sim y$ if and only if $x \sim y$ in $M_n(A)$ for some n. By [19, Proposition 2.1], the relation $\lesssim$, when restricted to projections, gives the usual Murray-von Neumann subequivalence. For the case of $\sim$, its restriction to projections gives the Murray-von Neumann equivalence when the algebra has stable rank one. Denote by $\langle x\rangle$ the $\sim$-equivalence class of x and define $\langle x\rangle + \langle y\rangle = \langle x' + y'\rangle$ where $x', y' \in M_\infty(A)_+$ satisfy $x' \sim x$, $y' \sim y$ and $x'y' = 0$, or by $\langle x\rangle + \langle y\rangle = \langle \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \rangle$. Define $S(A)$ to be the set of all $\sim$-equivalence classes in $M_\infty(A)_+$, and define a partial ordering on it by:

$$\langle x\rangle \leq \langle y\rangle \text{ if and only if } x \precsim y.$$

Thus $(S(A), \leq)$ is a partially ordered monoid.

For $\epsilon > 0$, denote by f_ϵ the continuous function from $\mathbb{R}$ to $\mathbb{R}$ which is 0 on $(-\infty, \epsilon]$, linear on $[\epsilon, 2\epsilon]$, and 1 on $[2\epsilon, \infty)$.

Lemma 1.1 *Let A be a C^*-algebra and $x, y \in A_+$. Then $x \precsim y$ if and only if there exists $\{r_j\}$ in A with $x = \lim_{j\to\infty} r_j y r_j^*$, if and only if for all $\epsilon > 0$, there exists $\delta > 0$ and $r \in A$ such that $f_\epsilon(x) = r f_\delta(y) r^*$. If $z \in A$, then $zz^* \sim z^*z$. Moreover, if $g, h \in C(\mathrm{Spec}(x))^+$, then $g(x) \precsim h(x)$ if $\mathrm{Coz}(g) \subseteq \mathrm{Coz}(h)$. Finally, the set $\{a \in A_+ \mid a \precsim x\}$ is a closed set of A_+ and, as a consequence, if $a \in A_+$ and $f_\epsilon(a) \precsim x$ for all $\epsilon > 0$, it follows that $a \precsim x$.*

Proof. The first conclusion is [19, Proposition 2.4], while the second and third are observed in [9, Section 1] and [19, Section 1] respectively. The fact that $\{a \in A_+ \mid a \precsim x\}$ is closed is noted in [5, Section 6]. Finally, if $f_\epsilon(a) \precsim x$ for all $\epsilon > 0$, note that $f_\epsilon(a) \sim a f_\epsilon(a)$ and that $\lim_{\epsilon\to 0} a f_\epsilon(a) = a$. Thus $a \precsim x$. $\square$

Let M be a monoid. All monoids in this paper will be abelian, written additively. We say that M is a **refinement monoid** provided that whenever $x_1, x_2, y_1, y_2 \in M$ satisfy $x_1 + x_2 = y_1 + y_2$, then $\sum_{j=1}^{2} z_{ij} = x_i$ and $\sum_{i=1}^{2} z_{ij} = y_j$ for some elements $z_{ij} \in M$. If $(M, \leq)$ is preordered, then we say that M is a **Riesz monoid** provided that M satisfies the **Riesz Decomposition Property**, that is, whenever $x, y_1, y_2 \in M$ satisfy $x \leq y_1 + y_2$, then there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leq y_i$ for all i. Finally, M is said to satisfy the **Riesz Interpolation Property** provided that, whenever $x_1, x_2, y_1, y_2 \in M$ such that $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in M$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$.

If M is the positive cone of a partially ordered abelian group, then these three properties are all equivalent, by [10, Proposition 2.1]. However, for general orderings, there is no relation among them.

A non-zero element $u \in M$ is called an **order-unit** provided that for any $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leq nu$. A monoid M is called **simple** if it is nonzero and every nonzero element is an order-unit. If A is a unital C^*-algebra, then $\langle 1_A\rangle$ is trivially an order-unit for $S(A)$.

Let M be a partially ordered abelian monoid. An **interval** over M is a nonempty subset $I \subseteq M$ which is upward directed and order-hereditary (i.e.: if $x \leq y$ and $y \in I$, then $x \in I$). Denote by $\Lambda(M)$ the set of all intervals of M. We can endow $\Lambda(M)$ with a natural abelian monoid structure, namely if $I, J \in \Lambda(M)$

$$I + J = \{z \in M \mid z \leq x + y \text{ for some } x \in I, y \in J\}.$$

It follows that if M is a Riesz monoid, then $I + J = \{x + y \mid x \in I, y \in J\}$.

We say that an interval I is **countably generated** if there exists a countable cofinal subset $X \subseteq I$ (that is, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq I$ such that for all $x \in I$, there exists $n \in \mathbb{N}$ such that $x \leq x_n$). Notice that in this context, we can write $I = \{y \in M \mid y \leq x_n \text{ for some } n\}$. If given intervals I and J, we have that $I \subseteq J$, then we say that I is **bounded** by J.

2 Structure of $S(A)$

The purpose of this section is to represent $S(A)$ as a monoid of intervals over $V(A)$, for σ-unital C^*-algebras with real rank zero and stable rank one. Subsequently, we prove that Riesz decomposition, refinement and interpolation hold in $S(A)$.

Definition 2.1 *For every sequence $\{v_n\}_n$ of positive elements with bounded norm, form the element $v = \sum_{n=1}^{\infty}(1/2^n)v_n \in A$. The element v will be called the* **standard element** *associated to the sequence $\{v_n\}$. Notice that in case the sequence $\{v_n\}_n$ consists of an increasing sequence of projections, then v can be expressed as an (infinite) sum of orthogonal projections, namely $v = v_1 + \sum_{n=2}^{\infty}(1/2^{n-1})(v_n - v_{n-1})$.*

Lemma 2.2 *Let A be a C^*-algebra and $b \in A_+$. Let e_b be the standard element associated to an increasing (and countable) approximate unit of $A_b = \overline{bAb}$. Then the class $u_b := \langle e_b\rangle$ is equal to $\langle b\rangle$, and hence u_b does not depend on the approximate unit. Moreover, u_b is an order-unit for $S(A_b)$. In particular, if A is σ-unital and e is the standard element associated to any increasing (and countable) approximate unit for A, it follows that $u = \langle e\rangle$ is an order-unit for $S(A)$.*

Proof. Notice first that the hereditary C^*-subalgebra of A generated by b, namely $A_b = \overline{bAb}$, is σ-unital ($\{f_{1/n}(b)\}$ is an approximate unit). Assume that $e_b = \sum_{n=1}^{\infty}(1/2^n)e_n$, where $\{e_n\}$ is an approximate unit for A_b. Then by [13, Exercises 3.5,3.6] $\overline{e_bA_be_b} = \overline{bAb}$. Then $e_b = \lim_{n\to\infty} b^3x_nb^3 = \lim_{n\to\infty}(b^3x_nb)bb$ for some $x_n \in A$. Therefore $e_b \precsim b$ in A_b. Similarly $b \precsim e_b$. Thus $u_b = \langle e_b\rangle = \langle b\rangle$, and the class of e_b does not depend on the chosen approximate unit.

For any positive element $x \in A_b$, note that $\overline{xA_bx} \subseteq A_b$, and therefore $\langle x\rangle \leq \langle b\rangle = u_b$ in $S(A_b)$. Now, if $x \in M_k(A_b)_+$, set $B = M_k(A_b)$ and $E_b^{(k)} = diag(e_b, \ldots, e_b) = \sum_{n=1}^{\infty}(1/2^n)E_n^{(k)}$, where $E_n^{(k)} = diag(e_k, \ldots, e_k)$. We have that the closure of $E_b^{(k)}BE_b^{(k)}$ equals B, so that $\langle a\rangle \leq \langle E_b^{(k)}\rangle = ku_b$. Thus u_b is an order-unit for $S(A_b)$. $\square$

Proposition 2.3 *Let $p_1 \le p_2 \le \dots$ and $q_1 \le q_2 \le \dots$ be sequences of projections in A. Let p and q be the standard elements associated to $\{p_n\}$ and $\{q_n\}$ respectively. Then $p \precsim q$ if and only if for all n, there exists m such that $p_n \precsim q_m$. In particular, $p \sim q$ if and only if for all n, there exists m such that $p_n \precsim q_m$ and $q_n \precsim p_m$. Under the further assumption that $sr(A) = 1$, $p \sim q$ if and only if there exist increasing sequences of projections, $\{r_n\}$ and $\{s_n\}$, such that $\sum_{n=1}^{\infty}(1/2^n)r_n \sim p$, $\sum_{n=1}^{\infty}(1/2^n)s_n \sim q$ and $r_n \sim s_n$ for all n.*

Proof. Suppose that for all n, there exists m such that $p_n \precsim q_m$. Let

$$a_n = p_1 + (1/2)(p_2 - p_1) + \dots + (1/2^{n-1})(p_n - p_{n-1}).$$

Then we have that:

$$a_n \sim p_1 \oplus (1/2)(p_2 - p_1) \oplus \dots \oplus (1/2^{n-1})(p_n - p_{n-1}) \sim$$
$$\sim p_1 \oplus (p_2 - p_1) \oplus \dots \oplus (p_n - p_{n-1}) \sim p_n \precsim q_m \precsim q,$$

whence $a_n \precsim q$ for all n, so that $p = \lim_{n\to\infty} a_n \precsim q$.

Conversely, if $p \precsim q$, then $p_n \precsim q$ for all n, because if $c_n = p_1 + 2(p_2 - p_1) + \dots + 2^{n-1}(p_n - p_{n-1})$, then $c_n{}^{\frac{1}{2}} p c_n{}^{\frac{1}{2}} = p_n$. Now using [19, Proposition 2.4], we have that for all $\epsilon > 0$ there exist $\delta > 0$ and $r \in A$ such that $f_\epsilon(p_n) = r f_\delta(q) r^* \precsim f_\delta(q)$. But, if $\epsilon > 0$ is sufficiently small, we then have $p_n = f_\epsilon(p_n) \precsim f_\delta(q)$.

Finally, if m is large enough, $\|qq_m - q\| < \delta$ (because it follows easily that $\lim_{k\to\infty} qq_k = q$; in fact, $qq_k = \sum_{n=1}^{k-1}(1/2^n)q_n + (\sum_{n=k}^{\infty}(1/2^n))q_k$). Now, by [19, Proposition 2.2], $f_\delta(q) \precsim q^{\frac{1}{2}} q_m q^{\frac{1}{2}} \precsim q_m$. Summarizing, for all n, there exists m such that $p_n \precsim q_m$.

Now observe that $p \sim q \Leftrightarrow p \precsim q$ and $q \precsim p$. So, assume that for a fixed n, there exist integers m_1, m_2 such that $p_n \precsim q_{m_1}$ and $q_n \precsim p_{m_2}$. Then, choosing $m = max\{m_1, m_2\}$, we have $p_n \precsim q_m$ and $q_n \precsim p_m$.

Suppose finally that $sr(A) = 1$. If, for all n, there exists m such that $p_n \precsim q_m$ and $q_n \precsim p_m$, by using Lemma 2.2 we may renumber if necessary and thus assume that $p_1 \precsim q_1 \precsim p_2 \precsim q_2 \dots$. We have that $p_1 \sim q_1' \le q_1 \precsim p_2$ for some projection q_1'. We will find a projection q_2' satisfying $q_1 \le q_2' \le q_2$ and $q_2' \sim p_2$. There exists a projection p_2' such that $p_2 \sim p_2' \le q_2$. Then we may write $q_1' \oplus q_1'' = q_1 \precsim p_2 \sim p_2' \le q_2$ for some projection q_1'', whence there exist projections q' and q'' satisfying $q_1 \sim q' \le p_2' \le q_2$, and $q' \oplus q'' = p_2'$. Now using cancellation of projections we have $q'' \le q_2 - q' \sim q_2 - q_1$ and so $q'' \sim q''' \le q_2 - q_1$, for some projection q'''. Take $q_2' = q_1 \oplus q'''$, and notice that $q_2' \le q_1 \oplus (q_2 - q_1) = q_2$, $q_1' \le q_1 \le q_2'$, and $q_2' = q_1 \oplus q''' \sim q' \oplus q'' = p_2' \sim p_2$. Continuing in this way we can construct projections q_n' satisfying $q_n' \le q_n \le q_{n+1}'$ and $p_n \sim q_n'$ for all $n \ge 1$. Choose $r_n = p_n$ and $s_n = q_n'$. Then $\sum_{n=1}^{\infty}(1/2^n)r_n \sim p$, $\sum_{n=1}^{\infty}(1/2^n)s_n \sim q$ and $r_n \sim s_n$ for all n. The reverse implication is immediate from the first part of the result. □

In [5, Definition 6.1.2], some different relations on positive elements are introduced. If $x, y \in M_\infty(A)_+$, say that $x \sim_s y$ if and only if there is an $a \in M_\infty(A)$ such that $M_\infty(A)_x = M_\infty(A)_{a^*a}$ and $M_\infty(A)_y = M_\infty(A)_{aa^*}$. It is easy to see that $\sim_s$ is an equivalence relation and that $x \sim_s y$ implies $x \sim y$. Denote by $\{x\}$ the $\sim_s$-equivalence class of $x \in M_\infty(A)_+$ and let $W_s(A)$ be the set of all such equivalence classes. Then $W_s(A)$ is a preordered monoid with structure defined by:

$$\{x\} + \{y\} = \{x' + y'\} \text{ where } x' \sim_s x, y' \sim_s y \text{ and } x'y' = 0,$$

$$\{x\} \leq \{y\} \text{ if and only if } x \precsim y.$$

Say that $x \approx y$ if and only if there exists a sequence $\{s_j\}$ in $M_\infty(A)$ such that $(s_j^* s_j)$ (resp. $(s_j s_j^*)$) is an approximate unit for $M_\infty(A)_x$ (resp. $M_\infty(A)_y$). As noted in [5, 6.1], if $x \sim_s y$, then $x \approx y$, and it is possibly well-known that $\approx$ implies $\sim$ (see [8, Section 3]), but for completeness we include a brief proof of these facts in the corollary below.

Both relations $\sim_s$ and $\approx$ give, when restricted to projections, the usual Murray-von Neumann equivalence and although $\approx$ seems more convenient for technical reasons than $\sim_s$, it fails to be transitive in general. There is an easy example to see this. Let A be a purely infinite simple C^*-algebra (which has real rank zero, by [24, Theorem 1]), and let p and q be any two nonzero projections. Take any strictly increasing sequence of projections in A, and construct an increasing sequence $p_1 \leq q_1 \leq p_2 \leq q_2 \leq \ldots$, with $p_i \sim p$ and $q_i \sim q$ for all i (for this we use the standard fact that for a purely infinite simple C^*-algebra A and for projections $p < q$, and $r \neq 0$ in A, there exists a projection $r' \sim r$ such that $p \leq r' \leq q$). Set $x = \sum_{i=1}^{\infty} (1/2^i) p_i$. Then $p \approx x$ and $q \approx x$. Since we can choose examples where $p \not\approx q$, this shows that $\approx$ need not be transitive.

But, as it is seen in the following corollary, if the algebra has real rank zero and stable rank one, then $\approx$, $\sim_s$ and $\sim$ coincide, and so $\approx$ is transitive in that case.

Corollary 2.4 *If A is a σ-unital C^*-algebra with real rank zero and stable rank one, and $x, y \in M_\infty(A)_+$, then $x \sim y$ if and only if $x \sim_s y$, if and only if $x \approx y$. In particular, $S(A) = W_s(A)$, and if we denote by $W(A)$ the monoid of $\approx$-equivalence classes of positive elements in $M_\infty(A)$, then $S(A) = W(A)$.*

Proof. We need to show that $\sim_s$ implies $\approx$, that $\approx$ implies $\sim$, and that $\sim$ implies $\sim_s$.

We will see that for a general C^*-algebra, $\sim_s$ implies $\approx$ and $\approx$ implies $\sim$. To see that $\sim_s$ implies $\approx$, it clearly suffices to show that $a^*a \approx aa^*$ for a general a. We have that $f_{1/j}(a^*a) = a^*ah_j = h_ja^*a$, for some h_j (by functional calculus). Take $s_j = ah_j^{\frac{1}{2}}$. Then $s_j{}^*s_j = h_j{}^{\frac{1}{2}}a^*ah_j{}^{\frac{1}{2}} = a^*ah_j = f_{1/j}(a^*a)$, and $s_js_j{}^* = ah_ja^*$. Finally, note that

$$s_js_j{}^*aa^* = ah_ja^*aa^* = af_{1/j}(a^*a)a^* \to aa^*$$

since $\lim_j f_{1/j}(a^*a)a^* = a^*$. Also $s_j{}^*s_ja^*a = f_{1/j}(a^*a)a^*a \to a^*a$.

Suppose now that $x \approx y$. We know that there exists a sequence (s_j) such that $(s_j{}^*s_j)$ (resp. $(s_js_j{}^*)$) is an approximate unit for $M_\infty(A)_x$ (resp. $M_\infty(A)_y$). We have that $s_j{}^*s_j = \lim_k xw_k^j x$, where $w_k^j \in M_n(A)$. For each j, we can choose $k(j) \in \mathbb{N}$ such that $\|xw_{k(j)}^j x - s_j{}^*s_j\| < 1/j$. Moreover, we have that $y \sim y^2 = \lim_j ys_js_j{}^*s_js_j{}^*y$.

Then

$$\|ys_jxw_{k(j)}^j xs_j{}^*y - y^2\| \leq \|ys_jxw_{k(j)}^j xs_j{}^*y - ys_js_j{}^*s_js_j{}^*y\| + \|ys_js_j{}^*s_js_j{}^*y - y^2\| \leq$$

$$\|y\|^2\|xw_{k(j)}^j x - s_j{}^*s_j\|\|s_j\|^2 + \|ys_js_j{}^*s_js_j{}^*y - y^2\| \leq$$

$$\|y\|^2(1/j) + \|ys_js_j{}^*s_js_j{}^*y - y^2\| \to 0,$$

when $j \to \infty$.

Therefore $y^2 = \lim_j (ys_j)x(w_{k(j)}^j xs_j{}^*y)$, and $y^2 \precsim x$ by [19, Proposition 2.4], whence $y \sim y^2 \precsim x$. Similarly $x \precsim y$.

Now suppose that $x \sim y$ and that A has real rank zero and stable rank one. By the proof of Proposition 2.3, we may choose increasing sequences of nonzero projections, $\{p_n\}$ and $\{q_n\}$, such that $p_n \sim q_n$ and that they form approximate units for the hereditary C^*-subalgebras A_x and A_y respectively. Set $p_0 = 0$ and $q_0 = 0$. Using cancellation on projections, $p_n - p_{n-1} \sim q_n - q_{n-1}$. Then there exist partial isometries w_n such that $w_nw_n^* = p_n - p_{n-1}$ and $w_n^*w_n = q_n - q_{n-1}$. Define $a = \sum_{n=1}^{\infty}(1/2^{\frac{n-1}{2}})w_n$. Then $aa^* = \sum_{n=1}^{\infty}(1/2^{n-1})(p_n - p_{n-1})$ and $a^*a = \sum_{n=1}^{\infty}(1/2^{n-1})(q_n - q_{n-1})$. Clearly $A_{aa^*} \subseteq A_x$. Set $t_n = p_1 + 2(p_2 - p_1) + \ldots + 2^{n-1}(p_n - p_{n-1})$. Then $t_naa^* = aa^*t_n = p_n$, whence $A_{aa^*} = A_x$. Similarly, $A_{a^*a} = A_y$. □

Suppose now that we are given an element $a \in M_\infty(A)_+$. Then we can define the following set:

$$I(a) = \{[p] \in V(A) \mid p \in \overline{aM_\infty(A)a}\}.$$

Lemma 2.5 *Let A be a C^*-algebra with real rank zero. Let $a, b \in M_\infty(A)_+$. Let $p_1 \leq p_2 \leq \ldots$(resp. $q_1 \leq q_2 \leq \ldots$) be an approximate unit for the hereditary C^*-subalgebra $M_\infty(A)_a$ (resp. $M_\infty(A)_b$). Then:*

1) *$I(a) = \{[p] \mid p \precsim a\} = \{[p] \mid p \precsim p_n$ for some $n\}$. In particular, $I(a)$ is a countably generated interval over $V(A)$.*

2) *$I(a) \subseteq I(b)$ if and only if $a \precsim b$.*

3) *If $ab = 0$, then $I(a + b) = I(a) + I(b)$.*

Proof. (1). We first prove that $I(a) = \{[p] \mid p \precsim a\}$. Clearly, if $[p] \in I(a)$, then $p \precsim a$. For the converse, if $[p] \in \{[q] \mid q \precsim a\}$, let B be a common matrix algebra over A such that $a, p \in B$ and $p \precsim a$ in B. Then for all $\epsilon > 0$, there exists $\delta > 0$, and $r \in B$ such that $f_\epsilon(p) = rf_\delta(a)r^*$, and since $f_\epsilon(p) = p$ if $0 < \epsilon < 1$, we get that $p = rf_\delta(a)r^*$. By [2, Theorem 9.3 (c)], there is a projection $q \in aBa$ such that $f_\delta(a) \in qBq$. In particular $p \precsim q$, whence there exists a projection $p' \in aBa$ with $p' \sim p$. Consequently, $[p] \in I(a)$.

By Lemma 2.2 and Proposition 2.3, it is clear that $p \lesssim a$ if and only if $p \lesssim p_n$ for some n (the algebra $pAp = \overline{pAp}$ has $\{p\}$ as an approximate unit).

(2). Suppose that $a \lesssim b$. Then, by Proposition 2.3, for all n, there exists m such that $p_n \lesssim q_m$. So, if $[p] \in I(a)$, then there exists n such that $p \lesssim p_n$ and therefore there exists m such that $p \lesssim p_n \lesssim q_m$. Thus $[p] \in I(b)$.

Conversely, if $I(a) \subseteq I(b)$, it is obvious that $[p_n] \in I(a) \subseteq I(b)$ for all n, so that there exists m with $p_n \lesssim q_m$. Therefore $a \lesssim b$ by Proposition 2.3.

(3). If $[p] \in I(a)$ and $[q] \in I(b)$, then note that $[p] + [q] = [p \oplus q]$ and since $p \lesssim a$, and $q \lesssim b$ this implies that $p \oplus q \lesssim a + b$. Therefore, $[p] + [q] \in I(a + b)$. Conversely, if $[e] \in I(a + b)$, then $e \lesssim a + b$. Since $\{p_n\}$ (resp. $\{q_n\}$) is an approximate unit for $\overline{aM_\infty(A)a}$ (resp. $\overline{bM_\infty(A)b}$), $\begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix}$ is an approximate unit for the hereditary C^*-algebra generated by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then there exists $n \in \mathbb{N}$ such that $e \lesssim \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix}$. By Riesz decomposition on projections ([23, Theorem 1.1]), $e = r_1 \oplus r_2$ for some projection r_i such that $r_1 \lesssim p_n$ and $r_2 \lesssim q_n$. Therefore $[r_1] \in I(a)$, $[r_2] \in I(b)$ and $[e] = [r_1] + [r_2] \in I(a) + I(b)$. $\square$

Definition 2.6 *Let A be a C^*-algebra. Define the set*

$$D(A) = \{[p] \in V(A) \mid p \textit{ is a projection in } A\}.$$

If A is σ-unital and if $\{e_n\}$ is an increasing approximate unit for A consisting of projections, then $D(A)$ can be described as a countably generated interval which has $\{[e_n]\}$ as a countable cofinal subset.

Proposition 2.7 *Let A be a σ-unital C^*-algebra with real rank zero. Then for any $a \in M_\infty(A)_+$, $I(a)$ is a countably generated interval of $V(A)$, bounded by $nD(A)$ for some positive integer $n \geq 1$. If we assume, further, that the stable rank of A is one, then every countably generated interval of $V(A)$ bounded by $nD(A)$ for some n is of the form $I(a)$ for some $a \in M_\infty(A)_+$.*

Proof. Using Lemma 2.5 (1), it is clear that $I(a)$ is a countably generated interval of $V(A)$. Let e be the standard element associated to some approximate identity for A consisting of projections. If $a \in M_t(A)$ for some t, then $I(a) \subseteq tI(e) = tD(A)$, whence $I(a)$ is bounded by $tD(A)$. Let I be any countably generated interval of $V(A)$ bounded by $nD(A)$ for some n, with generators $x_1, x_2, \ldots$. Comparing x_1, x_2, we can choose $x'_2 \in I$ such that $x'_2 \geq x_1, x_2$, because I is upward directed. It is clear that $x_1, x'_2, x_3, \ldots$ generate I as well. Therefore, by induction on n, there exist x'_n ($x'_1 = x_1$) such that $x'_n \leq x'_{n+1}$ for all n and I is generated by $x'_1, x'_2, \ldots$. So, given any countably generated interval, we can choose as generators an increasing sequence. Consider then I a countably generated interval with an increasing sequence of generators, say $x_i = [h_i]$, and assume that $I \subseteq nD(A)$ for some n. Since I is bounded by $nD(A)$, we can choose h_i in $M_n(A)$ for all i. Assume for simplicity that $h_i \in A$ for all i. We have that $h_1 \lesssim h_2 \lesssim h_3 \lesssim \ldots$. As

$h_1 \lesssim h_2$, there exists $h'_1 \leq h_2$ with $h_1 \sim h'_1$. Then $h_2 = h'_1 \oplus s_2$, and $s_2 \leq 1 - h'_1$ (in $\widetilde{A}$). Using cancellation on projections, we have that $1-h'_1 \sim 1-h_1$, whence $s_2 \sim s'_2 \leq 1-h_1$ for some $s'_2 \in A$ (because $s_2 \in A$). Take $r_1 = h_1, r_2 = r_1 \oplus s'_2$, and notice that $r_1 \leq r_2, r_2 \sim h_2$. Continuing by induction in the same way, we construct an increasing sequence of projections (in A) $\{r_n\}$ with $r_n \sim h_n$ for all n. Define $a = \sum_{n=1}^{\infty}(1/2^n)r_n \in A$. Now using Lemma 2.5 (1) it is clear that $I(a) = I$. □

Let M be a monoid. Let $\Lambda_\sigma(M)$ be the monoid (under addition) of countably generated intervals over M, (partially) ordered under set inclusion. Fix $D \in \Lambda_\sigma(M)$, and let $\Lambda_{\sigma,D}(M)$ be the hereditary submonoid of $\Lambda_\sigma(M)$ generated by D. In other words, the elements of $\Lambda_{\sigma,D}(M)$ are countably generated intervals over M bounded by nD for some n. If A is a σ-unital C^*-algebra and has an approximate unit $\{e_n\}$ consisting of projections, take e to be the standard element associated to $\{e_n\}$ and let $u = \langle e \rangle$. Then u is an order-unit for $S(A)$ by Lemma 2.2, and D is an order-unit for $\Lambda_{\sigma,D}(V(A))$. In case A is unital, the monoid $\Lambda_{\sigma,D}(V(A))$ consists of those countably generated intervals over $V(A)$ whose generators are bounded above by some element in $V(A)$.

Theorem 2.8 *Let A be a σ-unital C^*-algebra with real rank zero. Then there exists a normalized ordered monoid isomorphism from $S(A)$ to a submonoid of $\Lambda_{\sigma,D}(V(A))$. If, further, A has stable rank one, then $S(A)$ and $\Lambda_{\sigma,D}(V(A))$ are isomorphic as ordered monoids.*

Proof. Define
$$\Theta : S(A) \to \Lambda_{\sigma,D}(V(A))$$
by $\Theta(\langle a \rangle) = I(a)$. By Lemma 2.5 and Proposition 2.7, Θ is well-defined. Moreover,
$$\Theta(\langle a \rangle + \langle b \rangle) = I\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = I\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + I\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \Theta(\langle a \rangle) + \Theta(\langle b \rangle),$$
$\langle a \rangle \leq \langle b \rangle$ if and only if $\Theta(\langle a \rangle) = I(a) \subseteq I(b) = \Theta(\langle b \rangle)$, and $\Theta(u) = D$. Therefore, Θ is a normalized ordered monoid isomorphism from $S(A)$ to a submonoid of $\Lambda_{\sigma,D}(V(A))$. The last part of Proposition 2.7 shows that Θ is an order-isomorphism if $sr(A) = 1$. □

The next results show the structure of $S(A)$ as a partially ordered monoid. The first two are known, and we include some little re-statements.

Definition 2.9 *Let G be a directed partially ordered abelian group. If G^+ satisfies the Riesz decomposition property, then we say that G is a* **Riesz group** *or an* **interpolation group**.

Proposition 2.10 *[11, Lemma 2.6 (a)] Let G be a Riesz group and D a countably generated interval over G^+. Let I, J, K be elements in $\Lambda_{\sigma,D}(G^+)$ such that $I \subseteq J + K$. Then $I = I_1 + I_2$ for some intervals $I_i \in \Lambda_{\sigma,D}(G^+)$, such that $I_1 \subseteq J$ and $I_2 \subseteq K$.* □

Proposition 2.11 *[11, Proposition 2.5] Let G be a Riesz group and D a countably generated interval in G^+. Let I_1, I_2, J_1, J_2 be elements in $\Lambda_{\sigma,D}(G^+)$, such that $I_1 + I_2 = J_1 + J_2$. Then there exist intervals X_{ij} in $\Lambda_{\sigma,D}(G^+)$ such that $I_i = X_{i1} + X_{i2}$ for each i and $J_j = X_{1j} + X_{2j}$ for each j.* $\square$

Proposition 2.12 *Let M be a partially ordered monoid. Let D be a countably generated interval over M. Then the following conditions are equivalent:*

1) *M is an interpolation monoid;*

2) *$\Lambda(M)$ is an interpolation monoid;*

3) *$\Lambda_\sigma(M)$ is an interpolation monoid;*

4) *$\Lambda_{\sigma,D}(M)$ is an interpolation monoid.*

Proof. 1) $\Leftrightarrow$ 2) is essentially [22, Lemma 1.6].

4) $\Rightarrow$ 1). Let $x, y, z, t \in M$ such that $x, y \leq z, t$. For any $s \in M$, define $I(s) \in \Lambda_{\sigma,D}(M)$ by the rule $I(s) = \{w \in M \mid w \leq s\}$. It is clear that $I(x), I(y) \subseteq I(z), I(t)$, hence there exists an interval J in $\Lambda_{\sigma,D}(M)$ such that $I(x), I(y) \subseteq J \subseteq I(z), I(t)$. As $x, y \in J$ and J is upward directed, there exists $w \in J$ such that $x, y \leq w$. But $w \in I(z), I(t)$. Therefore, $x, y \leq w \leq z, t$, as wanted.

1) $\Rightarrow$ 3). Suppose we have $I_i, J_j \in \Lambda_\sigma(M)$ for $i, j = 1, 2$ such that $I_1, I_2 \subseteq J_1, J_2$. Take increasing sequences in M generating I_i, J_j, namely $\{x_{in}\}$ and $\{y_{jn}\}$ for $i, j = 1, 2$. As J_1, J_2 are upward directed, for x_{11} and x_{21} there exist elements $y_{1,m(1)} \in J_1$ and $y_{2,m(1)} \in J_2$ satisfying $x_{11}, x_{21} \leq y_{1,m(1)}, y_{2,m(1)}$. By the interpolation property assumed on M, there exists $t_1 \in M$ such that $x_{11}, x_{21} \leq t_1 \leq y_{1,m(1)}, y_{2,m(1)}$. Assume that for $n \geq 1$, a sequence of elements $t_1 \leq \ldots \leq t_n$ in M satisfying $x_{1n}, x_{2n} \leq t_n \leq y_{1,m(n)}, y_{2,m(n)}$, for $y_{1,m(n)} \in J_1$ and $y_{2,m(n)} \in J_2$, has been constructed. As $x_{1,n+1}, x_{2,n+1}, t_n \in J_1, J_2$, there exist elements $y_{1,m(n+1)} \in J_1$ and $y_{2,m(n+1)} \in J_2$ such that $x_{1,n+1}, x_{2,n+1}, t_n \leq y_{1,m(n+1)}, y_{2,m(n+1)}$. Using the interpolation property, there exists $t_{n+1} \in M$ such that $x_{1,n+1}, x_{2,n+1}, t_n \leq t_{n+1} \leq y_{1,m(n+1)}, y_{2,m(n+1)}$. Set $K = \{w \in M \mid \exists n \text{ such that } w \leq t_n\}$. Then K is nonempty, hereditary and upward directed. Moreover, as for each n, $x_{1n}, x_{2n} \leq t_n \leq y_{1,m(n)}, y_{2,m(n)}$, we have that $I_1, I_2 \subseteq K$ and $K \subseteq J_1, J_2$. It follows that $K \in \Lambda_\sigma(M)$ is an interpolation element for $I_1, I_2 \subseteq J_1, J_2$.

3) $\Rightarrow$ 4) is clear. $\square$

To conclude this section, we summarize the previous results in the context of C^*-algebras.

Theorem 2.13 *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. Then the following conditions hold:*

1) *$S(A)$ satisfies the Riesz decomposition property.*

2) *$S(A)$ is a refinement monoid.*

3) $S(A)$ *satisfies the interpolation property.*

4) $S(A)$ *is upper monotone* σ*-complete, that is, every bounded ascending sequence of elements in* $S(A)$ *has a supremum in* $S(A)$.

Proof. Notice that $V(A) = K_0(A)^+$ since $sr(A) = 1$. By [23, Theorem 1.1], $V(A)$ has Riesz decomposition, and by [10, Proposition 2.1] it also has interpolation. By Theorem 2.8, $S(A) \cong \Lambda_{\sigma,D}(V(A))$. Apply then Propositions 2.10, 2.11, 2.12 to get (1), (2) and (3). For (4), let $I_1 \subseteq I_2 \subseteq I_3 \dots$ be a bounded increasing sequence of intervals in $\Lambda_{\sigma,D}(V(A))$. Then $I = \cup_{n=1}^{\infty} I_n$ is an interval and, in fact, since each I_n is countably generated, then so is I. Note that $I \subseteq tD$ for some positive integer t because $I_n \subseteq tD$ for all n. Therefore I is the order-supremum of the sequence $\{I_n\}$, hence $\Lambda_{\sigma,D}(V(A))$ is an upper monotone σ-complete monoid, and so is $S(A)$. □

Using the Theorem, it can be proved that $S(A)$ satisfies the following weaker form of countable interpolation: whenever $\{x_i\}_{i\in\mathbb{N}}$ and y_j for $j = 1, \dots n$ are elements in $S(A)$ satisfying $x_i \leq y_j$ for each i and j, then there exists $z \in S(A)$ such that $x_i \leq z \leq y_j$ for every i and j.

3 Some applications

By using the representation of $S(A)$ as countably generated intervals over $V(A)$, in this section we characterize the lower semicontinuous dimension functions on A as σ-normal states on $S(A)$. We also show that the (FCQ), the stable (FCQ) and the (FCQ+) are equivalent for unital simple C^*-algebras with real rank zero, and that they hold if the monoid $V(A)$ has strict unperforation. It is finally shown that $S(A)$ being Riesz determines to some extent the real rank of the algebra. The techniques used to prove this fact give a generalization of [19, Theorem 7.2] in two different directions.

Definition 3.1 *Let* M *be a preordered monoid with order-unit* u*. A* **state** *on* (M, u) *is a positive monoid morphism* $s : M \to \mathbb{R}$ *such that* $s(u) = 1$*. We denote by* $St(M, u)$ *the set of states on* (M, u)*. A state* $s \in St(M, u)$ *is said to be* σ**-normal** *if whenever we have an increasing sequence* $\{a_n\}$ *in* M *and* $a = \sup_M a_n$ *exists, then* $s(a) = \sup_n s(a_n)$.

Let A be a σ-unital C^*-algebra and $s \in St(S(A), u)$. Then s induces a function d_s on $M_\infty(A)$ defined by:

$$d_s(a) = s\langle a^*a\rangle, \text{ for } a \in M_\infty(A).$$

If $a \geq 0$, then $d_s(a) = s\langle a\rangle$. Denote by $LDF(A)$ the set of **lower semicontinuous dimension functions** in A, as in [7].

Lemma 3.2 *Let* A *be a* σ*-unital* C^**-algebra with real rank zero and stable rank one. Let* $\langle x_1\rangle \leq \langle x_2\rangle \leq \langle x_3\rangle \leq \dots$ *be an increasing sequence in* $S(A)$*. Then* $\langle x\rangle = \sup_{S(A)}\langle x_n\rangle$ *if and only if* $I(x) = \cup_{n=1}^{\infty} I(x_n)$.

Proof. Assume first that $\langle x\rangle = \sup_{S(A)}\langle x_n\rangle$. Then $\cup_{n=1}^{\infty} I(x_n) \subseteq I(x)$ because $x_n \precsim x$ for all n (see Lemma 2.5 (2)). Since $\cup_{n=1}^{\infty} I(x_n)$ is a countably generated interval over $V(A)$ bounded by $I(x)$, we conclude from Proposition 2.7 that there exists $y \in M_\infty(A)_+$ such that $I(y) = \cup_{n=1}^{\infty} I(x_n)$. Then $x_n \precsim y$ for all n and since $\langle x\rangle$ is the order-supremum of $\langle x_n\rangle$, we get that $x \precsim y$. Therefore $I(x) \subseteq I(y) = \cup_{n=1}^{\infty} I(x_n)$, whence equality holds.

Conversely, if $I(x) = \cup_{n=1}^{\infty} I(x_n)$, then $x_n \precsim x$ for all n. If there is $y \in M_\infty(A)_+$ such that $x_n \precsim y$ for all n, then $I(x) = \cup_{n=1}^{\infty} I(x_n) \subseteq I(y)$, that is, $x \precsim y$. Thus $\langle x\rangle = \sup_{S(A)}\langle x_n\rangle$. $\square$

Proposition 3.3 *Let A be a σ-unital C^*-algebra and $s \in St(S(A), u)$. If s is σ-normal, then d_s is lower semicontinuous. Under the further assumptions that A has real rank zero and stable rank one, the converse is also true.*

Proof. We will need the following facts: $\lim_{\epsilon\to 0} x f_\epsilon(x) = x$ and $x f_\epsilon(x) \sim f_\epsilon(x)$ for all $x \in M_\infty(A)_+$. Note first that a dimension function d is lower semicontinuous if and only if $d(x) = \sup_{\epsilon>0} d(f_\epsilon(x))$ for all $x \in M_\infty(A)_+$. For, assume first that $d \in LDF(A)$. If $x \in M_n(A)_+$ for some n, then $x f_\epsilon(x) \precsim x$, and thus $d(x f_\epsilon(x)) \le d(x)$. Consequently:

$$\limsup_{\epsilon>0} d(x f_\epsilon(x)) \le d(x).$$

On the other hand, $d(x) \le \liminf_{\epsilon>0} d(x f_\epsilon(x))$ by lower semicontinuity. Thus $d(x) = \sup_{\epsilon>0} d(x f_\epsilon(x)) = \sup_{\epsilon>0} d(f_\epsilon(x))$. For the converse, use [7, Proposition I.1.5].

Thus $d_s \in LDF(A)$ if and only if $d_s(x) = \sup_{\epsilon>0} d_s(f_\epsilon(x))$ for all $x \in M_\infty(A)_+$, if and only if $s\langle x\rangle = \sup_{\epsilon>0} s\langle f_\epsilon(x)\rangle$ for all $x \in M_\infty(A)_+$, that is, $s\langle \sup_{\epsilon>0} x f_\epsilon(x)\rangle = \sup_{\epsilon>0} s\langle x f_\epsilon(x)\rangle$ for all $x \in M_\infty(A)_+$. And since $\sup_{\epsilon>0}\langle f_\epsilon(x)\rangle = \langle x\rangle$ (see Lemma 1.1), the first part of the result follows.

Suppose now that d_s is lower semicontinuous, that A has real rank zero and stable rank one. If $D = D(A)$, then $S(A) \cong \Lambda_{\sigma,D}(V(A))$, by Theorem 2.8. Note that if for $y \in M_\infty(A)_+$, there is an approximate unit consisting of an increasing sequence of projections $\{p_n\}$ for the hereditary C^*-algebra generated by y, then $I(y) = \cup_{n=1}^{\infty} I(p_n)$, by Lemma 2.5 (1).

Suppose that $\langle x\rangle = \sup_{S(A)}\langle x_n\rangle$ with $\langle x_1\rangle \le \langle x_2\rangle \le \dots$ an increasing sequence in $S(A)$. Then $I(x) = \cup_{n=1}^{\infty} I(x_n)$ by Lemma 3.2. For each n, take $\{p_{n,k}\}$ to be an approximate unit of (increasing) projections for $M_\infty(A)_{x_n}$. Then $I(x) = \cup_n I(x_n) = \cup_n \cup_k I(p_{n,k})$. Let $\{q_m\}$ be an approximate unit of (increasing) projections for $M_\infty(A)_x$. Then $\lim_{m\to\infty} q_m x = x$. Now, for all m, there exist indices n and k such that $q_m \precsim p_{n,k} \precsim x_n$; hence for all m, there exists n such that $q_m \precsim x_n$. Since d_s is lower semicontinuous, $s\langle x\rangle \le \liminf_m s\langle q_m x\rangle \le \liminf_m s\langle q_m\rangle \le \sup_n s\langle x_n\rangle$. On the other hand $s\langle x_n\rangle \le s\langle x\rangle$ for all n since $\langle x\rangle = \sup_{S(A)}\langle x_n\rangle$. Therefore $s\langle x\rangle = \sup_n s\langle x_n\rangle$ and thus s is a σ-normal state. $\square$

Definition 3.4 *Let M be a monoid. For $x, y \in M$, we say that $x \leq^* y$ provided that there exists $0 \neq z \in M$ such that $x + z = y$. We say that M is* **strictly unperforated** *if whenever $nx \leq^* ny$ for some natural number n, it follows that $x \leq^* y$. If $(M, \leq)$ is already preordered, we say that $x < y$ if $x \leq y$ and $x \neq y$. Finally, $(M, \leq)$ is said to be* **almost unperforated** *provided that whenever $(m+1)x \leq my$ for $m \in \mathbb{N}$ and $x, y \in M$, it follows that $x \leq y$.*

In [19, Theorem 5.2(a)], Rørdam gives a proof of the **Comparability Question** for positive elements in a simple unital C^*-algebra tensored with a UHF-algebra. We show that answering the comparability question for positive elements is equivalent to answering the comparability question for projections for simple unital C^*-algebras with real rank zero. As a consequence, if a simple unital C^*-algebra has real rank zero and strict unperforation on the monoid of isomorphism classes of projections, then a positive answer to the (FCQ+) will follow.

Lemma 3.5 *Let A be a simple unital C^*-algebra with $S(A)$ almost unperforated. If $x, y \in M_\infty(A)_+$ satisfy $d(x) < d(y)$ for all $d \in LDF(A)$, then $x \precsim y$.*

Proof. See the proof of [19, Theorem 5.2 (a)]. □

Definition 3.6 *Let M be a monoid. An* **atom** *of M is a nonzero element $a \in M$ such that there is no $b \in M$ in M satisfying $0 \leq^* b \leq^* a$. We say that M is an* **atomic monoid** *if each element of M can be written as a sum of atoms. We say that M is* **conical** *if the set M^* of nonzero elements is closed under addition.*

The following Lemma is known:

Lemma 3.7 *Let M be a simple conical refinement monoid with order-unit u. Then the following conditions are equivalent:*

1) *M is strictly unperforated;*

2) *If $nx \leq^* ny$ for some $n \in \mathbb{N}$ and $x, y \in M$ such that $x, y \leq u$, then $x \leq^* y$;*

3) *For every nonzero $x \in M$ such that $x \leq u$, there exists $n \in \mathbb{N}$ such that, if $y \in M$ and $ny \leq u$, then $y \leq x$.*

Proof. If M is atomic, then by [1, Lemma 1.6] it is isomorphic to the infinite cyclic monoid. Thus the conditions of the Lemma hold trivially in that case, hence we may assume that M is nonatomic.

1) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 3). Let $x \in M - \{0\}$ and suppose that $x \leq u$. Since M is simple, there exists $n \in \mathbb{N}$ such that $u \leq (n-1)x \leq^* nx$. Then $u \leq^* nx$. Suppose that $y \in M$ satisfies $ny \leq u$. Then $ny \leq^* nx$, whence $y \leq^* x$.

3) $\Rightarrow$ 1). If $a + u = u$ for some $a \in M^*$, then $x \leq^* y$ for all $x, y \in M^*$, by [1, Proposition 1.4 (b)].

Assume that $x + u = u$ implies $x = 0$ for $x \in M$. Then by [1, Corollary 1.8] M is cancellative. Therefore M equals the positive cone of a Riesz group. By [3, Theorem 4.2], M is strictly unperforated. □

Condition 3) in the Lemma above was labelled as weak comparability in [1] (see also [14]).

Definition 3.8 *Let A be a (unital) C^*-algebra. A (normalized)* **quasi-trace** *is a function $\tau : A \to \mathbb{C}$ satisfying*

a) $\tau(1) = 1$.

b) $0 \leq \tau(xx^*) = \tau(x^*x)$, *for all* $x \in A$.

c) $\tau(a + ib) = \tau(a) + i\tau(b)$, *for all* $a, b \in A_{sa}$.

d) τ *is linear on abelian C^*-subalgebras of A.*

e) τ *extends to functions from each $M_n(A)$ to $\mathbb{C}$ satisfying $b) - d)$.*

The set of all quasi-traces is denoted by $QT(A)$.

If $\tau \in QT(A)$ and we let $d_\tau(x) = \lim_{\epsilon \to 0} \tau(f_\epsilon(x))$ for $x \in M_\infty(A)_+$, then $d_\tau \in LDF(A)$ and in fact, if $d \in LDF(A)$, then there exists $\tau \in QT(A)$ such that $d = d_\tau$. It turns out that this correspondence is a bijection (see [7, Section II.2]).

Definition 3.9 *Let A be a unital simple C^*-algebra. We say that the (FCQ) holds for A (or that A has strict comparability) provided that whenever p and q are projections in A such that $\tau(p) < \tau(q)$ for each quasi-trace τ, it follows that $p \prec q$. Similarly, we say that the (FCQ+) holds for A provided that whenever $x, y \in M_\infty(A)_+$ satisfy $d(x) < d(y)$ for all $d \in LDF(A)$, then $x \lesssim y$.*

Although some of the arguments in the Corollary below are standard, we include them for completeness.

Corollary 3.10 *Let A be a unital simple C^*-algebra with real rank zero. The following conditions are equivalent:*

1) *The (FCQ) holds for A;*

2) *The (FCQ) holds for $M_n(A)$ for all $n \geq 1$;*

3) *The (FCQ+) holds for A.*

Proof. 1) $\Rightarrow$ 3). Let $M = V(A)$ and $u = [1_A]$. Then M is simple and conical. By [1, Lemma 2.3], M is refinement. We show that M is strictly unperforated.

Suppose that $nx \leq^* ny$ for $n \in \mathbb{N}$ and $x, y \leq u$. Write $x = [p]$ and $y = [q]$ for some projections p, q in A. Therefore $np \prec nq$ in $M_n(A)$, that is, $np \sim r < nq$ for some projection $r \in M_n(A)$. Let $\tau \in QT(A)$. The set $I = \{z \in A \mid \tau(zz^*) = 0\}$ is a closed two-sided ideal (see, e.g., [7, Theorem II.2.2, Theorem I.1.17]), hence it is zero by the simplicity of A. From this, we get that $n\tau(p) < n\tau(q)$, hence $\tau(p) < \tau(q)$. Then $p \prec q$ by hypothesis, that is, $x \leq^* y$. By Lemma 3.7, M is strictly unperforated.

Now observe that $\Lambda_{\sigma,D}(M)$ is almost unperforated. Suppose that for $m \in \mathbb{N}$ and $I, J \in \Lambda_{\sigma,D}(M)$, we have that $(m+1)I \subseteq mJ$. Then, if $a \in I$, there exists $b \in J$ such that $(m+1)a \leq mb \leq^* (m+1)b$ (using that J is upward directed). Since M is strictly unperforated we conclude that $a \leq^* b$, and hence $I \subseteq J$.

By Theorem 2.8, $S(A)$ is order-isomorphic to a submonoid of $\Lambda_{\sigma,D}(M)$, and therefore $S(A)$ is almost unperforated. Finally, the (FCQ+) holds for A by Lemma 3.5.

3) $\Rightarrow$ 2). Let $p, q \in M_n(A)$ be projections, and suppose that $\tau(p) < \tau(q)$ for every quasi-trace τ. Let d_τ be the lower semicontinuous dimension function corresponding to τ, and note that $d_\tau(p) = \tau(p) < \tau(q) = d_\tau(q)$. Therefore $p \prec q$.

2) $\Rightarrow$ 1) is clear. □

Corollary 3.11 *Let A be a unital simple C^*-algebra with real rank zero. Assume that $V(A)$ is strictly unperforated. If $x, y \in M_\infty(A)_+$ satisfy $d(x) < d(y)$ for all $d \in LDF(A)$, then $x \precsim y$.* □

Using the same kind of techniques, it is possible to give a characterization of the real rank zero condition for certain C^*-algebras A in terms of the monoid $S(A)$. To achieve this, some preliminary notions and results are needed.

Proposition 3.12 *(P. Ara) Let A be a unital C^*-algebra with stable rank one, and let $x \in M_\infty(A)_+$. Then $\langle x \rangle = \langle p \rangle$, for p a projection, if and only if 0 is an isolated point in* $\mathrm{Spec}(x)$ *or* $0 \notin \mathrm{Spec}(x)$.

Proof. The "if" condition is clear. Suppose that $\langle x \rangle = \langle p \rangle$, for p a projection and that 0 is a non-isolated point of $\mathrm{Spec}(x)$. There exist $\delta > 0$ and a projection $q \sim p$ such that $f_\delta(x)q = q = qf_\delta(x)$. To see the latter, apply [19, Proposition 2.4] to the fact that $p \precsim x$, and so if $0 < \epsilon < 1$, there exists $\delta' > 0$ and $r \in A$ such that $p = f_\epsilon(p) = rf_{\delta'}(x)r^* = vv^*$, where $v = rf_{\delta'}(x)^{\frac{1}{2}}$. Then $q = v^*v = f_{\delta'}(x)^{\frac{1}{2}}r^*rf_{\delta'}(x)^{\frac{1}{2}}$ is a projection equivalent to p and if $\delta = \delta'/2$, it follows that $f_\delta(x)q = q$.

Moreover, as 0 is not an isolated point in $\mathrm{Spec}(x)$, it is clear that $f_\delta(x)$ is not a projection. Therefore $f_\delta(x) \neq q$. Notice that:

$$q = f_\delta(x)^{\frac{1}{2}}qf_\delta(x)^{\frac{1}{2}} \leq f_\delta(x)^{\frac{1}{2}}1_A f_\delta(x)^{\frac{1}{2}} = f_\delta(x).$$

Now, as $x \precsim q$, we have that $uf_\delta(x)u^* \in qAq$ for some $u \in U(A)$, by [19, Proposition 2.4 (v)], hence $uf_\delta(x)u^* \leq \|f_\delta(x)\|uu^* \leq 1$ and so $uf_\delta(x)u^* = quf_\delta(x)u^*q \leq q$. But $q + (f_\delta(x) - q) = f_\delta(x)$, and this implies

$$uqu^* + u(f_\delta(x) - q)u^* \leq q,$$

where $u(f_\delta(x) - q)u^* > 0$. Thus $uqu^* < q$, a contradiction (because $sr(A) = 1$ and in particular A is stably finite). □

Definition 3.13 *Let $(M, \leq, u)$ be a positively ordered monoid (that is, for all $x \in M$, $x \geq 0$) with order-unit, and let $\Phi : M \to \mathrm{Aff}(St(M,u))^+$ be the natural map. We say that M satisfies* **condition (D)** *provided that $\Phi(M)$ is dense in $\mathrm{Aff}(St(M,u))^+$. A unital ring R satisfies condition (D) if the positive cone of its Grothendieck group $(K_0(R), [R])$ satisfies condition (D).*

The following result is contained essentially in [15].

Theorem 3.14 *Let A be a stably finite, simple, infinite dimensional, unital C^*-algebra with real rank zero. Then A satisfies condition (D).*

Proof. By [2, Theorem 9.3], a C^*-algebra has real rank zero if and only if it is an exchange ring, in the sense of [21]. Therefore the hypothesis of [15, Corollary 4.6] are fulfilled. □

Definition 3.15 *([6, Definition 6.1.1]) Let A be a (unital) C^*-algebra. We say that A satisfies* **condition (PT)** *provided that projections separate quasi-traces, that is, whenever $\tau, \tau' \in QT(A)$ are different, then there exists a projection $p \in M_\infty(A)$ such that $\tau(p) \neq \tau'(p)$.*

M. Rørdam, in [19, Theorem 7.2], gives a criterion to decide when the real rank of a stably finite simple C^*-algebra with finitely many extremal quasi-traces tensored with a UHF-algebra is zero. A similar proof can be used in more general circumstances. We sketch the argument for the reader's convenience.

Theorem 3.16 *Let A be a (unital) simple infinite dimensional C^*-algebra with stable rank one and $S(A)$ almost unperforated. Then*

$$RR(A) = 0 \Leftrightarrow A \text{ satisfies conditions (D) and (PT)} .$$

Proof. ($\Rightarrow$) That A satisfies condition (D) follows from Theorem 3.14. For condition (PT), see [7] or [11, 12.1].

($\Leftarrow$) Assume that A satisfies both conditions (D) and (PT). Given $x \in A_+$, we will find projections $p_j \in A$ such that:

$$f_{\delta_1}(x) \leq p_1 \leq f_{\delta_2}(x) \leq p_2 \leq f_{\delta_3}(x) \leq \dots$$

where $\delta_j = 16^{-j}2^{-j}$. Clearly this implies that A has real rank zero.

It suffices then to show that for each $\delta > 0$, there exists a projection p such that $f_{2\delta}(x) \leq p \leq f_{\delta/16}(x)$. Fix $\delta > 0$. If $\mathrm{Spec}(x) \cap (\delta/4, \delta/2) = \emptyset$, then $f_{\delta/2}(x)$ is a projection and we can choose $p = f_{\delta/2}(x)$. If $\mathrm{Spec}(x) \cap (\delta/4, \delta/2) \neq \emptyset$, then we have that

$$f_\delta(x) \leq f_{\delta/2}(x) < f_{\delta/4}(x) \leq f_{\delta/8}(x).$$

Therefore, since $f_{\delta/2}(x)$ and $f_{\delta/4}(x)$ commute, we conclude from the simplicity of A that $\tau(f_{\delta/2}(x)) < \tau(f_{\delta/4}(x))$ for each $\tau \in QT(A)$.

Now consider the functions $f, g \in \mathrm{Aff}QT(A)$, defined by:

$$f(\tau) = \tau(f_{\delta/2}(x)), \quad g(\tau) = \tau(f_{\delta/4}(x)), \text{ for } \tau \in QT(A).$$

It is clear that f, g are affine continuous functions and that $f \ll g$. There exists $h \in \mathrm{Aff}QT(A)$ such that $f \ll h \ll g$. Then we can choose an $\epsilon > 0$ with $f \ll h - \epsilon \ll h + \epsilon \ll g$. Recall from [8, Theorem 3.3] that the natural map from $QT(A)$ to $St(K_0(A), [1_A])$ given by evaluation on quasi-traces is surjective. The condition (PT) assumed on A ensures that this map is in fact an homeomorphism of compact convex sets. Therefore $\mathrm{Aff}QT(A)$ and $\mathrm{Aff}(St(K_0(A), [1_A]))$ are isomorphic, and consequently we denote by Φ' the natural representation of $K_0(A)^+$ in $\mathrm{Aff}QT(A)$. It follows that $\Phi'(K_0(A))^+$ is dense in $\mathrm{Aff}QT(A)^+$, because A satisfies condition (D). Then we get a projection $p \in M_\infty(A)_+$ with $\|h - \Phi'([p])\| < \epsilon$. Therefore, $f \ll \Phi'([p]) \ll g$.

This means that $\tau(f_{\delta/2}(x)) < \tau(p) < \tau(f_{\delta/4}(x))$ for all $\tau \in QT(A)$.

Let d_τ be the lower semicontinuous dimension function associated to τ, and notice that $f_\epsilon(f_\delta(x)) \leq f_{\delta/2}(x)$ and that $f_{\delta/4}(x) \leq f_\epsilon(f_{\delta/8}(x))$ for all $0 < \epsilon < 1$. Hence, $\tau(f_\epsilon(f_\delta(x))) \leq \tau(f_{\delta/2}(x))$ and $\tau(f_{\delta/4}(x)) \leq \tau(f_\epsilon(f_{\delta/8}(x)))$. Thus

$$d_\tau(f_\delta(x)) \leq \tau(f_{\delta/2}(x)) < \tau(p) < \tau(f_{\delta/4}(x)) \leq d_\tau(f_{\delta/8}(x)).$$

Note that $d_\tau(q) = \tau(q)$ for every projection q, and using Lemma 3.5 we get

$$f_\delta(x) \precsim p \precsim f_{\delta/8}(x).$$

Now the proof ends as in [19, Proof of Theorem 7.2]. □

Definition 3.17 *Let (G, u) be a partially ordered abelian group with order-unit. A state s on (G, u) is called a* **discrete state** *if and only if $s(G)$ is a cyclic subgroup of $\mathbb{R}$.*

Note that any additive subgroup of $\mathbb{R}$ is either a cyclic group or a dense subset of $\mathbb{R}$ (for a proof, see [10, Lemma 4.21]). Now with all these ingredients we are able to prove:

Theorem 3.18 *Let A be a unital simple finite C^*-algebra with $V(A)$ strictly unperforated. The following conditions are equivalent:*

1) *$RR(A) = 0$;*

2) *$S(A)$ is Riesz, almost unperforated, A satisfies condition (PT), $sr(A) = 1$.*

Proof. Since a finite-dimensional C^*-algebra satisfies both conditions of the statement, we may assume that A is infinite-dimensional.

1) ⇒ 2)

By [1, Theorem 2.4], A has stable rank one. As in the proof of Corollary 3.10, it follows that $S(A)$ is almost unperforated. Thus the Theorem above applies and we get that A satisfies condition (PT). Finally, by Theorem 2.13 (1), $S(A)$ is a Riesz monoid.

2) ⇒ 1)

Assume that A is simple, $sr(A) = 1$ and $S(A)$ is Riesz. If we have $[p] \leq [q] + [r]$ for $[p], [q], [r] \in V(A)$, then $\langle p \rangle \leq \langle q \rangle + \langle r \rangle$ in $S(A)$. Then $\langle p \rangle = \langle p_1 \rangle + \langle p_2 \rangle$ for some $p_1, p_2 \in M_\infty(A)_+$ such that $\langle p_1 \rangle \leq \langle q \rangle$ and $\langle p_2 \rangle \leq \langle r \rangle$. Now observe that $0 \in \mathrm{Spec}(p)$ is an isolated point and, as $\langle p \rangle = \langle p_1 \oplus p_2 \rangle$, we have that 0 is an isolated point of $\mathrm{Spec}(p_1 \oplus p_2)$, by Proposition 3.12. Therefore, 0 is an isolated point of both $\mathrm{Spec}(p_1), \mathrm{Spec}(p_2)$, hence we may assume that $p_i, i = 1, 2$ are projections. So we obtain that $[p] = [p_1] + [p_2]$ with $[p_1] \leq [q]$ and $[p_2] \leq [r]$. Therefore $K_0(A)$ is a simple interpolation group.

Suppose that there is an atom $[e] \in V(A)$; we may assume that $e \in A$. We first show that there exists a nonzero element $a \in eAe$ such that $\langle a \rangle < \langle e \rangle$. If $a \sim e$ for all $0 \neq a \in eAe$, then in particular $e \precsim a$, and it follows from [19, Proposition 2.4] that there exist elements $r, s \in A$ such that $e = ras$, whence a is invertible in eAe, because by hypothesis A is finite and so each corner is finite. Therefore $eAe = \mathbb{C}$. Since A is infinite-dimensional, this is not the case.

Thus there exists $a \neq 0$ such that $\langle a \rangle < \langle e \rangle$. As A is simple, there exists $n \in \mathbb{N}$ such that $\langle 1 \rangle \leq n\langle a \rangle$. By Riesz decomposition in $S(A)$ together with the fact that 1 is a projection, we get $\langle 1 \rangle = \sum_{i=1}^{n} \langle e_i \rangle$ for some projections e_i satisfying $e_i \precsim a$ for all i, again by Proposition 3.12. But then $\langle e_i \rangle < \langle e \rangle$, whence $[e_i] \leq^* [e]$, and this implies that $e_i = 0$ for all i, a contradiction.

Thus $V(A)$ is atomless. By [10, Proposition 14.3], there are no discrete states on $(K_0(A), u)$, and therefore no extremal discrete states. By [15, Theorem 3.5], $K_0(A)$ satisfies condition (D). Apply finally Theorem 3.16. □

Corollary 3.19 *Let A be a unital simple C^*-algebra with $sr(A) = 1$, $S(A)$ almost unperforated. The following conditions are equivalent:*

1) $RR(A) = 0$*;*

2) $S(A)$ *is Riesz, A satisfies condition (PT).*

Proof. Apply Theorem 3.16 and the proof of Theorem 3.18. □

It is also possible to generalize [19, Theorem 7.2]. Denote by K the C^*-algebra of compact operators over some infinite-dimensional separable Hilbert space.

Theorem 3.20 *Let D be a stably finite simple unital C^*-algebra, and let B be an (infinite-dimensional) UHF-algebra. Then $RR(B \otimes D) = 0$ if and only if $D \otimes K$ satisfies condition (PT).*

Proof. Set $A = B \otimes D$. By [19, Lemma 5.1], $S(A)$ is almost unperforated and by [18, Corollary 6.6], the stable rank of A is one. We now show that A satisfies condition (D).

Note that $V(A) \subseteq S(A)$ because $sr(A) = 1$, and therefore $V(A)$ is strictly unperforated, as follows. Suppose that $nx \leq^* ny$ for some $n \in \mathbb{N}$ and $x, y \in V(A)$. There exists a nonzero element $z \in V(A)$ such that $nx + z = ny$. Since A is simple, there exists $k \in \mathbb{N}$ such that $x \leq kz$. Note that $(kn + 1)x = knx + x \leq knx + kz = kny$, hence $x \leq^* y$, because $S(A)$ is almost unperforated. Therefore $K_0(A)$ is weakly unperforated (in the sense of [4, Section 6.7.1], [12, Section 2.1] or [11, Section 8]). Also, B is an infinite-dimensional UHF-algebra, whence $K_0(A) \cong K_0(B) \otimes K_0(D)$, and $K_0(B)$ is a dense subgroup of $\mathbb{R}$. By the proof of [10, Theorem 7.9], $\Phi(K_0(A)^+)$ is dense in $\mathrm{Aff}(St(K_0(A),[1_A]))^+$, where $\Phi : K_0(A) \to \mathrm{Aff}(St(K_0(A),[1_A]))$ is the natural map.

Let τ_0 denote the unique trace on B. We claim that A satisfies condition (PT) if and only if $D \otimes K$ satisfies condition (PT). Since B is a UHF-algebra, there is a sequence of integers $\{n_i\}$ such that $A \cong \varinjlim M_{n_i}(D)$.

Assume that A satisfies condition (PT) and let τ_1, τ_2 be different quasi-traces on D. Then τ_i extend uniquely to quasi-traces $\overline{\tau_i}$ on $\varinjlim M_{n_i}(D)$ such that $\overline{\tau_i}_{|M_{n_i}(D)} = \tau_i/n_i$. Since A satisfies condition (PT), it follows that $\varinjlim M_{n_i}(D)$ satisfies condition (PT), and hence there is a projection $p \in M_{n_i}(D)$ for some i such that $\overline{\tau_1}(p) \neq \overline{\tau_2}(p)$. Therefore $\tau_1(p) \neq \tau_2(p)$. Consequently, $D \otimes K$ satisfies condition (PT).

Conversely, suppose that $D \otimes K$ satisfies condition (PT), and let τ_1, τ_2 be different quasi-traces on A. Then we may write $\tau_i = \tau_0 \otimes \tau_i'$ for $i = 1, 2$, where τ_i' are different quasi-traces on D. Hence there exists a projection $p \in M_\infty(D)$ such that $\tau_1'(p) \neq \tau_2'(p)$, and thus $\tau_1(1 \otimes p) \neq \tau_2(1 \otimes p)$. Therefore A satisfies condition (PT), and therefore the claim is proved.

Now Theorem 3.16 applies and we conclude that $RR(A) = 0$ if and only if A satisfies condition (PT), and it follows from the claim above that the latter is equivalent to $D \otimes K$ satisfying condition (PT). □

4 Structure of $K_0^*(A)$

We turn now our attention to the Grothendieck group of the monoid $S(A)$, which is denoted by $K_0^*(A)$ (see [9]). This section is devoted to studying the decomposition properties that $K_0^*(A)$ inherits from those of $S(A)$. In particular, for the algebras under consideration, $K_0^*(A)$ is always a Riesz group, and it is lattice-ordered if $K_0(A)$ is.

For a preordered monoid $(M, \leq)$, we denote by $G(M)$ its Grothendieck group. Recall that $G(M)$ can be constructed as follows. Define an equivalence relation on M by $x \sim y$ if and only if there exists $z \in M$ such that $x + z = y + z$. Set $M_c = M/\sim$ and denote the equivalence classes of the elements of M by $[x]$. Define an addition by $[x] + [y] = [x + y]$ for $x, y \in M$, and take $[x] \leq [y] \Leftrightarrow x + z \leq y + z$ for some $z \in M$ as an ordering. Notice that with this structure, M_c is order-cancellative, and it is called the cancellative monoid associated to M. If u is an order-unit for M, then $[u]$ will be an order-unit for M_c, which will be denoted again by u. Observe that if M is either algebraic or partially ordered, then so is M_c.

Now, by adjoining formal inverses to the elements of M_c we can define an abelian group, denoted by $G(M)$. It is clear that $G(M) = \{x - y \mid x, y \in M_c\}$. Define an order on $G(M)$ by taking as positive cone:

$$G(M)^+ = \{x - y \mid x, y \in M_c \text{ and } y \leq x\}.$$

Note that $G(M)$ is partially ordered if M is, and that for $a, b, c, d \in M$,

$$[a] - [b] \leq [c] - [d] \text{ in } G(M) \Leftrightarrow a + d + e \leq b + c + e \text{ in } M \text{ for some } e \in M .$$

Recall the definition of a Riesz group given in 2.9.

Definition 4.1 *Let G be a partially ordered abelian group. We say that G is* **unperforated** *if whenever $nx \geq 0$ for some $n \in \mathbb{N}$ and $x \in G$, it follows that $x \geq 0$. If G is an unperforated Riesz group then G is said to be a* **dimension group**.

The interpolation property proved in Section 2 for the monoid allows to check interpolation on the group, as follows.

Lemma 4.2 *Let $(M, \leq)$ be a preordered monoid. If M is an interpolation monoid, then $G(M)$ is an interpolation group. If M is unperforated, then so is $G(M)$.*

Proof. Let a_1, a_2, b_1, b_2 be elements in $G(M)$ such that $a_i \leq b_j$ for all i and j. There exist elements $z, x_i, y_j \in M$ such that $a_i = [x_i] - [z]$ and $b_j = [y_j] - [z]$. Therefore, by adding $[z]$ to the inequality we get

$$[x_i] \leq [y_j] \text{ for all } i, j;$$

hence there exists $t \in M$ such that

$$x_i + t \leq y_j + t \text{ for all } i \text{ and } j.$$

By hypothesis, there exists $x \in M$ interpolating the above inequality. Consider the element $e = [x] - [z + t] \in G(M)$. Then it is easy to check that e satisfies $a_i \leq e \leq b_j$ for all i, j.

Now assume that M is unperforated and that $na \geq 0$ for some $n \in \mathbb{N}$ and $a \in G(M)$. There exist $x, y \in G(M)$ such that $a = [x] - [y]$, and since $na \geq 0$ there is $s \in M$ satisfying

$$ny + s \leq nx + s.$$

Thus by adding $(n-1)s$ to the inequality we get $n(y+s) \leq n(x+s)$, and therefore $y+s \leq x + s$ because M is unperforated. Consequently $a \geq 0$ and so $G(M)$ is unperforated. □

Corollary 4.3 *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. Then $K_0^*(A)$ is a Riesz group. If, further, $V(A)$ is unperforated, then $K_0^*(A)$ is a dimension group.*

Proof. Clearly $K_0^*(A)$ is a partially ordered abelian group. Since A is σ-unital, $S(A)$ has an order-unit u (by 2.2), whence $[u]$ is an order-unit for $K_0^*(A)$. Hence $K_0^*(A)$ is directed. By Theorem 2.13 and Lemma 4.2, $K_0^*(A)$ satisfies the Riesz interpolation property, whence $K_0^*(A)$ is a Riesz group.

Assume, further, that $V(A)$ is unperforated. This implies that $S(A)$ is unperforated, as follows. By Theorem 2.8, we may identify $S(A)$ with $\Lambda_{\sigma,D}(V(A))$. Assume that

$nI \subseteq nJ$ for some $I, J \in S(A)$ and $n \in \mathbb{N}$. If $x \in I$, then $nx \in nJ$, so that $nx \leq ny$ for some $y \in J$. Therefore $x \leq y$, and so $x \in J$; hence $I \subseteq J$. By Lemma 4.2 $K_0^*(A)$ is unperforated, hence it is a dimension group. □

As a consequence of this fact we give a partial answer to a conjecture of Blackadar and Handelman, stated in [7, Section II.4]. Let A be a unital C^*-algebra. A normalized state s on $S(A)$ (that is, $s\langle 1_A \rangle = 1$) is called a **dimension function**, and the set of dimension functions is denoted by $DF(A)$ (see [7], [19]). Note that $DF(A) = St(K_0^*(A), [1_A])$.

Corollary 4.4 *Let A be a unital C^*-algebra with real rank zero and stable rank one. Then $DF(A)$ is a Choquet simplex.*

Proof. By Corollary 4.3, $K_0^*(A)$ is a Riesz group, and therefore $DF(A)$, being the state space of $K_0^*(A)$, is a Choquet simplex, by [10, Theorem 10.17]. □

The structure of $K_0^*(A)$ is richer under some additional assumptions. Let M be a monoid in which every pair of elements has an infimum. Let $x, y \in M$. We say that x, y have a translation-invariant infimum if $\inf\{x+z, y+z\} = \inf\{x, y\} + z$ for any $z \in M$.

Lemma 4.5 *Let A be a C^*-algebra. Assume that every pair of elements in $S(A)$ has a translation-invariant infimum. Then $K_0^*(A)$ is lattice-ordered.*

Proof. The first step consists of showing that every pair of elements in $K_0^*(A)^+$ has an infimum. Suppose that $[a] - [b], [c] - [d] \in K_0^*(A)^+$, that is $a + s \geq b + s$ and $c + t \geq d + t$ for some $s, t \in S(A)$. Then consider the elements $w = b + d + s + t$ and $z = \inf\{a + d + s + t, c + b + s + t\}$, and notice that $w \leq z$; further,

$$z + b \leq a + d + s + t + b = a + w,$$

$$z + d \leq c + b + s + t + d = c + w.$$

Hence, $[z] - [w] \leq [a] - [b], [c] - [d]$.

Now, to see that $[z] - [w]$ is the greatest lower bound, suppose that for $[x] - [y] \in K_0^*(A)^+$ we have $[x] - [y] \leq [a] - [b], [c] - [d]$. Then $x + b + u \leq y + a + u$ and $x + d + v \leq y + c + v$ for some $u, v \in S(A)$. Then

$$x + w + u + v = x + b + d + s + t + u + v \leq (y + u + v) + (a + d + s + t),$$

and similarly

$$x + w + u + v \leq (y + u + v) + (c + b + s + t).$$

Notice that

$$\inf\{(y + u + v) + (a + d + s + t), (y + u + v) + (c + b + s + t)\} = (y + u + v) + z,$$

by hypothesis. It follows then that $x+w+u+v \leq y+z+u+v$ and thus $[x]-[y] \leq [z]-[w]$, as wanted.

Finally, as $K_0^*(A)$ is directed, apply [10, Proposition 1.5] to obtain that $K_0^*(A)$ is lattice-ordered. □

Definition 4.6 *[22, 1.3] Let $(M, \leq)$ be a preordered monoid. We say that M satisfies the* **interval axiom** *if whenever $a, b, c, d \in M$ satisfy $d \leq a + c, b + c$ then there exists $x \in M$ such that $x \leq a, b$ and $d \leq x + c$.*

Lemma 4.7 *Let $(M, \leq)$ be an algebraically ordered monoid and assume that M is Riesz and cancellative. Then M satisfies the interval axiom.*

Proof. Let $a, b, c, d \in M$ and assume that $d \leq a + c, b + c$. By Riesz decomposition $d = d_1 + d_2$ with $d_1 \leq a$ and $d_2 \leq c$. Let $d_2' \in M$ such that $d_2 + d_2' = c$. Now, as $d \leq b + c$, we get $d_1 + d_2 \leq b + d_2 + d_2'$, whence it follows from cancellation that $d_1 \leq b + d_2'$. Apply Riesz decomposition to obtain $d_1 = d_{11} + d_{12}$ with $d_{11} \leq b$ and $d_{12} \leq d_2'$. Take $x = d_{11}$ and notice that by construction $x \leq a, b$ and $d = d_{11} + d_{12} + d_2 \leq x + d_2' + d_2 = x + c$. □

The following Proposition was inspired by [22, Lemma 1.7]:

Proposition 4.8 *Let $(M, \leq)$ be an interpolation preordered monoid satisfying the interval axiom. Then $\Lambda_{\sigma,D}(M)$ satisfies the interval axiom.*

Proof. Let $I, J, K, L \in \Lambda_{\sigma,D}(M)$ and assume that $L \subseteq I + K, J + K$. Choose a countable cofinal sequence $\{x_n\}_n$ for L. As K is directed, there exist elements $y_{11} \in I, y_{21} \in J$ and $z_1 \in K$ such that $x_1 \leq y_{11} + z_1, y_{21} + z_1$. Now there exists $w_1 \in M$ satisfying $w_1 \leq y_{11}, y_{21}$ and $x_1 \leq w_1 + z_1$. Proceeding similarly with x_2, we may clearly assume that $x_2 \leq y_{12} + z_2, y_{22} + z_2$ with $y_{i1} \leq y_{i2}$ for $i = 1, 2$, with $y_{12} \in I, y_{22} \in J$ and $z_2 \in K$ (using the directedness of both I and J). Again we use the interval axiom for M to get $w_2' \in M$ such that $x_2 \leq w_2' + z_2$ and $w_2' \leq y_{12}, y_{22}$. Now observe that $w_1, w_2' \leq y_{12}, y_{22}$. By the interpolation property assumed on M, we can find $w_2 \in M$ satisfying $w_1, w_2' \leq w_2 \leq y_{12}, y_{22}$. By construction $w_2 \geq w_1$ and $x_2 \leq w_2 + z_2$. Thus we can proceed inductively to get an increasing sequence of elements $\{w_n\}$ in M such that $x_n \leq w_n + z_n$ for $z_n \in K$ and $w_n \leq y_{1n}, y_{2n}$, for some $y_{1n} \in I$ and $y_{2n} \in J$. Define $L' = \{x \in M \mid x \leq w_n \text{ for some } n\}$. By the previous inequalities we get that $L' \in \Lambda_{\sigma,D}(M)$ and $L' \subseteq I, J$, while $L \subseteq L' + K$, and the result follows. □

Corollary 4.9 *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. If every pair of elements in $S(A)$ has an infimum, then $K_0^*(A)$ is lattice-ordered. In particular, this holds if $K_0(A)$ is a lattice-ordered group.*

Proof. For the first conclusion, we only need to see that each pair of elements in $S(A)$ has a translation-invariant infimum and then apply Lemma 4.5. As $RR(A) = 0$ and $sr(A) = 1$ the monoid $V(A)$ satisfies the interval axiom, by Lemma 4.7. Now represent $S(A)$ as the monoid of intervals $\Lambda_{\sigma,D}(V(A))$, and use Proposition 4.8 to conclude that $S(A)$ satisfies the interval axiom. Set $M = S(A)$ and let $x, y \in M$, $z = \inf\{x, y\}$. Take any $w \in M$, and suppose that for some $d \in M$ we have $d \leq x + w, y + w$. As M satisfies the interval axiom, there exists $s \in M$ such that $s \leq x, y$ and $d \leq s + w$. It follows that $s \leq z$ and $d \leq z + w$. Hence $z + w = \inf\{x + w, y + w\}$.

Now assume that $K_0(A)$ is a lattice-ordered group. Then, in particular, $V(A)$ is a lattice. If $[e], [f] \in V(A)$, let us denote by $[e] \wedge [f]$ the infimum of $[e]$ and $[f]$. As before, $S(A) \cong \Lambda_{\sigma,D}(V(A))$. Let $I, J \in \Lambda_{\sigma,D}(V(A))$ with countable cofinal subsets $\{x_n\}$ and $\{y_n\}$ respectively. Then $\{x_n \wedge y_n\}$ is a countable cofinal subset for $I \cap J$ and so $I \cap J \in \Lambda_{\sigma,D}(V(A))$, showing that every pair of elements in $\Lambda_{\sigma,D}(V(A))$ has an infimum. $\square$

Acknowledgments

It is a pleasure to thank Pere Ara for his guidance and his support, and also Ken Goodearl for his advice and all his inspiring suggestions.

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