Shape and Conley index of plane continua

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(Joint work with José M. R. Sanjurjo)

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Definitions

- M, C^∞ compact n-dimensional manifold .
- $f : M \to \mathbb{R}$, C^∞ function

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Let $p \in Crit(f)$, we say that p is *non-degenerated* if $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j} \neq 0$. We say that f is a *Morse function* if each one of its critical points is non-degenerated. We define the *Morse index* of a non-degenerated critical point p as the number of negative eigenvalues of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$.

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Morse inequalities

Morse inequalities point out the relationship between the topology of the manifold and the Morse functions defined on it.

Theorem (Morse inequalities)

Let $f : M \to \mathbb{R}$ be a Morse function defined on a manifold. Let c_i be the total number of critical points which have index i and β_i the i-dimensional Betti number of the manifold M. Then:

- $\beta_i \leq c_i$,
- $\sum_{i=0}^{n} (-1)^i c_i = \chi(M)$.

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The torus and the heigh function

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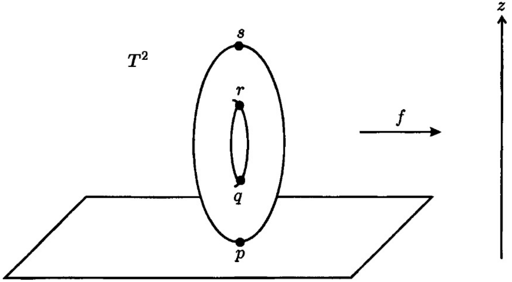

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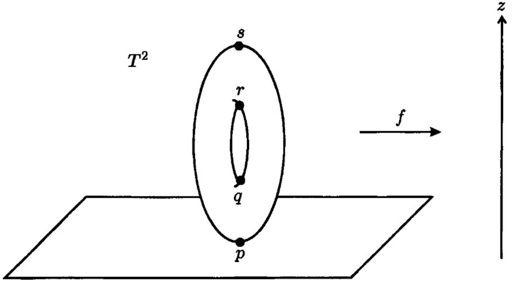

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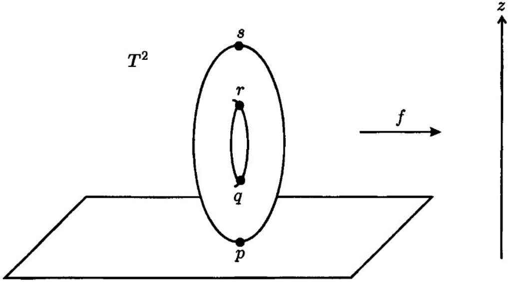

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Dynamical interpretation

The study of Morse functions is equivalent to the study of the ODE

$$\dot{x} = -\nabla f(x).$$

If each critical point of f is non-degenerated, it means that each critical point $-\nabla f$ is *hyperbolic*. In addition, the *Morse index* of each one agrees with the dimension of its *unstable manifold*. We infer that the Morse inequalities point out the relationship between the dynamics defined by the vector field $-\nabla f$ in a neighbourhood of each critical point and the topology of the manifold M.

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Conley's problem

- Let $\varphi : X \times \mathbb{R} \to X$ be a flow defined on a locally compact metric space.
- Let $K \subset X$ be an isolated compact invariant subset.
- Is possible to define an index which provides some information about the dynamics in a neighbourhood of K and such that it generalises in a natural way the Morse index?
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- Let K be an isolated invariant compactum,
- (N, L) an *isolating block* of K.

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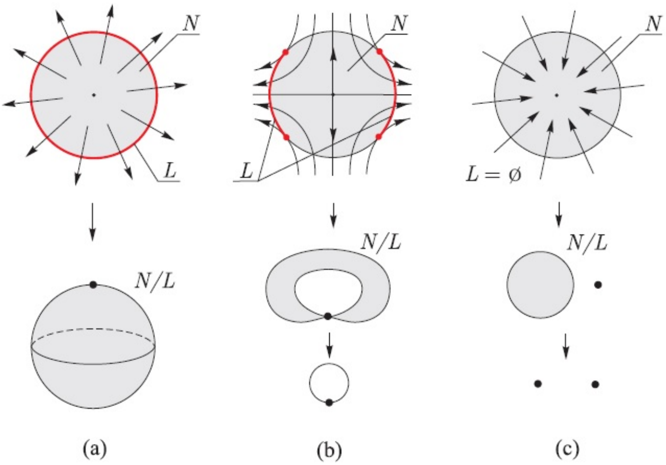
Some examples



The Conley index for a source, a saddle and a sink.

Properties

- K admits a base of isolating blocks,
- $h(K)$ does not depend on the choice of the isolating block,
- $K \neq \emptyset$ if $h(K) \neq \{*\}$,
- If $K = K_1 \cup K_2$ with $K_1 \cap K_2 = \emptyset$, $h(K) = h(K_1) \vee h(K_2)$,
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Conley index of hyperbolic critical points

Theorem

Let p be a hyperbolic critical point of a smooth flow defined on a manifold. Then $h(\{p\}) = (\mathbb{S}^k, \{*\})$ where k is the Morse index of p.

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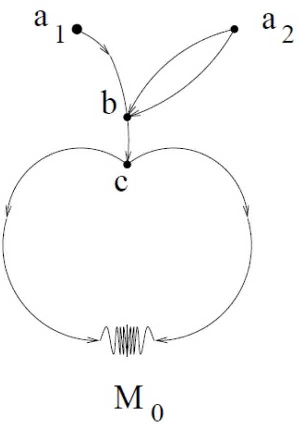
Example



$$M_3 = \{ a_1 , a_2 \}$$

$$M_2 = \{ b \}$$

$$M_1 = \{ c \}$$

$$M_0 = \{ \mid \}$$

The Morse equation

The *Poincaré polinomial* of a topological space X (we suppose spaces with good local behaviour, i.e., compact manifolds, finite CW-complexes, polyhedra...) is defined

$$p_t(X) = \sum_q \beta_q t^q,$$

where β_q denotes the q-dimensional Betti number of X.

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Let K be an isolated invariant compactum of a flow φ and $\{M_1, M_2, \dots M_k\}$ a Morse decomposition of K. Then there is a polinomial $q(t)$ with non-negative integer coefficients such that:

$$\sum_i p_t(h(M_i)) = p_t(h(K)) + (1+t)q(t).$$

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The isolating block problem

Although the Conley index of an isolating invariant compactum K does not depend on the choice of the isolating block N it would be useful to avoid this calculation. In this way, Robbin and Salamon defined a topology in the *unstable manifold* of K, $W^u(K) = \{x \in X \mid \lim_{t \to -\infty} d(xt, K) = 0\}$ which is called the *intrinsic topology* and they proved that the *shape* of the Conley index agrees with the *shape* of the Alexandrov compactification of $W^u(K)$ endowed with the *intrinsic topology*. This approach presents two problems:

1. The original definition of the intrinsic topology involves isolating blocks.
2. This topology is very far to be intuitive and easy to understand. In addition it does not agree in general with the topology as a subespace of the phase space (which is known as the extrinsic topology).

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A result of José M. R. Sanjurjo shows that the intrinsic topology agrees with the extrinsic topology if and only if K is a global repeller in $W^u(K)$.

Our goal is to introduce a way of calculation the Conley index which does not involve neither isolating blocks nor the intrinsic topology. This new method, which will be explained next, is based on the study of the flow in a part of the unstable manifold. Until now, we only could present a complete treatment of this method on $\mathbb{R}^2$. In spite of this, it gave us a total characterization of the conley index of plane flows.

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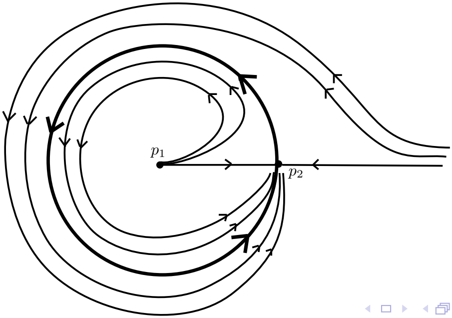
Parallelizable flows

It seems clear that if we choose a suitable compact section S of the flow restrictred to $W^u(K) \setminus K$ it is *parallelizable*. This means that the set $S \times \mathbb{R}$ is homeomorphic to $S\mathbb{R}$ and the homeomorphism is provided by the flow. However this is not true at all, for instance the mendelson attractor provides an example (with $W^s(K)$).



The initial part of $W^u(K) \setminus K$

In spite of the flow may not be parallelizable in $W^u(K) \setminus K$, it is possible to restrict ourselves to a suitable situation.

Definition

Let S be a section of the flow restricted to $W^u(K) \setminus K$. We define the S-initial part of $W^u(K) \setminus K$ as the subset

$$I_S^u(K) = \{x \in W^u(K) \setminus K \mid \text{there exists } t > 0 \text{ such that } xt \in S\}.$$

We say that a section S is initial provided that it is compact and $\omega^*(S) \subset K$. Under this assumptions we have that $I_S^u(K)$ is homemorphic to $S \times (-\infty, 0]$.

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If N is an isolating block and we denote by N^- the negatively invariant part of N, then $N^- \setminus K$ agrees with $I^u_{n^-}(K)$, where $n^- = N^- \cap L$.
If $X = \mathbb{R}^2$ It is possible to prove that K admits a basis of isolating blocks with the extra property of being compact surfaces with boundary which satifies that L is a 1-manifold with boundary. By combining this with the fact that all initial sections are homeomorphic (the flow provides such homeomorphisms) we can state the next result

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Let φ be a flow defined on $\mathbb{R}^2$ and K and isolated invariant continua. Then, each initial section S of $W^u(K) \setminus K$ has the homotopy type of a finite disjoint union of circles and points.

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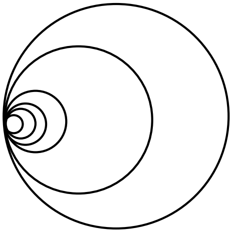
Compact invariant plane continua

Our aim is to calculate the Conley index of an invariant plane continua by using the data provided by the dynamics in the initial part of its unstable manifold (say the initial section) and the topology of our plane continua K. Particularly we are interested in the number of connected components in which our continua disconnects the plane. Because of K is isolated and invariant, this number must be finite.



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Let K be a non-empty isolated invariant continuum of the flow $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and S an initial section of $W^u(K) \setminus K$.

1. *If $u \neq 0$ and $u - u_c \leq n$ then the Conley index of K is the pointed homotopy type of $\left(\bigvee_{i=1,\ldots,k} \mathbb{S}^1_i, *\right)$, where $k = n + u_c - 2$,*
2. *if $u - u_c = n$ then K is a repeller and its Conley index is the pointed homotopy type of $\left(\mathbb{S}^2 \bigvee \left(\bigvee_{i=1,\ldots,n-1} \mathbb{S}^1_i\right), *\right)$,*
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3. *If $u = 0$ then K is an attractor and its Conley index is the pointed homotopy type of $\left(\bigvee_{i=1,..,n-1} \mathbb{S}^1_i \cup \{\bullet\}, \bullet\right)$, where the $\mathbb{S}^1_i$ are pointed $1-$spheres based on $*$ and $\bullet$ denotes a point not belonging to $\bigvee_{i=1,..,n-1} \mathbb{S}^1_i$.*

The main result

Theorem

Let K be a non-empty isolated invariant continuum of the flow $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and S an initial section of $W^u(K) \setminus K$.

1. *If $u \neq 0$ and $u - u_c \leq n$ then the Conley index of K is the pointed homotopy type of $\left(\bigvee_{i=1,\ldots,k} \mathbb{S}^1_i, *\right)$, where $k = n + u_c - 2$,*
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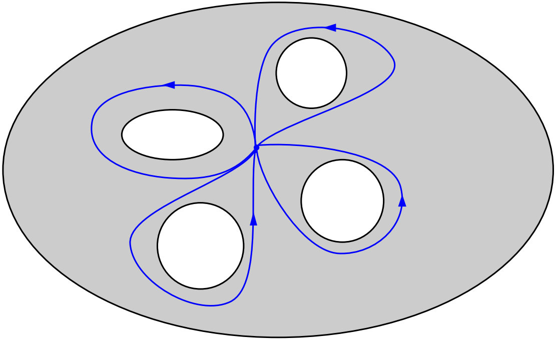
Sketch of the proof

If $\mathbb{R}^2 \setminus K$ has n connected componentens we can choose an isolating block homeomorphic to a disc with $n-1$ holes.



Case 1

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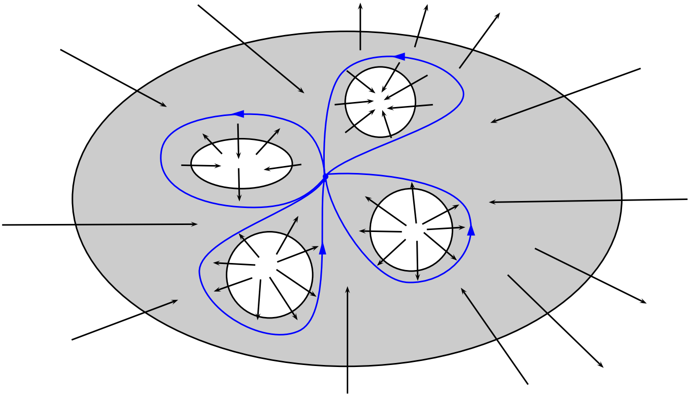
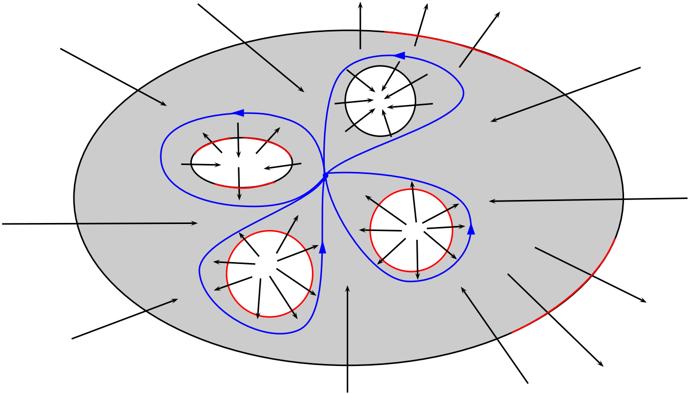


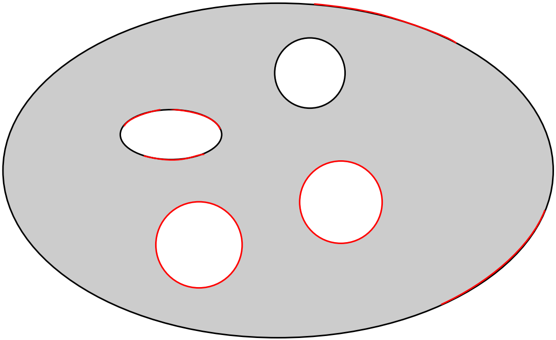
Case 1

L must have u_c components which are intervals and $u - u_c$ which are circumpherences.



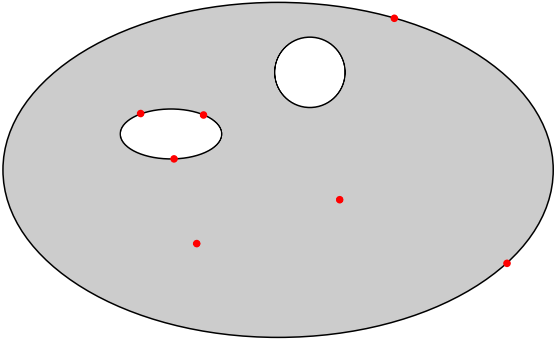
Case 1

So we have to collapse the red sets to a point.



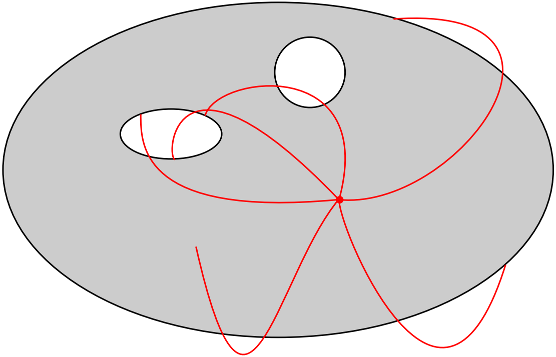
Case 1

At first, we collapse each one to a different point. We observe that each circular component fills the hole.



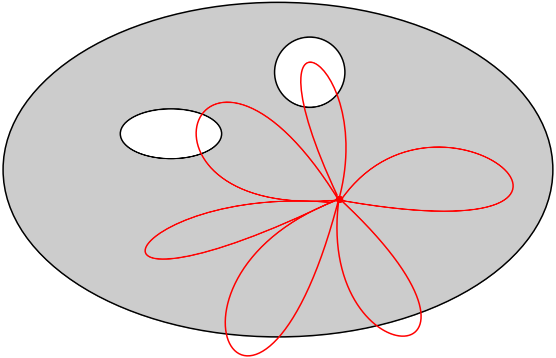
Case 1

We obtain a disk with $n - u + u_c - 1$ holes and $u - 1$ one dimensional “handles” attached.



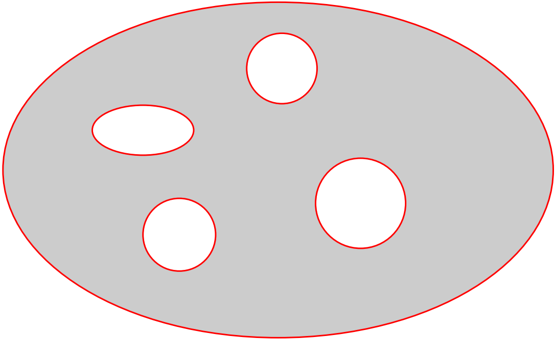
Case 1

So we obtain a topological space with the homotopy type of a wedge of $n + u_c - 2$ circunpherences.



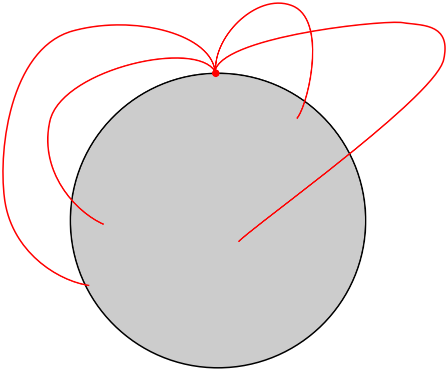
Case 2

If $u - u_c = n$, then L must be the whole boundary



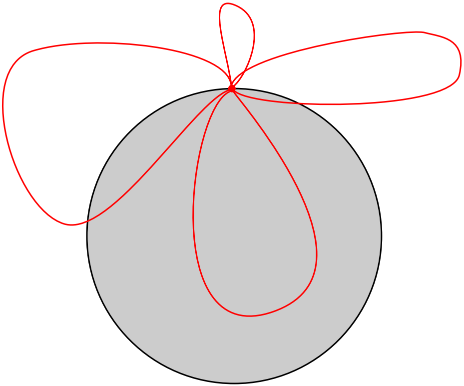
Case 2

If we collapse the whole boundary to a point, we obtain a 2-sphere with $n-1$ one dimensional handles attached.



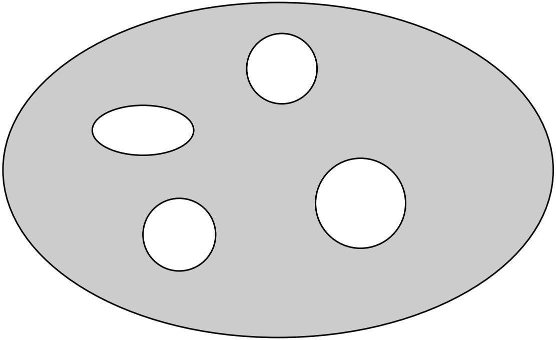
Case 2

It is exactly a wedge of a 2-sphere with $n-1$ circumpherences.



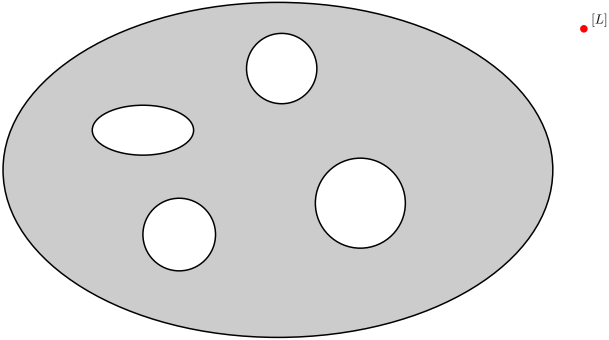
Case 3

If $u = 0$, then $L = \varnothing$.



Case 3

So the empty set collapses to a point and we obtain a disjoint union of a disk with $n-1$ holes and a point.



Definition

Let $\Omega \subset \mathbb{R}^n$ be a *bounded open* subset and $f : \overline{\Omega} \to \mathbb{R}^n$ map which has no zeros in $\partial\Omega$.

The *Brouwer degree* of f with respect to the open set Ω, as the *only* integer $\deg(f, \Omega)$ which satisfies

1. Non-triviality: if $0 \in \Omega$, then $\deg(id, \Omega) = 1$,
2. existence: if $\deg(f, \Omega) \neq 0$, then $f^{-1}(0) \cap \Omega \neq \varnothing$,
3. aditivity: if Ω_1 and Ω_2 are two open disjoint subsets contained in Ω and f does not have zeros in $\Omega \setminus (\Omega_1 \cup \Omega_2)$, then $\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2)$,
4. homotopical invariance: if $F : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is a map which satifies that $F(x, t) \neq 0$ for each $(x, t) \in \partial\Omega \times [0, 1]$, then

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Brouwer degree and the Conley index

Theorem

Let $X : \Omega \to \mathbb{R}^n$ be a vector field and φ the flow induced by X. If K is an isolated invariant compactum and N an isolating neighbourhood of K, then

$$\deg(X, \operatorname{int}(N)) = (-1)^n \chi(h(K)).$$

Corollary

Let $X : \Omega \to \mathbb{R}^2$ be a vector field and φ the flow induced by X. If K is an isolated invariant compactum and N an isolating neighbourhood of K, then

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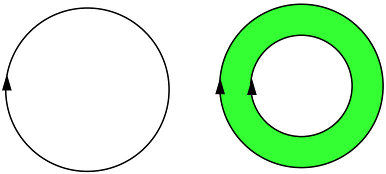
Application

The last formula allows us to infer the next result.

Theorem

Let K be a non-empty isolated invariant continuum which does not contain equilibrium points. Then it must be a non-saddle set which disconnects the plane into two connected components. Moreover it must be a limit cycle or an annulus.



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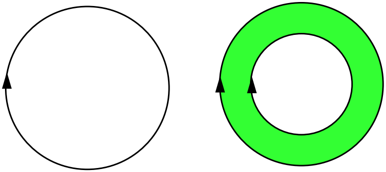

Thank you for your attention!!