STABLE FINITENESS OF GROUP RINGS IN ARBITRARY CHARACTERISTIC

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Dedicated to S. Car

Abstract. We show that every (discrete) group ring $D[G]$ of a free-by-amenable group G over a division ring D of arbitrary characteristic is stably finite, in the sense that one-sided inverses in all matrix rings over $D[G]$ are two-sided. Our methods use Sylvester rank functions and the translation ring of an amenable group.

Introduction

In the late 1960's, Kaplansky [8] showed that over a field K of characteristic 0, the (discrete) group algebra $K[G]$ is directly finite for all groups G. Alternative proofs of this were given shortly after by Montgomery [11] and Passman [15]; see also [16, Chapter 2]. We recall that a ring R is *directly finite* (resp. *stably finite*) if one-sided inverses in R (resp. in all matrix rings $M_n(R)$) are two-sided : $xy = 1 \implies yx = 1$. (Von Neumann finite and 1-finite are other synonyms for directly finite.) Direct finiteness of $K[G]$ in characteristic $p > 0$ has, however, remained an open problem. We show that over a division ring D of *any* characteristic, and for any free-by-amenable group G, the group ring $D[G]$ is stably finite (Theorem 3.4).

The key to Kaplansky's proof in characteristic 0 is showing that every non-trivial idempotent in the complex group algebra $\mathbb{C}[G]$ has a real trace (coefficient of $1 \in G$) strictly between 0 and 1. (One then elegantly concludes that $xy = 1 \implies \mathrm{tr}(yx) = \mathrm{tr}(xy) = 1 \implies yx = 1$.) In turn, this fact is established by embedding $\mathbb{C}[G]$ in the weak closure of its action on the Hilbert space $L^2(G)$. Montgomery's proof uses instead the uniform closure, but Passman's proof takes place entirely inside $\mathbb{C}[G]$ itself. (Notice, however, that even in characteristic 0, the above techniques do not work if K is not commutative.) Our general characteristic proof is more geometric than analytic. We work with $D[G]$ as a subring of the so-called translation ring associated with the Cayley graph of a group (not necessarily the same group G).

In its simplest form, when G is a finitely generated amenable group and D is a division ring, our idea is to show that $D[G]$ faithfully embeds in some stably finite factor ring of

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the translation ring $\mathbb{T}(G, D)$ of G over D. The stable finiteness of this factor is in turn to be deduced from the existence of a faithful Sylvester rank function on its (finite) matrix rings. However, this technique can't work in the non-amenable case, as evidenced by the fact that if a group G contains a free subgroup of rank two, then the translation ring $\mathbb{T}(G, R)$ of G over any non-zero ring R has no (non-zero) directly finite factor rings (Theorem 4.1). To get around this in the general case when G is an extension of a free normal subgroup H by an amenable group G/H (which we can assume is finitely generated), we need to replace the coefficient ring D by the group ring $R = D[H]$, and work with the crossed product $R * (G/H)$ ($\cong D[G]$) as a subring of the translation ring $\mathbb{T}(G/H, R)$ of the amenable group G/H over R. The stable finiteness of $D[G]$ is then deduced from the following stronger result (again obtained by passing to a suitable factor ring of the translation ring) (Theorem 3.2): *If $R * G$ is a crossed product of a finitely generated amenable group G over a ring R which admits a G-faithful Sylvester rank function, then $R * G$ too admits a faithful Sylvester rank function.* The critical role of amenability in all of this is that it enables one to extend Sylvester rank functions on a ring R to Sylvester rank functions on the translation ring $\mathbb{T}(G, R)$.

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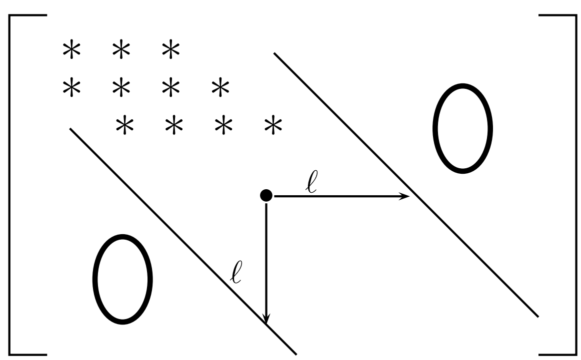
1. Preliminaries

Let R be a (unital) ring and let (X, d) be a discrete metric space. Following Gromov [5, p. 262] we define the *translation ring* $\mathbb{T}(X, R)$ of X over R to be the ring of all square matrices $(a(x, y))$, indexed by $X \times X$ and with entries from R, such that $a(x, y) = 0$ whenever $d(x, y) > l$ for some constant l depending on the matrix. The least such l is called the *bandwidth* of the matrix. Of particular interest is the case where $d : X^2 \to \mathbb{Z}^+$ and the space (X, d) is *uniformly discrete* in the sense that all balls of a given finite radius have a uniformly bounded finite size. The simplest such example is $X = \mathbb{N}$ with the Euclidean metric and $R = K$ is a field. Then $\mathbb{T}(X, K)$ is just the algebra of all $\omega \times \omega$ matrices over K with constant bandwidth in the classical sense, i.e. of the form below



(In [6, 7, 14] this algebra was called the growth algebra $G(0)$ since it was the first of a whole spectrum of growth algebras $G(r)$ for r in the unit interval $[0,1]$.)

An important class of translation rings over discrete metric spaces arises from connected graphs: if Γ is a connected graph, we take $X = V(\Gamma)$ to be the vertex set and $d(x,y)$ to be the minimum of the lengths of the paths joining x and y. In turn, the specialization of this which is central for us is to take $\Gamma = \Gamma(G,S)$, the Cayley graph of a finitely generated group G with respect to a finite generating set S. We denote the corresponding translation ring by $\mathbb{T}(G,R)$. Recall that with the Cayley graph, $X = V(\Gamma) = G$ and that there is an edge from x to y precisely when $x = h^{\pm 1}y$ for some h in S, so that the corresponding metric is just the word metric and X is uniformly discrete. In particular the closed ball $B(1,n)$ centered on 1 with radius n is simply $\{h_1^{\pm 1} \cdots h_t^{\pm 1} : t \leq n \text{ and } h_1, \ldots, h_t \in S\}$.

Note that for a finitely generated group G, the translation ring $\mathbb{T}(G,R)$ does not depend on the particular choice of the finite set of generators. The translation ring is big enough to contain the group ring $R[G]$, and also any crossed product $R * G$; see Lemma 3.1.

Originally the concept of an amenable group G arose in ergodic theory, and was defined in terms of the existence of an invariant mean or invariant measure (e.g. every continuous action of G on a compact space has a G-invariant measure). For our purposes, it is more appropriate to adopt an equivalent definition in terms of the asymptotic behaviour of boundaries of finite symmetric subsets of G, formulated by Følner [4] in the 1950's (see also [1, Theorem F.6.8]). (A subset S of a group G is called *symmetric* if S is closed under inverses.)

Definition 1.1. *A group G is called amenable if for each finite symmetric subset S of G and positive real number ε, there exists a finite non-empty subset A of G with*

$$|\partial_S A| \leq \varepsilon |A| \, .$$

Here $\partial_S A = \{a \in A : Sa \nsubseteq A\}$ is the S-boundary of A.

Remark 1.2. This definition of amenable group is not totally symmetric, since the set $\partial_S A$ is a "left boundary" for A. However, the symmetry of the condition is evident once we consider the sets A^{-1} for "right boundaries".

The class of amenable groups is closed under subgroups, factor groups, extensions and directed unions, and contains every abelian group (consequently, every solvable group) and every finite group. An amenable group cannot contain a free subgroup on 2 generators. Although Olshanskii in the 1970's constructed a non-amenable group which has no such subgroup, at least it is true that every finitely generated non-amenable group must have exponential growth (in the Gelfand-Kirillov sense). Just recently an example of a finitely presented, non-amenable group not containing a free subgroup of rank 2 has been constructed by Olshanskii and Sapir [13], thereby settling a long-standing conjecture. A general reference for amenable groups is [18].

We now describe the type of rank function that we will use throughout the paper. Let R be a ring. Adopting the terminology of [19, Page 97], we say that a function ρ which assigns a non-negative real number to each finite (but not necessarily square) matrix a is a *Sylvester rank function* if the following conditions hold:

(S1) $\rho(z) = 0$, where z is any zero matrix (hereafter we denote zero matrices by 0);
(S2) $\rho((1)) = 1$, where (1) is 1×1;
(S3) $\rho(ab) \leq \min\{\rho(a), \rho(b)\}$ for all matrices a and b which can be multiplied;
(S4) $\rho \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \rho(a) + \rho(b)$ for all matrices a, b;
(S5) $\rho \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \geq \rho(a) + \rho(c)$ for all matrices a, b, c of appropriate sizes.

A Sylvester rank function ρ is said to be *faithful* if $\rho(a) \neq 0$ for all non-zero matrices a. Note that $\rho(a) \geq \rho(a_{ij})$ by (S3), where $a = (a_{ij})$, and so ρ is faithful if and only if $\rho(a) \neq 0$ for every a in R (identified with a 1×1 matrix). Note that if we have an (injective) homomorphism φ from R to a division ring D, then we obtain a (faithful) Sylvester rank function ρ on R by the rule $\rho(a) = \mathit{rank}_D(\varphi(a))$. Malcolmson proved in [10] that, conversely, given a Sylvester rank function ρ on R taking integer values, there exists a division ring D and an epimorphism in the category of rings $\varphi : R \to D$ such that ρ is induced by φ. (Sylvester rank functions taking integer values are called algebraic rank functions in [10].)

Assume that R admits a Sylvester rank function ρ. If e and f are equivalent idempotent matrices over R, so that $e = ab$ and $f = ba$ for some (finite) matrices a and b over R, then, by (S3) we have $\rho(e) = \rho(ab) = \rho(abab) \leq \rho(ba) = \rho(f)$ and by symmetry $\rho(f) \leq \rho(e)$, so that $\rho(e) = \rho(f)$. Moreover, if e and f are orthogonal idempotent finite matrices over R, then $\rho(e+f) = \rho(e)+\rho(f)$ since the matrices $\begin{pmatrix} e+f & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ are conjugate. It follows that a ring R admitting a faithful Sylvester rank function must be stably finite. Indeed, if a and b are $n \times n$ matrices over R such that $ab = I_n$, where I_n is the $n \times n$ identity matrix, then by (S2), (S4) and the above observations on idempotents, we have $\rho(ba) = \rho(ab) = \rho(I_n) = n$ and so $\rho(I_n - ba) = 0$, which gives $I_n - ba = 0$ because ρ is faithful.

Notice that for any Sylvester rank function ρ, we have $\rho(a+b) \leq \rho(a) + \rho(b)$. For we always have $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \end{pmatrix}$ and can then apply (S3) and (S4).

In Section 2, when we come to extend Sylvester rank functions to a translation ring, we need to have a notion of a "limit" that exists for all bounded sequences of real numbers (a_n). The only properties we require of this limit is that it should agree with the usual one when that exists, that the limit of a sum be the sum of the limits, and that non-negative sequences have non-negative limits. To achieve this, we fix a *free* (also called non-principal) ultrafilter ω on $\mathbb{N}$ and take $\lim_{\omega} a_n$, the limit along that ultrafilter.

(Recall that this limit is l if, by definition, for each $\epsilon > 0$, the set $\{n \in \mathbb{N} : |l - a_n| < \epsilon\}$ belongs to ω.) Different choices of the ultrafilter can result in different limits. A general reference for the theory of filters and convergence is [2, Chapter I: §6].

2. Translation rings associated with amenable groups

Let G be an amenable finitely generated group and let X be its Cayley graph with respect to a given finite set $\mathrm{Gen}(G)$ of generators of G. Let R be a ring and let $\mathbb{T} = \mathbb{T}(X, R) = \mathbb{T}(G, R)$ be the corresponding translation ring.

Assume that R has a Sylvester rank function ρ. We now prepare to show (Theorem 2.3) that ρ can be extended to a Sylvester rank function on $\mathbb{T}(G, R)$. Let a be any element in $\mathbb{T} = \mathbb{T}(G, R)$. For a finite subset $A \subseteq X$, we define the *normalized rank* $\rho_A(a)$ by

$$\rho_A(a) = \frac{\rho[e(A)\, a\, e(A)]}{\rho[e(A)]} = \frac{\rho[e(A)\, a\, e(A)]}{|A|}\,,$$

where $e(A)$ is the diagonal idempotent in $\mathbb{T}$ such that

$$e(x,x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A\,. \end{cases}$$

Note that $0 \leqslant \rho_A(a) \leqslant 1$ and that $\rho_A(a + b) \leqslant \rho_A(a) + \rho_A(b)$.

We denote by S_n the closed n-sphere in X centered at 1, that is, $S_n = B(1, n) = \{h_1^{\pm 1} \cdots h_t^{\pm 1} : t \le n \text{ and } h_1, \ldots, h_t \in \mathrm{Gen}(G)\}$. The following result is equivalent to amenability for *finitely generated* groups; see [5, 0.5.A].

Proposition 2.1. *There is an increasing chain of finite subsets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ such that*

(a) $X = \bigcup_{n=1}^{\infty} A_n$, *and*

(b) $|\partial_{S_n} A_n| \leqslant \dfrac{|A_n|}{2^n\, |S_n|}$.

For the remainder of the paper, we fix the chain $A_1 \subseteq A_2 \subseteq \ldots$ constructed in Proposition 2.1. The following computations will be useful. Let us denote $e(A_k)$ simply by e_k and $e(S_k A_k)$ simply by s_k.

Lemma 2.2. *Let R be a ring with a Sylvester rank function ρ. Then the following properties hold:*

(1) *For all $k \ge 1$ we have $\rho(s_k - e_k) \le 2^{-k} \rho(e_k)$.*
(2) *If $a \in \mathbb{T}$ has bandwidth at most k, then $e_k a = e_k a s_k$ and $a e_k = s_k a e_k$.*

Proof. (1) We have $S_k A_k = A_k \bigcup S_k(\partial_{S_k} A_k)$ so that $|S_k A_k| \leqslant |A_k| + |S_k|\, |\partial_{S_k} A_k|$. Note that by construction $e_k \le s_k$ (as idempotents), hence we have an orthogonal decomposition $s_k = (s_k - e_k) + e_k$. As we have observed in Section 1, ρ is additive on orthogonal sums of idempotents, whence $\rho(s_k) = \rho(s_k - e_k) + \rho(e_k)$. This fact, coupled with the fundamental property of the sequence $\{A_k\}$ (condition (b) in Proposition 2.1) yields

$$\rho(s_k - e_k) = |S_k A_k| - |A_k| \leqslant |\partial_{S_k} A_k|\, |S_k| \leqslant \frac{|A_k|}{2^k}\,,$$

as desired.

(2) We will prove only the statement corresponding to $e_k a$.

Note that

$$(e_k a)(z,t) = \begin{cases} 0 & \text{if } z \notin A_k \\ a(z,t) & \text{if } z \in A_k\,. \end{cases}$$

Since a has bandwidth at most k we have $a(z,t) = 0$ if $d(z,t) > k$ and so $(e_k a)(z,t) \neq 0$ implies $t \in S_k A_k$. We conclude that $e_k a = e_k a s_k$. □

Let X be any uniformly discrete metric space and let $n \in \mathbb{N}$. For any ring R there is an obvious ring isomorphism

$$\mathbb{T}(X, M_n(R)) \xrightarrow{\cong} M_n(\mathbb{T}(X,R))\,.$$

The isomorphism sends a matrix $a = (a(x,y))_{(x,y)\in X\times X}$, with $a(x,y)$ in $M_n(R)$ to the matrix a' in $M_n(\mathbb{T}(X,R))$ such that $a'_{ij} = (a(x,y)_{ij})_{(x,y)\in X\times X}$. It will be convenient to identify $M_n(\mathbb{T}(X,R))$ with $\mathbb{T}(X, M_n(R))$.

We now return to our earlier situation where X is the Cayley graph of a finitely generated amenable group G and R is a ring with a Sylvester rank function ρ. Let $a \in \mathbb{T}(X, M_n(R))$. For a finite subset $A \subseteq X$, we define the *normalized rank* $\rho^n_A(a)$ by

$$\rho^n_A(a) = \frac{\rho[e^n(A)\, a\, e^n(A)]}{|A|}\,,$$

where $e^n(A)$ is the diagonal idempotent in $\mathbb{T}(X, M_n(R))$ such that

$$e^n(A)(x,x) = \begin{cases} I_n & \text{if } x \in A \\ 0 & \text{if } x \notin A\,. \end{cases}$$

(The ranks are computed viewing $e^n(A)\, a\, e^n(A)$ as a finite matrix over R.) We have $0 \leqslant \rho^n_A(a) \leqslant n$ and $\rho^n_A(a+b) \leqslant \rho^n_A(a) + \rho^n_A(b)$ for all a, b in $\mathbb{T}(X, M_n(R))$. Note that $\rho^n_A = (\rho^n)_A$, where ρ^n corresponds to the (unnormalized) rank function ρ^n induced by ρ on matrices over $M_n(R)$. Set $\rho^n_k := \rho^n_{A_k}$ for all k, n (where $\{A_k\}$ is the sequence of sets fixed according to Proposition 2.1). We will write ρ_k for ρ^1_k.

Let ω be a free ultrafilter on $\mathbb{N}$. Define ρ_ω on $\mathbb{T}$ by

$$\rho_\omega(a) = \lim_\omega \rho^n_k(a)$$

for a in $M_n(\mathbb{T}(G,R)) = \mathbb{T}(G, M_n(R))$. Note that the limit along the ultrafilter exists because $0 \leq \rho^n_k(a) \leq n$ for all k. If a is a non-square matrix over $\mathbb{T}(G,R)$, then we define $\rho_\omega(a) = \rho_\omega \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, where the zero matrices are chosen to make the matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ square.

Theorem 2.3. *Let G be a finitely generated amenable group and let R be a ring with a Sylvester rank function ρ. For any free ultrafilter ω on $\mathbb{N}$, the function ρ_ω is a Sylvester rank function on $\mathbb{T}(G,R)$ extending ρ.*

Proof. Set $\mathbb{T} = \mathbb{T}(G,R)$. It is clear that ρ_ω extends ρ if we identify R with its diagonal copy in $\mathbb{T}$. Let us check properties (S1)-(S5). By the extension property, (S1) and (S2) are obviously satisfied.

(S4) By completing with suitable zero matrices we can assume that a and b are square matrices. Assume that $a \in M_n(\mathbb{T})$ and $b \in M_m(\mathbb{T})$. Setting $e_k^n = e_{A_k}^n$, we have

$$\rho[e_k^{n+m} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} e_k^{n+m}] = \rho \begin{pmatrix} e_k^n a e_k^n & 0 \\ 0 & e_k^m b e_k^m \end{pmatrix} = \rho(e_k^n a e_k^n) + \rho(e_k^m b e_k^m)\,.$$

Consequently we get

$$\begin{aligned} \rho_\omega \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &= \lim_\omega \rho_k^{n+m} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= \lim_\omega \frac{\rho[e_k^{n+m} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} e_k^{n+m}]}{|A_k|} \\ &= \lim_\omega \frac{\rho(e_k^n a e_k^n)}{|A_k|} + \lim_\omega \frac{\rho(e_k^m b e_k^m)}{|A_k|} \\ &= \rho_\omega(a) + \rho_\omega(b)\,. \end{aligned}$$

Property (S5) is proved in a similar way to (S4).

(S3) It is here that we use the amenability of G. Since $M_n(\mathbb{T}) \cong \mathbb{T}(G, M_n(R))$ for all n, we can assume without loss of generality that $a, b \in \mathbb{T}(G,R)$. By condition (2) in Lemma 2.2, we have $e_k a b e_k = e_k a s_k b e_k = e_k a e_k b e_k + e_k a(s_k - e_k) b e_k$ for all $k \geq 1$. Therefore

$$\rho(e_k a b e_k) \leq \rho[(e_k a e_k)(e_k b e_k)] + \rho(s_k - e_k) \leq \rho(e_k a e_k) + \frac{|A_k|}{2^k}\,,$$

where the last inequality follows from condition (1) in Lemma 2.2. We conclude that

$$\rho_k(ab) \leq \rho_k(a) + 2^{-k}\,,$$

and so

$$\rho_\omega(ab) = \lim_\omega \rho_k(ab) \leq \lim_\omega \rho_k(a) + \lim_\omega 2^{-k} = \rho_\omega(a)\,.$$

Similarly $\rho_\omega(ab) \leq \rho_\omega(b)$. □

3. Crossed products and group algebras

In this section we will apply the results of Section 2 to obtain stable finiteness of group algebras of free-by-amenable groups. For this we need to consider crossed products. We will use the notation in [17]. We recall that a crossed product $R*G$ of a group G over a ring R is an associative ring that contains R and has as an R-basis the set $\overline{G}$, a copy of G. Thus every element of $R*G$ is uniquely a finite sum $\sum_{x\in G}\overline{x}r_x$ with r_x in R. The product in $R*G$ is determined by the rules

$$\overline{x}\,\overline{y}=\overline{xy}\tau(x,y)$$

for some map τ of $G\times G$ into the group of units of R, and

$$r\overline{x}=\overline{x}r^{\sigma(x)}$$

where $\sigma: G\to \mathrm{Aut}(R)$. By [17, Lemma 1.3], if $R*G$ is a crossed product and N is a normal subgroup of G then $R*G=(R*N)*(G/N)$, where the latter is some crossed product of the group G/N over the ring $R*N$.

Lemma 3.1. *Let $R*G$ be a crossed product of a finitely generated group G over a ring R. Then the regular representation θ embeds the crossed product $R*G$ in the translation ring $\mathbb{T}(G,R)$.*

Proof. Recall that the crossed product $R*G$ is a free right R-module with basis $\{\overline{x}:x\in G\}$. Consider the regular matrix representation θ of $R*G$ (under left multiplication) relative to the basis $X=\overline{G}$. For r in R we have $\theta(r)(\overline{x})=\overline{x}r^{\sigma(x)}$, so $\theta(r)\in\mathbb{T}(G,R)$ (with bandwidth 0). Fix $g=h_1^{\pm1}\cdots h_t^{\pm1}\in G$ (with the h's from the fixed generating set $\mathrm{Gen}(G)$). For any y in G, the y^{th} column of $\theta(\overline{g})$ has a single non-zero entry $(\tau(g,y))$ at position (x,y) for $x=gy$. Since $d(x,y)\le t$, this shows that $\theta(\overline{g})\in\mathbb{T}(G,R)$ with bandwidth at most t. Since $\theta(R*G)$ is generated by $\theta(R)$ and $\theta(\overline{G})$, the result follows. □

Given a crossed product $R*G$, we say that a Sylvester rank function ρ on R is *G-faithful* if

$$\inf\{\rho(r^{\sigma(x)}):x\in G\}>0$$

for all non-zero r in R. In particular, notice that this condition is fulfilled by any faithful Sylvester rank function ρ which is also *G-invariant*, in the sense that $\rho(a^{\sigma(x)})=\rho(a)$ for all finite matrices a over R and for all x in G.

Theorem 3.2. *Let $R*G$ be a crossed product of a finitely generated amenable group G over a ring R. Assume that R admits a G-faithful Sylvester rank function. Then $R*G$ admits a faithful Sylvester rank function. In particular, $R*G$ is stably finite.*

Proof. As we have already observed in Section 1, any ring having a faithful Sylvester rank function is stably finite, so the second assertion will follow from the first.

By Lemma 3.1 the regular representation θ embeds $R*G$ in the translation ring $\mathbb{T}=\mathbb{T}(G,R)$ of the Cayley graph X of G. Fix a free ultrafilter ω on $\mathbb{N}$ and define the

Sylvester rank function ρ_ω on $\mathbb{T}$ as in Section 2. Let I_ω be the set of elements a in $\mathbb{T}$ such that $\rho_\omega(a) = 0$. Clearly I_ω is an ideal of $\mathbb{T}$, and for a in $M_n(\mathbb{T})$ we have $\rho_\omega(a) = 0$ if and only if $a \in M_n(I_\omega)$. Define a Sylvester rank function $\overline{\rho}_\omega$ on $\mathbb{T}/I_\omega$ by

$$\overline{\rho}_\omega(\overline{a}) = \rho_\omega(a)$$

for all finite matrices $\overline{a}$ over $\mathbb{T}/I_\omega$, where a is a lift of $\overline{a}$. Clearly $\overline{\rho}_\omega$ is a faithful Sylvester rank function.

It is enough to check that $\theta(R * G) \bigcap I_\omega = 0$, for then $R * G$ will embed in $\mathbb{T}/I_\omega$ and will therefore admit a faithful Sylvester rank function.

Let $0 \neq \sum_{i=1}^{m} \overline{g}_i r_i \in R * G$. We can write

$$g_i = h_{i_1}^{\pm 1} \cdots h_{i_{t(i)}}^{\pm 1},$$

where $h_{i_1}, \ldots, h_{i_{t(i)}} \in \mathrm{Gen}(G)$. Since ρ is G-faithful, there exist $c_i > 0$ such that $\rho(r_i^{\sigma(x)}) \geq c_i$ for all x in G. Set $c = \max\{c_i : i = 1, \ldots, m\}$.

Let $k > \max\{t(i) : i = 1, \ldots, m\}$. Let $A'_k = A_k \backslash \partial_{S_k} A_k$ and note that $S_k A'_k \subseteq A_k$. In particular $g_i x \in A_k$ for all $i = 1, \ldots, m$ and all x in A'_k. By using Proposition 2.1, we compute that

$$|A'_k| = |A_k| - |\partial_{S_k} A_k| \geq |A_k| - \frac{|A_k|}{2^k\,|S_k|}.$$

Choose a subset A''_k of A'_k with $|A''_k| \geq |A'_k|/m^2$ and with the "separating" property $\{g_i x \mid 1 \leqslant i \leqslant m\} \bigcap \{g_i y \mid 1 \leqslant i \leqslant m\} = \emptyset$ for all distinct x, y in A''_k. For example, choose A''_k maximal with respect to this separating property. (Note that if B has the separating property but $m^2|B| < |A'_k|$, then there is some y in A'_k not of the form $g_j^{-1} g_i x$, with x in B, and then $B \cup \{y\}$ also has the separating property.) Put $e''_k = e(A''_k)$, and note that $e''_k = e''_k e_k = e_k e''_k$. Therefore we have

$$\rho(e_k \theta\Big(\sum_{i=1}^{m} \overline{g}_i r_i\Big) e_k) \geq \rho(e_k \theta\Big(\sum_{i=1}^{m} \overline{g}_i r_i\Big) e''_k).$$

For x_α in A''_k, the action of $\theta\Big(\sum_{i=1}^{m} \overline{g}_i r_i\Big)$ on the basis element $\overline{x}_\alpha$ (in the regular representation) is

$$\theta\Big(\sum_{i=1}^{m} \overline{g}_i r_i\Big)(\overline{x}_\alpha) = \sum_{i=1}^{m} \overline{g_i x_\alpha}\,\tau(g_i, x_\alpha) r_i^{\sigma(x_\alpha)}.$$

We infer from this computation, the fact that $A''_k \subseteq A'_k$, and the separating property of the family A''_k, that the matrix $e_k \theta\Big(\sum_{i=1}^{m} \overline{g}_i r_i\Big) e''_k$ looks like blocks of $m \times 1$ column matrices $C(x_\alpha) = (\tau(g_i, x_\alpha) r_i^{\sigma(x_\alpha)})^T$ for x_α ranging over A''_k, with the blocks positioned over disjoint rows for different x_α. By (S5) we have $\rho(C(x_\alpha)) \geq \max\{\rho(\tau(g_i, x_\alpha) r_i^{\sigma(x_\alpha)}) :$

$i = 1, \dots, m\} \geq c$, where the last equality follows from the facts that $\tau(g_i, x_\alpha)$ is a unit of R and $\max\{\rho(r_i^{\sigma(x_\alpha)}) : i = 1, \dots, m\} \geq \max\{c_i : i = 1, \dots, m\} = c$.

Using the observations above we get

$$\begin{aligned}\rho(e_k\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big)e_k) &\geq \rho(e_k\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big)e_k'') \\ &= \rho(\bigoplus_{x_\alpha \in A_k''} C(x_\alpha)) \\ &= \sum_{x_\alpha\in A_k''} \rho(C(x_\alpha)) \qquad \text{(by (S4))} \\ &\geq c|A_k''| \geq \frac{c\,|A_k'|}{m^2} \geq \frac{c\,|A_k|}{m^2} - \frac{c\,|A_k|}{2^k\,|S_k|\,m^2}\,.\end{aligned}$$

Therefore

$$\rho_k(\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big)) \geq \frac{c}{m^2} - \frac{c}{2^k\,|S_k|\,m^2}\,.$$

Taking limits along the ultrafilter ω, we get

$$\rho_\omega(\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big)) = \lim_\omega \rho_k((\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big))) \geq \lim_\omega(\frac{c}{m^2} - \frac{c}{2^k\,|S_k|\,m^2}) = \frac{c}{m^2} \neq 0\,.$$

It follows that

$$\theta\Big(\sum_{i=1}^m \overline{g}_i r_i\Big) \notin I_\omega\,.$$

□

Remark 3.3. Observe that the Sylvester rank function ρ_ω in the proof of Theorem 3.2 induces a G-invariant Sylvester rank function ρ'_ω on R via $\rho'_\omega(a) = \rho_\omega(\theta(a))$ for all finite matrices a over R. This new function will agree with the original ρ only in case ρ itself is G-invariant.

We can now establish our principal theorem.

Theorem 3.4. *Let D be any division ring and let G be a group having a free normal subgroup H such that G/H is amenable. Then the group ring $D[G]$ admits a faithful Sylvester rank function. In particular, $D[G]$ is stably finite.*

Proof. Note that a subgroup of a free-by-amenable group is also free-by-amenable. Hence, we only have to prove the result for finitely generated free-by-amenable groups, since $D[G]$ is stably finite if this is true of each subring $D[H]$ with H a finitely generated subgroup.

We have $D[G] = D[H] * (G/H)$. It is well-known that $D[H]$ is a *fir* (see [3, Theorem 5.3.9]), and so it can be embedded in a division ring. Fix an embedding of $D[H]$ into a division ring L and consider the faithful Sylvester rank function ρ_L on matrices over $D[H]$ given by $\rho_L(a) = rank_L(a)$. Since $\rho_L(r) = 1$ for all non-zero elements r in $D[H]$ we trivially have that ρ_L is G-faithful. The result now follows from Theorem 3.2. $\square$

Remarks 3.5. (i) Of course the proof of Theorem 3.4 shows that every crossed product $R * G$ of an amenable group G over a domain R which can be embedded in a division ring is stably finite. If one is only interested in the stable finiteness of $R * G$, there is a somewhat shorter proof which does not use ultrafilters. We sketch this proof for the convenience of the reader. Since R can be embedded in a division ring L we get a faithful Sylvester rank function on R by setting $\rho(a) = rank_L(a)$ for every finite matrix a over R. Set $I = \{x \in \mathbb{T} : \lim_{k\to\infty} \rho_k(x) = 0\}$. By the arguments in the proof of Theorem 3.2, $\theta(R*G) \bigcap I = 0$. Suppose that a and b are $n \times n$ matrices over $R*G$ such that $ab = I_n$. By using Lemma 2.2 one can see that all the entries of $I_n - ba$ belong to I and so $I_n - ba = 0$. (Incidentally I is an ideal of $\mathbb{T}(G, R)$; in fact $I = \bigcap_\omega I_\omega$.)

(ii) The previous observation shows that a more general result is available, namely in the setting that G is a group having a normal subgroup H such that G/H is amenable and such that, for the division ring D, the group ring $D[H]$ is embeddable in a division ring. This last property is known to be satisfied by other classes of groups properly containing the class of free groups. For example, all group rings over division rings of torsion-free one-relator groups can be embedded in division rings, as shown in [9, Theorem 3]. (Evidently, this class contains the class of free groups.) As the result of Mal'cev and Neumann shows (see, e.g. [16, Chapter 13, Theorem 2.11]), every group algebra over a field K of an ordered group can be embedded in a K-division algebra. (The class of ordered groups also extends that of free groups; see [16, Chapter 13, Corollary 2.8].)

(iii) It was observed in [12, page 597] that for a given ring R, the class of groups G such that $R[G]$ is stably finite is residually closed, in the sense that if we have a family $\{H_\alpha\}$ of normal subgroups of G closed under finite intersections, with $\bigcap_\alpha H_\alpha = 1$, and with all rings $R[G/H_\alpha]$ stably finite, then $R[G]$ is also stably finite. Combining this fact with Theorem 3.4 we obtain that $D[G]$ is stably finite for every division ring D and every residually amenable group G.

4. The translation ring of a free group

In this short section, we will prove that the translation ring associated with any finitely generated group containing a free group of rank 2 has no (non-zero) stably finite factors. For non-amenable groups G, this explains why our techniques require us to consider the translation ring over a factor group of G (and over a bigger coefficient ring).

Let G be a finitely generated group and let H be a finitely generated subgroup. For any ring R, the translation ring $\mathbb{T}(H, R)$ is naturally contained in $\mathbb{T}(G, R)$ as follows. First take a system of generators $\mathrm{Gen}(G)$ containing $\mathrm{Gen}(H)$. Let $\mathcal{T}$ be a right transversal of H in G, so that $G = \bigcup_{x_\alpha \in \mathcal{T}} Hx_\alpha$. For a in $\mathbb{T}(H, R)$ define $\varphi(a)$ in $\mathbb{T}(G, R)$ as the linear map

$$\varphi(a)(hx_\alpha) = a(h)x_\alpha\,,$$

or in terms of matrices $\varphi(a)(hx_\alpha, kx_\beta) = \delta_{\alpha,\beta}a(h,k)$ for all h, k in H. It is easy to check that φ provides an isomorphism from $\mathbb{T}(H,R)$ onto a subring of $\mathbb{T}(G,R)$.

Theorem 4.1. *Let G be a finitely generated group containing a free subgroup of rank 2 and let R be any non-zero ring. Then $\mathbb{T}(G,R)$ has no directly finite factor rings.*

Proof. Let H be a free subgroup of rank 2 of G. Since $\mathbb{T}(H,R)$ is embedded in $\mathbb{T}(G,R)$ it suffices to prove that there are orthogonal idempotents e_1 and e_2 in $\mathbb{T}(H,R)$ such that each e_i is equivalent to 1. Let x, y be free generators of H. We will use the reduced expressions of words w in H with respect to x, y. Define R-linear maps a, b, c, d on $R[H]$ by specifying the following rules for their actions on H:

$$a(w) = \begin{cases} xw & \text{if } w \text{ does not start with } x^{-1} \\ yw & \text{if } w \text{ starts with } x^{-1} \end{cases} \quad ;$$

$$b(w) = \begin{cases} x^{-1}w & \text{if } w \text{ starts with } x \\ y^{-1}w & \text{if } w \text{ starts with } yx^{-1} \\ 0 & \text{otherwise} \end{cases} \quad ;$$

$$c(w) = \begin{cases} x^{-1}w & \text{if } w \text{ does not start with } x \\ y^{-1}w & \text{if } w \text{ starts with } x \end{cases} \quad ;$$

$$d(w) = \begin{cases} xw & \text{if } w \text{ starts with } x^{-1} \\ yw & \text{if } w \text{ starts with } y^{-1}x \\ 0 & \text{otherwise} \end{cases} \quad .$$

Clearly $a, b, c, d \in \mathbb{T}(H,R)$. Moreover $1 = ba = dc$, and ab and cd are orthogonal idempotents, as desired. □

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