Matching for discontinuous interval maps

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joint with

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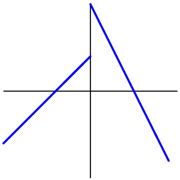
explaining observations in a paper by

V. Botella-Soler, J. A. Oteo, J. Ros, and P. Glendinning

Madrid, July 2014

The map T_β

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + 2 & \text{if } x \le 0, \\ T_\beta^+(x) = \beta - 2x & \text{if } x \ge 0. \end{cases}$$



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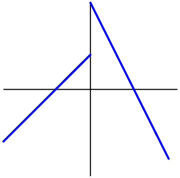

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Figure: Invariant density for the T_β: left $\beta = \frac{1}{2}(\sqrt{5} + 1)$ right: $\beta = \sqrt[3]{7}$.

Markov Partitions and Entropy

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$$\Pi_{i,j} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j, \\ 0 & \text{if } P_j \cap T(P_i) = \emptyset, \\ \text{No other possibility, because } \{P_i\} \text{ is Markov} \end{cases}$$

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The topological entropy is

$$h_{top}(T) = \log \sigma$$

for σ the leading eigenvalue of Π.

Markov partitions and Entropy

Scale Π by the slopes $t_i = |DT_{|P_i}|$ to obtain a matrix

$$A_{i,j} = \frac{1}{t_i}\Pi_{i,j}.$$

Then $\ell_i = |P_i|$ and $\rho_i = \frac{d\mu}{dx}_{|P_i}$ satisfy $\sum_i \rho_i \ell_i = 1$ and

$$\begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T A = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T \quad \text{and} \quad A \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}$$

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Rokhlin's formula gives the metric entropy:

$$h_\mu(T) = \sum_{i=1}^{N} \max\{\log(t_i), 0\}\mu(P_i)$$

Not Markov but Matching

For the family T_β, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

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The pre-matching set is

$$\{T^j(0^-)\}_{j=0}^{\kappa_- -1}\} \cup \{T^j(0^-)\}_{j=0}^{\kappa_+ -1}\};$$

The pre-matching partition are the complementary domains of the prematching set; it plays the role of Markov partition.

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Definition: The matching index is $\Delta = \kappa_+ - \kappa_-$.

Theorem: On every parameter interval where matching occurs, topological and metric entropy

$$h_\mu(T_\beta) \text{ and } h_{top}(T_\beta) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta > 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta < 0, \end{cases}$$

as function of β.

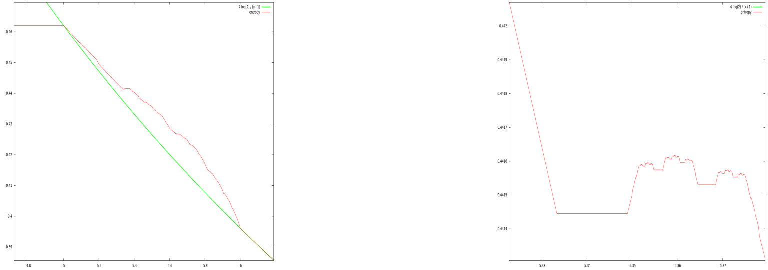
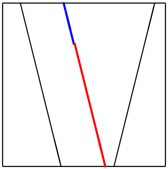


Figure: Entropy $h_\mu(T_\beta)$ for $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

Entropy seems constant on the parameter interval $[2, 5]$; it is filled with countably many intervals on which $\Delta = 0$.

Remarks on the Proof

Let F be the first return map to a nice interval $J \ni T_{\beta}^{\kappa_+(0^+)}$. The return time is τ.



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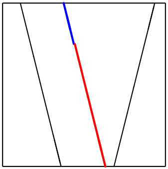

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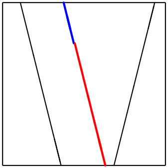

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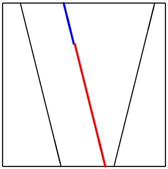

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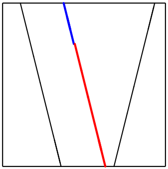

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- The periods of periodic points in J change by Δ if κ_+ is used instead of κ_-. This proportion decreases as β increases. Topological entropy is the exponential growth rate

$$h_{top}(T_\beta) = \lim_n \frac{1}{n} \#\{n\text{-periodic points}\},$$

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Observations towards the proof:

- Let $r_n(x) = \#\{0 \leq i < n : T^n(x) > 0\}$. If $r_m(0^-) = r_n(0^+)$ then $T^m(0^-) - T^n(0^+)$ are a multiple of 2 apart.

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- Therefore, if $T^m(0^-) \in J_\beta$, either $T^m(0^-)$ or $T^{m+1}(0^-)$ will match with $\mathrm{orb}(0^+)$.
- Hence we need to estimate the measure of the set of β such that $\mathrm{orb}(0^-)$ avoids J_β, and in particular is not dense.

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Theorem: The non-matching set E has Hausdorff dimension 1. The left neighborhood of $\beta = 6$ is responsible for this:

$$\dim_H(E \setminus (6 - \varepsilon, 6)) < 1 \text{ for every } \varepsilon > 0.$$

Hausdorff dimension proof

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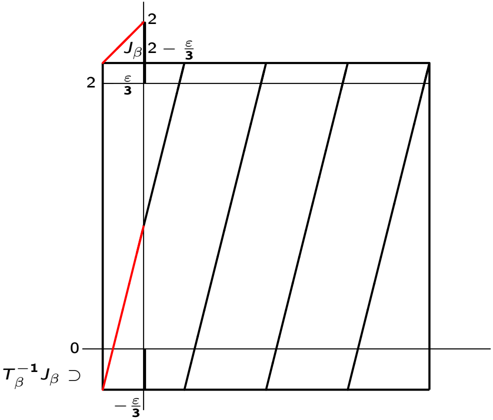
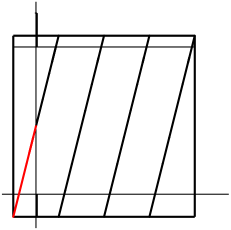


Up to the interval $[-\frac{\varepsilon}{3}, 0]$ which moves directly into J_β, this is a *quadrupling map*.

Hausdorff dimension proof

Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.



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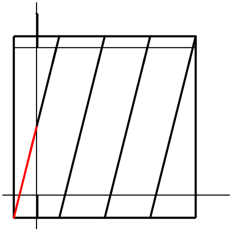

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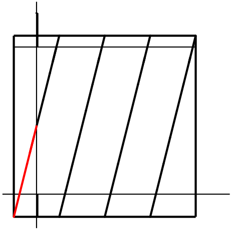

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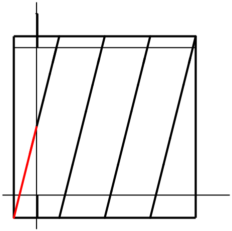

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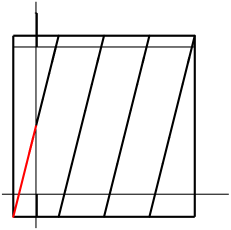

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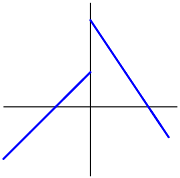
- In fact, $orb(0^-) \subset K_\varepsilon$ iff $orb(0^+) \subset K_\varepsilon$.
- $\dim_H\{\beta : \mathrm{orb}(0^-) \in K_\varepsilon\} = \dim_H(K_\varepsilon)$.

Other slopes

Generalize to slope s

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + s & \text{if } x \leq 0, \\ T_\beta^+(x) = \beta - sx & \text{if } x \geq 0. \end{cases}$$



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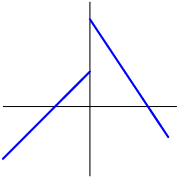

For $s = \frac{1}{2}(\sqrt{5}+1)$ and $\sqrt{2}+1$ and some other, large intervals of matching has been observed.

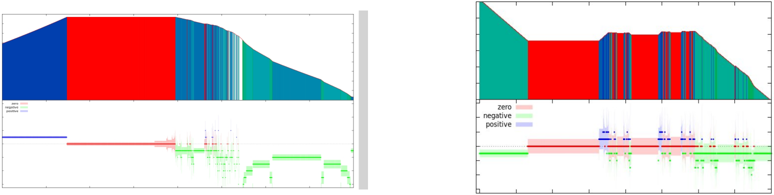

Figure: $h_\mu(T_\beta)$ for $s = \frac{\sqrt{5}+1}{2}$, $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

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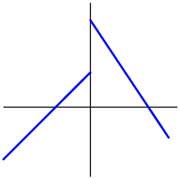

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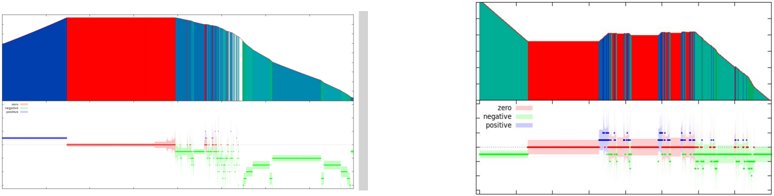

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Note that these slopes are quadratic Pisot numbers.

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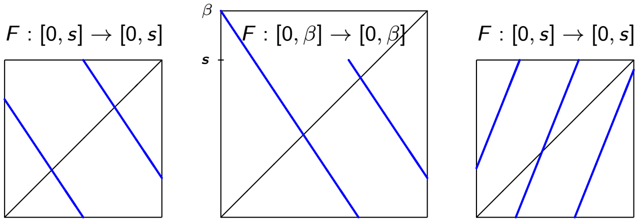

matching   any matching??   typical matching?

Figure: Return map F for $\beta < s$, $s < \beta < 3 + \sqrt{5}$, and $\beta > 3 + \sqrt{5}$.

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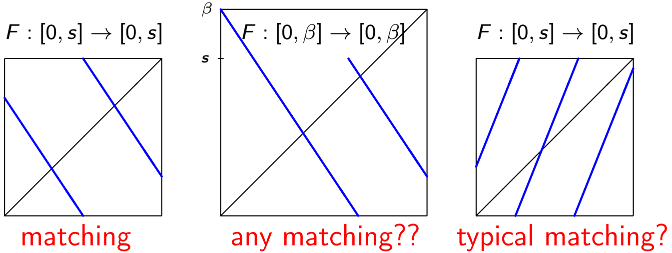

Figure: Return map F for $\beta < s$, $s < \beta < 3 + \sqrt{5}$, and $\beta > 3 + \sqrt{5}$.

F acts affinely on H. Restricted to $\mathrm{orb}(0^\pm)$, we need to iterate

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \tau_n \\ 0 \end{pmatrix}$$

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where $\tau_n(0^{\pm})$ is the branch number containing $F^n(0^{\pm})$, starting with

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Question: Does this happen Lebesgue typically for $s = \frac{\sqrt{5}+1}{2}$?

C. Bonanno, C. Carminati, S. Isola, G. Tiozzo, *Dynamics of continued fractions and kneading sequences of unimodal maps*, Discrete Contin. Dyn. Syst. **33** (2013), no. 4, 1313–1332.

V. Botella-Soler, J. A. Oteo, J. Ros, P. Glendinning, *Families of piecewise linear maps with constant Lyapunov exponents*, J. Phys. A: Math. Theor. **46** 125101

C. Carminati, G. Tiozzo, *Tuning and plateaux for the entropy of α-continued fractions*, Nonlinearity **26** (2013), no. 4, 1049–1070.

H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426

H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity, **21** (2008), 1207–1225.