Hochschild Cohomology for Involutive A_∞-algebras

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Motivation of the Problem

Kevin Costello in 2007 classifies oriented Open-closed TCFTs and computes its homology by stating that it is the Hochschild homology of the open state sector of the theory.

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Our project in Swansea seeks to generalize Costello's theorem to a G-equivariant setting. Therefore, a good knowledge of both Hochschild homology and cohomology is basic.

Main Results

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Proposition

For an involutive associative algebra A and an involutive A-bimodule M, the following quasi-isomorphism holds:

$$\Sigma^{-1}\operatorname{Der}^{+}(\widehat{T}\Sigma^{-1}M^{\star},\widehat{T}\Sigma^{-1}A^{\star}) \cong \mathcal{R}\operatorname{Hom}_{iA-Bimod}(A,M).$$

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Proposition

For an involutive A_∞-algebra A and an involutive A_∞-bimodule M we have: $C^{\bullet}(A,M) \cong \operatorname{Hom}_{\overline{iA-Bimod}}(A,M)$.

Involutive Algebras

Involutive Algebras

An involutive $\mathbb{K}$-algebra A is an algebra over a field $\mathbb{K}$ endowed with a $\mathbb{K}$-linear map (an involution) $^* : A \to A$ satisfying:

1. $(a^*)^* = a$;
2. $(a \cdot b)^* = b^* \cdot a^*$ for every $a, b \in A$.

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Given two involutive A-bimodules M, N, a morphism between them is a morphism of $M \xrightarrow{f} N$ that preserves the involution.

Involutive Algebras

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Let us denote, for an involutive A-bimodule M, the space of involution-preserving maps $M \xrightarrow{d} A$ satisfying the Leibniz rule

$$d(x \cdot y) = d(x) \cdot y + (-1)^{|x| \cdot |d|} \cdot x \cdot d(y)$$

as $\mathrm{Der}^{+}(\widehat{T}\Sigma^{-1}M^{*}, \widehat{T}\Sigma^{-1}A^{*})$.

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We denote by $\mathrm{Hom}^{+}_{\mathbb{K}-\mathtt{Mod}}(A, M)$ the space of homomorphisms $f : A \to M$ which preserve involutions.

Involutive Algebras

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For an involutive A-bimodule M, giving a derivation m in $\mathrm{Der}^{+}(\widehat{T}\Sigma^{-1}M^{*}, \widehat{T}\Sigma^{-1}A^{*})$ is equivalent to giving a map

$$\overline{m} \in \bigoplus_{n} \mathrm{Hom}_{\mathbb{K}}^{+}(A^{\otimes n}, M),$$

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which yields the following isomorphism:

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Let us observe that $\mathrm{Hom}^+_{\mathbb{K}-\mathtt{Mod}}(A, M)$ can be endowed with the following involution: $f^\star(x) = -f(x^\star)$.

Involutive Algebras

Involutive Algebras

Lemma

For an involution-preserving morphism f, the morphism

$$\begin{aligned} df(a_0 \otimes \ldots \otimes a_n) &= a_0 f(a_1 \otimes \ldots \otimes a_n) \\ &+ \sum_{i=0}^{n-1} (-1)^i f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\ &+ (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}) a_n \end{aligned}$$

is involution-preserving.

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$$a^* = (a_0 \otimes \cdots \otimes a_{n+1})^* = a_{n+1}^* \otimes \cdots \otimes a_0^*,$$

$\mathrm{Bar}_n(A)$ becomes an iA-bimodule which can be given the structure of chain complex with a map $\mathrm{Bar}_n(A) \xrightarrow{b_n} \mathrm{Bar}_{n-1}(A)$:

$$b_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}.$$

Involutive Bar Resolution

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Lemma

For an involutive $\mathbb{K}$*-algebra* A*, the map*

$$b_n : \mathrm{Bar}_n(A) \to \mathrm{Bar}_{n-1}(A)$$

is involution-preserving.

Involutive Bar Resolution

Lemma

For an involutive $\mathbb{K}$-algebra A, the map

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Lemma

For an involutive $\mathbb{K}$-algebra A, $\mathrm{Bar}(A)$ is an involutive projective resolution for A.

Involutive A_∞-algebras

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An involutive A_∞-algebra is an involutive graded space A endowed with maps

$$b_n : (SA)^{\otimes n} \to SA,\ n \geq 1,$$

of degree 1 such that the identity below holds:

$$\sum_{i+j+l=n} b_{i+j+l} \circ (\mathrm{Id}^{\otimes i} \otimes b_j \otimes \mathrm{Id}^{\otimes l}) = 0,\ \forall n \geq 1.$$

Morphisms of Involutive A_∞-algebras

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A morphism of involutive A_∞-algebras $f : A_1 \to A_2$ is given by an a series of homogeneous involution-preserving maps of degree zero $f_n : (SA_1)^{\otimes n} \to SA_2$, $n \geq 1$, such that

$$\sum_{i+j+l=n} f_{i+l+1} \circ (\mathrm{Id}^{\otimes i} \otimes b_j \otimes \mathrm{Id}^{\otimes l}) = \sum_{i_1+\cdots+i_s=n} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s})$$

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Composition of morphisms of A_∞-algebras is given by

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The identity on SA is defined as $f_1 = \mathrm{Id}$ and $f_n = 0$ for $n \geq 2$.

A_∞-quasi-isomorphisms

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We say that $f : A_1 \to A_2$ is an A_∞-quasi-isomorphism if f_1 is a quasi-isomorphism.

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Proposition

Let A be an involutive A_∞-algebra, V a complex and $f_1 : A \to V$ a quasi-isomorphism of complexes. Then V admits a structure of involutive A_∞-algebra such that f_1 extends to an A_∞-quasi-isomorphism $f : A \to V$.

Modules and Bimodules Over Involutive A_∞-algebras

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If M is a graded $\mathbb{K}$-module, an involutive left-module structure for M over an involutive A_∞-algebra A is an involution-preserving differential on $BA \otimes M$ over BA compatible with the differential on BA.

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An involutive bimodule structure for M over an involutive A_∞-algebra A is an involution-preserving differential on the bi-comodule $BA \otimes M \otimes BA$ over BA compatible with the differential on BA.

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The differential on $BA \otimes M$ is given by a series of maps, asked to be involution-preserving, b_n^M:

$$b_n^M : A^{\otimes(n-1)} \otimes M \to M.$$

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For an involutive bimodule the picture is

$$b_n^M : A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(j-1)} \to M.$$

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All these maps must satisfy the identity:

$$\sum_{i+j+l=n} b_{i+j+l}^M \circ (\mathrm{Id}^{\otimes i} \otimes b_j^M \otimes \mathrm{Id}^{\otimes j}) = 0.$$

Morphisms Between Bimodules

Morphisms Between Bimodules

A morphism of involutive A_∞-bimodules $f : L \to M$ is a given by a collection of maps $f_{i,j} : A^{\otimes(i-1)} \otimes L \otimes A^{\otimes(j-1)} \to M$ satisfying, for $a \in A^{\otimes(i-1)}, l \in L, a' \in A^{\otimes(j-1)}$:

$$f_{i,j}((a,l,a')^\star) = (f_{i,j}(a,l,a'))^\star$$

and certain compatibility conditions.

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For an involutive A_∞-algebra A we define $\overline{iA - \texttt{Bimod}}$ a category with objects involutive A-bimodules and where $\mathrm{Hom}_{\overline{iA-\texttt{Bimod}}}(M, N)$ is:

$$\underline{\mathrm{Hom}}^n(BA \otimes M, BA \otimes N) := \prod_{i \in \mathbb{Z}} \mathrm{Hom}((BA \otimes M)^i, (BA \otimes N)^{i+n}).$$

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The differential sends $\{f_i\}_i$ to $\{m^N \circ f^i - (-1)^n f^{i+1} \circ m^M\}_i$.

The Involutive Hochschild Cochain Complex

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Let us denote with $iA - \texttt{Bimod}$ the category of involutive A-bimodules; since $\operatorname{Bar}(A)$ is an involutive resolution for A:

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$$\begin{aligned}\mathcal{R}\operatorname{Hom}_{iA-\texttt{Bimod}}(A, M) &\cong \operatorname{Hom}_{iA-\texttt{Bimod}}(\operatorname{Bar}(A), M) \\ &\cong \operatorname{Hom}^{+}_{\mathbb{K}-\texttt{Mod}}(A^{\bullet}, M).\end{aligned}$$

Main Result for Involutive Algebras

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Lemma

The right derived functor is well defined: given two involutive projective resolutions $P \to A \leftarrow Q$ *and a left exact functor* $iA - \mathtt{Bimod} \xrightarrow{\mathcal{F}} iA - \mathtt{Bimod}$: $\mathcal{R}_n(A) = \mathrm{H}^n(\mathcal{F}(P)) \cong \mathrm{H}^n(\mathcal{F}(Q))$.

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Proposition

For an involutive associative algebra A *and an involutive* A*-bimodule* M*, the complex* $\Sigma^{-1}\operatorname{Der}^+(\widehat{T}\Sigma^{-1}M^*, \widehat{T}\Sigma^{-1}A^*)$ *is quasi-isomorphic to* $\mathcal{R}\operatorname{Hom}_{iA-\texttt{Bimod}}(A, M)$.

The Involutive Hochschild Cochain Complex

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The Hochschild cochain complex of an involutive A_∞-algebra A with coeficients on an involutive A_∞-bimodule M is defined as the $\mathbb{K}$-vector space

$$C^n(A, M) := \prod_{n \geq 0} \mathrm{Hom}^+_{\mathbb{K}-\mathtt{Mod}}((SA)^{\otimes n}, M).$$

Technicalities

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Lemma

For an involutive A_∞-algebra A there is a natural involution-preserving A_∞-quasi-isomorphism, then a homotopy equivalence, of involutive A_∞-bimodules $B(A, A, A) \to A$.

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Lemma

Let B be an A_∞-algebra. If P, Q are homotopy equivalent as involutive B-bimodules then, for every involutive B-bimodule A, the following quasi-isomorphism holds:

$$\mathrm{Hom}_{\overline{iB-Bimod}}(P, A) \cong \mathrm{Hom}_{\overline{iB-Bimod}}(Q, A).$$

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Proof.

$$\prod_{n\geq 0} \mathrm{Hom}^+_{\mathbb{K}-\mathtt{Mod}}((SA)^{\otimes n}, M) \cong \mathrm{Hom}_{\overline{iA-\mathtt{Bimod}}}(B(A,A,A),M) \cong \mathrm{Hom}_{\overline{iA-\mathtt{Bimod}}}(A,M).$$

□

Diolch yn fawr

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