Looking for Morse functions on symmetric spaces

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Orthogonal Lie groups

- Let $\mathbb{K}$ be one of the algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternions).

$$O(n,\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} \colon AA^* = \mathrm{I}\}$$

  is the compact Lie group of orthogonal (resp. unitary, symplectic) matrices
- Let G be the connected component of the identity ($SO(n)$, $U(n)$ or $Sp(n)$).
- The Lie algebra of G is formed by the skew-symmetric (resp. skew-Hermitian) matrices,

$$\mathfrak{g} = \{X \in \mathbb{K}^{n\times n} \colon X + X^* = 0\}.$$

- The Riemannian metric induced on $G \subset \mathbb{K}^{n\times n}$ by the usual inner product $\langle X, Y\rangle = \Re\,\mathrm{Tr}(X^*Y)$ is bi-invariant

Compact symmetric spaces

- Let $\sigma\colon G \to G$ be an involutive automorphism and

$$K = G^\sigma = \{B \in G \colon \sigma(B) = B\}$$

- We shall assume that σ is the restriction of an involutive automorphism $\sigma : \mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}$ of unital algebras.
- Also, $\sigma(X^*) = \sigma(X)^*$ for all $X \in \mathbb{K}^{n\times n}$.

⋆ *These conditions are not too restrictive; for instance, all the compact irreducible Riemannian symmetric spaces in Cartan's classification fullfill them.*

- The homogeneous space G/K (null torsion and parallel curvature) is called a **globally symmetric compact space.**
- The embedding $\gamma([B]) = B\sigma(B)^{-1}$, $\gamma\colon G/K \hookrightarrow G$, is an isometry (up to the constant 2).

Proposition

Assume that G/K is connected. Then the image $M = \gamma(G/K)$ of γ is the connected component N_I of the identity of the submanifold

$$N = \{B \in G\colon \sigma(B) = B^{-1}\}.$$

- The manifold M will be called the **Cartan model** of the symmetric space G/K.
- The isometric action of G induced by γ on M is given by $I_B^M(A) = BA\sigma(B)^{-1}$, for $B \in G$, $A \in M$.

Espacios simétricos irreducibles, compactos y simplemente conexos:

Tipo	Modelo Cartan	dim	$\sigma(X)$
AI	$SU(n)/SO(n)$	$(n-1)(n+2)/2$	$\overline{X}$
AII	$SU(2n)/Sp(n)$	$(n-1)(2n+1)$	$-J\overline{X}J$
AIII	$SU(p+q)/SU(p)\times SU(q)$	$2pq$	$\mathrm{I}_{p,q}X\,\mathrm{I}_{p,q}$
BDI	$SO(p+q)/SO(p)\times SO(q)$	pq	$\mathrm{I}_{p,q}X\,\mathrm{I}_{p,q}$
DIII	$SO(2n)/U(n) \quad [n\geq 4]$	$n(n-1)$	$-J\overline{X}J$
CI	$Sp(n)/U(n) \quad [n\geq 3]$	$n(n+1)$	$-\mathbf{i}X\mathbf{i}$
CII	$Sp(p+q)/Sp(p)\times Sp(q)$	$4pq$	$\mathrm{I}_{p,q}X\,\mathrm{I}_{p,q}$

Example

The Lie group G itself is a symmetric space defined by the automorphism

$$\begin{array}{rccc} \sigma: & G \times G & \to & G \times G \\ & (B_1, B_2) & \mapsto & (B_2, B_1) \end{array}$$

- The fixed point set is the diagonal Δ.
- The diffeomorphism $G \cong (G \times G)/\Delta$ is given by $B \cong [(B, \mathrm{I})]$.
- $N = \{(B, B^{-1}) \in G \times G\}$

Proposition

For any point $A \in M$, the tangent space is

$$T_A M = \{Y \in \mathbb{K}^{n \times n} \colon YA^* + AY^* = 0, \, \sigma(Y) = Y^*\}.$$

- The height function $h_X \colon \mathbb{K}^{n\times n} \to \mathbb{R}$ with respect to an hyperplane perpendicular to $X^* \in \mathbb{K}^{n\times n}$ $(X \neq 0)$ is given, up to a constant, by

$$h_X(Y) = \langle X^*, Y\rangle = \Re\,\mathrm{Tr}(XY).$$

- Let $h_X^M \colon M \to \mathbb{R}$ be the restriction of h_X to the Cartan model $M \subset G \subset \mathbb{K}^{n\times n}$ of the symmetric space G/K.

  We denote $\widehat{X} := X^* + \sigma(X)$. Notice that $\sigma(\widehat{X}) = \widehat{X}^*$.

Proposition

The gradient of h_X^M at any point $A \in M$ is the projection of $\mathrm{grad}\, h_X$ onto $T_A M$, that is,

$$(\mathrm{grad}\, h_X^M)_A = \frac{1}{4}\left(\widehat{X} - A\sigma(\widehat{X})A\right).$$

Remark.– Instead of height, one can consider the *distance* to X^*. Since

$$|A - X^*|^2 = a h_X^M(A) + b, \quad a, b \in \mathbb{R},$$

both functions have the same critical points in M.

Geometrically, these are the points where the line $\overrightarrow{AX^*}$ is perpendicular to $T_A M$.

Proposition

The Hessian $H(h_X^M)_A \colon T_AM \to T_AM$ *of the height function* $h_X^M \colon M \to \mathbb{R}$ *is given by*

$$H(h_X^M)_A(W) = -\frac{1}{4}\left(A\sigma(\widehat{X})W + W\sigma(\widehat{X})A\right).$$

An easy computation shows that:

(i) A is a critical point of h_X^M if and only if the matrix $\widehat{X}^*A$ is Hermitian;

(ii) $W \in T_AM$ iff WA^* is skew-Hermitian and $\sigma(W) = W^*$;

(iii) W is in the kernel of the Hessian if in addition the matrix $\widehat{X}^*W$ is Hermitian.

Example (The complex Grassmannian $U(2)/(U(1)\times U(1))$)

It is defined by the automorphism $\sigma(A)=\mathrm{I}_{1,1}A\mathrm{I}_{1,1}$, where $\mathrm{I}_{1,1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The Cartan model is an $S^2\subset U(2)\cong S^3\times S^1$,

$$M=\left\{\begin{pmatrix}s & -\overline{z}\\ z & s\end{pmatrix}, (s,z)\in\mathbb{R}\times\mathbb{C}\colon s^2+|z|^2=1\right\}.$$

- Let us take on M h_X^M with $X=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Then $\widehat{X}=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$ and the critical points are the two poles $\pm\mathrm{I}$.
- The tangent space is $T_{\varepsilon\mathrm{I}}M=\left\{W=\begin{pmatrix}0 & z\\ -\overline{z} & 0\end{pmatrix}, z\in\mathbb{C}\right\}$ and $(Hh_X^M)_{\varepsilon\mathrm{I}}(W)=(-\varepsilon/2)W$, so **$h_X^M$ is a Morse function on M.**

- The gradient of $h_X^G \colon G \to \mathbb{R}$ at $A \in G$ is

$$(\operatorname{grad} h_X^G)_A = \frac{1}{2}(X^* - AXA).$$

- The Hessian $\mathrm{H}(h_X^G)_A \colon T_A G \to T_A G$ is given by

$$\mathrm{H}(h_X^M)_A(W) = -\frac{1}{2}\left(AXW + WXA\right).$$

⋆ A similar computation is valid *mutatis mutandi* for the gradient flow and the local structure of the critical set in the group G. In all formulae it is enough to substitute $\widehat{X}$ by $2X^*$.

Corollary

Let $M \subset G$ be the Cartan model of the symmetric space G/K. Then the critical set in M of the height function h_X^M is

$$\Sigma(h_X^M) = \Sigma(h^G_{\sigma(\widehat{X})}) \cap M.$$

- So $\Sigma(h_X^G) \cap M \subset \Sigma(h_X^M)$.
- Notice that $X^* = AXA \Rightarrow \widehat{X} = A\sigma(\widehat{X})A$ when $\sigma(A) = A^*$.

Corollary

If $\sigma(X) = X^*$, then the critical points of the height function h_X^M on the symmetric space M verify that $\Sigma(h_X^M) = \Sigma(h_X^G) \cap M$.

- Ramanujam [*J. Differ. Geom.*, 1969] stated that

  > *the critical submanifolds of G/K are shown to be the intersection of the space G/K and the critical submanifolds of G.*

- Dynnikov-Veselov [*St. Petersbg. Math. J.*, 1997] wrote that

  > *symmetric spaces [...] are invariant by the gradient flow of the height function on the corresponding Lie groups*

  and that

  > *the restricted flow coincides with the gradient flow of the [restricted] function.*

- As we have just seen, this kind of result only holds in particular –although important– cases, but is no longer true for a generic height function on a symmetric space:

– When $X = \mathrm{I}$ the critical points of the height function h_X^M are just the points of $\Sigma(h_X^G)$ that belong to M.

– The same result is true when X is a real diagonal matrix for some symmetric spaces (studied by Duan [Birkhäuser, 2005] and Dynnikov-Veselov [*St. Petersbg. Math. J.*, 1997]).

As a rule, the preceding result no longer holds:

Example ($Sp(1)/U(1)$)

- It is defined by the automorphism $\sigma(X) = -\mathbf{i}X\mathbf{i}$.
- The Cartan model M is the sphere $S^2 \subset Sp(1) = S^3$ formed by

$$M = \{q = s + \mathbf{j}z \colon s \in \mathbb{R}, z \in \mathbb{C}, \text{ with } s^2 + |z|^2 = 1\}$$

Notice that q has a null $\mathbf{i}$-coordinate.

Now we consider the height function h_X with $X = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

- On the group $G = Sp(1)$, the critical points of h_X^G are

$$\Sigma(h_X^G) = \{\pm \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\}.$$

These two points are not in M because they have a non-null $\mathbf{i}$-coordinate.

- Nevertheless, there are points of M that are critical points for the height function restricted to the Cartan model $M \subset Sp(1)$ of $Sp(1)/U(1)$.

- This time, the condition for a point $q \in M$ to be critical for h_X^M is
$$\widehat{X} = q\sigma(\widehat{X})q,$$
where $\widehat{X} = X^* + \sigma(X) = -2(\mathbf{j} + \mathbf{k})$. So, from $-2\overline{q}(\mathbf{j} + \mathbf{k}) = 2(\mathbf{j} + \mathbf{k})q$ we obtain that
$$\Sigma(h_X^M) = \{\pm\frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k})\}.$$

- The generalized Cayley transform allows to linearize the gradient flow of any height function on a symmetric space.
- Let $A \in G$, that is, $AA^* = \mathrm{I}$. We consider the open set of matrices

$$\Omega(A) = \{X \in \mathbb{K}^{n\times n} \colon A + X \text{ is invertible}\}.$$

Definition (Gómez-Macías-PS, *Ann. Global Anal. Geom.*, 2011)

The *Cayley tansform centered at A* is the map $c_A \colon \Omega(A) \to \Omega(A^*)$ defined by

$$c_A(X) = (\mathrm{I} - A^*X)(A+X)^{-1} = (A+X)^{-1}(\mathrm{I} - XA^*).$$

Its most interesting property is that it is a diffeomorphism, with $c_A^{-1} = c_{A^*}$.

Lemma

If $A \in G$ then

$$c_{\sigma(A)} \circ \sigma = \sigma \circ c_A$$

on $\Omega(A)$, or, equivalently, $\sigma \circ c_{\sigma(A)} = c_A \circ \sigma$ on $\Omega(\sigma(A))$.

Theorem

Let $M \subset G$ be the Cartan model of the symmetric space G/K. Let $A \in M$. Then

$$\Omega_M(A) := \Omega(A) \cap M$$

is a **contractible** *open subspace of M.*

Theorem

Let h_X^M be an arbitrary height function on the symmetric space M. Let A be a critical point. Then the solution of the gradient equation

$$4\alpha' = \widehat{X} - \alpha\sigma(\widehat{X})\alpha,$$

with initial condition $\alpha_0 \in \Omega_M(A)$, is the image by the Cayley transform c_{A^*} of the curve

$$\beta(t) = \exp(\frac{-t}{4}A^*\widehat{X})\beta_0 \exp(\frac{-t}{4}\widehat{X}A^*),$$

where $\widehat{X} = X^* + \sigma(X)$ and $\beta_0 = c_A(\alpha_0) \in T_{A^*}M$.

We can obtain an explicit formula for the solution making use of the property $A^* \exp(\widehat{X} A^*) A = \exp(A^* \widehat{X})$:

$$\begin{aligned} \alpha(t) &= A\big(\sinh(\frac{t}{4} A^* \widehat{X}) + \cosh(\frac{t}{4} A^* \widehat{X}) A^* \alpha_0\big) \\ &\quad \times \big(\cosh(\frac{t}{4} A^* \widehat{X}) + \sinh(\frac{t}{4} A^* \widehat{X}) A^* \alpha_0\big)^{-1}. \end{aligned}$$

* *This formula, for the particular case of a Lie group G, the classical Cayley transform c_{I} and the particular height function h_D^G where D is a real diagonal matrix is due to Dynnikov and Vesselov* [St. Petersbg. Math. J., *1997].*

- If one restricts to the case $X = D$ a real diagonal matrix such that $\sigma(D) = D$, then the gradient flow of h_X^G will be tangent to the symmetric space M, embedded into G.
- That means that the restricted flow coincides with the gradient flow of the height function restricted to M, with respect to the induced metric.
- But usually the symmetric space will not be invariant under the gradient flow. In fact, the gradient flow of h_X^G could even be tranverse to M.

Example

Let us consider again $Sp(1)/U(1)$. Take the critical point $A = (1/\sqrt{2})(\mathbf{j}+\mathbf{k}) \in M$. The gradient flow line of h_X^M passing through $\alpha_0 = 1$ is given by

$$\alpha^M(t) = \operatorname{sech} t\sqrt{2} - \mathbf{j}(\tanh t\sqrt{2})\frac{1-\mathbf{i}}{\sqrt{2}} \in M.$$

On the other hand, the flow line of h_X^G in the group G passing through the same point $\alpha_0 = 1$ is

$$\alpha^G(t) = \operatorname{sech}(t\sqrt{3}) - \tanh(t\sqrt{3})\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

where we have chosen the Cayley transform corresponding to the critical point $A = (1/\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k})$. Notice that $\alpha^G(t) \notin M$ for $t \neq 0$.

- For Lie Groups, h_X^G is Morse or Bott-Morse depending on the matrix X singular values.
- Let $X = UDV^*$ be the singular value decomposition (SVD) of X, then

$$(\mathrm{grad}\, h_X)_A = V(\mathrm{grad}\, h_D)_{V^*AU}U^*.$$

Analogously,

$$(\mathrm{H}h_X)_A(Y) = V(\mathrm{H}h_D)_{V^*AU}(V^*YU)U^*.$$

- The study of Morse-Bott functions can be considerably simplified by means of the *singular value decomposition*.
- Also there is an interesting relationship between polar forms and critical points.
- We shall show that there exist decompositions conformed to Cartan model.

SVD decompositions and polar form

- Given $Y \in \mathbb{K}^{n\times n}$, there exist orthogonal (resp. unitary, symplectic) matrices U, V such that $Y = UDV^*$ where

$$D = \begin{pmatrix} 0_{n_0} & & & \\ & t_1 I_{n_1} & & \\ & & \ddots & \\ & & & t_k I_{n_k} \end{pmatrix} \in \mathbb{K}^{n\times n},$$

  for $0, t_1^2, \ldots, t_k^2$ the real and non-negative eigenvalues of the Hermitian positive-semidefinite matrix YY^*.
- So we have the *left polar decomposition* $Y = S\Omega$, where $\Omega = UV^*$ is orthogonal and $S = UDU^*$ is H.p.-s. (S is the only H.p.-s. square root of $YY^* = UD^2U^*$).

- Each $A \in \Sigma(h_Y^G)$ determines a decomposition $Y = (YA)A^*$ of Y, where $\Sigma = YA$ is Hermitian but no necessarily positive semidefinite (*almost polar*).
- $\Sigma = W \Delta W^*$, where

$$\Delta = D \cdot \begin{pmatrix} 0_{n_0} & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_k \end{pmatrix} \text{ and } E_i = \begin{pmatrix} \varepsilon_i^1 & & \\ & \ddots & \\ & & \varepsilon_i^{n_i} \end{pmatrix}, \ \varepsilon_i^j = \pm 1.$$

Then the value of the function at the critical point A is

$$h_Y^G(A) = \Re \operatorname{Tr}(\Sigma) = t_1 \operatorname{Tr} E_1 + \cdots + t_k \operatorname{Tr} E_k.$$

Proposition

The critical point $A \in G$ is a global maximum of h_Y^G if and only if the decomposition $(YA)A^*$ is a true polar decomposition (i.e. the Hermitian matrix $\Sigma = YA$ is positive-semidefinite).

- When $Y = SA^*$ is a polar decomposition, A maximizes the distance of Y to the orthogonal matrices. In the same way, it maximizes the function $\Re \operatorname{Tr}(YB)$, $B \in G$.

- Remember that the critical points of h_X^M are the critical points of $h^G_{\sigma(\widehat{X})}$ that lie in M.
- Given $\widehat{X} = UDV^*$ an SVD decomposition we shall assume that $\sigma(D)$ is positive-semidefinite

  *For symmetric spaces in Cartan's classification, either $\sigma(D) = D$ or $\sigma(D) = -JDJ$, with $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Theorem (adapted polar decomposition)

Let $Y \in \mathbb{K}^{n\times n}$ be a matrix such that $\sigma(Y) = Y^*$. Assume that $\sigma(D)$ is positive semi-definite for the matrix D of singular values of Y. Then there exists a polar decomposition $Y = S\Omega$ such that $\sigma(\Omega) = \Omega^*$ and $\sigma(S) = \Omega^* S\Omega$.

Corollary

There exists a right polar decomposition $Y = \Omega S'$ such that $\sigma(\Omega) = \Omega^*$ and $\sigma(S') = \Omega S' \Omega^*$.

Corollary (adapted SVD)

There exists a singular value decomposition $Y = UDV^*$ such that the matrix $\Theta = U^* \sigma(V)$ verifies $\sigma(\Theta) = \Theta^*$.

- Under mild hypothesis, the study of any height function h_X^M can be reduced to the particular case $h_D^{M'}$ where
  - D is a real non-negative diagonal matrix and
  - $M' \subset G$ is a symmetric space diffeomorphic to M.

- Let $\widehat{X} = UDV^*$ be an adapted SVD. Then, from

$$\sigma(V)\sigma(D)\sigma(V)^* = \sigma(S) = \Omega S \Omega^* = UDU^*$$

it follows that $\sigma(D) = \Theta^* D \Theta$.
Now, we have that $\sigma'(D) = D$, so

$$\widehat{D}' = 2D \text{ and } \sigma'(\widehat{D}') = 2D.$$

Proposition

Assume $M = N$. Then the point $A \in M$ is a critical point of h_X^M if and only if $U^*AV \in M'$ is a critical point of $h_D^{M'}$, where $\widehat{X} = UDV^*$ is an adapted SVD.

Lemma

Let A be a critical point of h_D^G, that is, $D = ADA$. Then A can be decomposed into boxes, of size $n_0, n_1, \ldots, n_k$ respectively,

$$A = \begin{pmatrix} A_0 & & & 0 \\ & A_1 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix}$$

such that $A_0 A_0^* = I$ and $A_i^2 = I$, $A_i = A_i^*$, for $1 \leq i \leq k$.

- Recall that $\Sigma(h_D^M) = \Sigma(h_D^G) \cap M$
- Then h_D^G is a Morse function if and only if $\dim S^M(A) = 0$, which is equivalent to $n_0 = 0$ and $n_1 = \cdots = n_k = 1$.

Theorem

$\Sigma(h_X^G) \cong \Sigma(h_X) \cong O(n_0, \mathbb{K}) \times \Sigma(n_1) \times \cdots \times \Sigma(n_k)$, where $\Sigma(n_i)$ is the disjoint union of $G_p^{n_i}$, $0 \leq p \leq n_i$.

Corollary

The height function h_X^G in the Lie group G is a Morse function if and only if the singular values of the matrix X are positive and pairwise different.

- Let $G = Sp(2)$ and $\sigma(A) = -\mathbf{i}A\mathbf{i}$.
- $G/K = Sp(2)/U(2)$ (complex structures on $\mathbb{H}^2$ which are compatible with the hermitian product, i.e., $\mathcal{J} \in Sp(2)$ such that $\mathcal{J}^2 = -\mathrm{I}$).
- $M = N = \{A \in Sp(2) \colon \sigma(A) = A^*\}$.
  Explicitly, it is formed by the diagonal matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad |\alpha| = |\delta| = 1, \Re(\alpha\mathbf{i}) = \Re(\delta\mathbf{i}) = 0,$$

jointly with the matrices

$$\begin{pmatrix} \alpha & -\mathbf{i}\bar{\beta}\mathbf{i} \\ \beta & \beta\bar{\alpha}\mathbf{i}\beta^{-1}\mathbf{i} \end{pmatrix}, \quad \beta \neq 0, |\alpha|^2 + |\beta|^2 = 1, \Re(\alpha\mathbf{i}) = 0.$$

Let us take $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathbb{H}^{2\times 2}$ with $x = 1 + \mathbf{j}$ and $y = \mathbf{i} + \mathbf{j}$.

– First, we study the function h_X^G **on the Lie group G.**

- $X = UDV^* = \frac{1}{\sqrt{2}} X \,\mathrm{diag}(\sqrt{2}, \sqrt{2})\mathrm{I}$. Then

$$\Sigma(h_X^G) \cong \Sigma(h_D^G) \cong \Sigma(2) = G_0^2 \sqcup G_1^2 \sqcup G_2^2.$$

Two points and $Sp(2)/(Sp(1) \times Sp(1)) \cong S^4$.

The three components are $\{\pm \mathrm{I}\}$ and the sphere

$$\left\{ \begin{pmatrix} s & \bar{\beta} \\ \beta & -s \end{pmatrix} : s \in \mathbb{R}, s^2 + |\beta|^2 = 1 \right\},$$

which are the orbits by the adjoint action of I, $-\mathrm{I}$ and $\pm P = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively.

- Finally, the critical set of h_X^G is

$$\Sigma(h_X^G) = V\Sigma(h_D^G)U^* = \{\pm U^*\} \sqcup G_1^2 U^*$$

- Notice that $\Sigma(h_X^G) \cap M = \emptyset$.

The restriction h_X^M:

- Let us remember that $\Sigma(h_X^M) = \Sigma(h^G_{\sigma(\widehat{X})}) \cap M$.
- We have
$$\sigma(\widehat{X}) = \widehat{X}^* = \begin{pmatrix} x_0 & 0 \\ 0 & y_0 \end{pmatrix},$$
where $x_0 = \bar{x} - \mathbf{i}x\mathbf{i} = 2\mathbf{j}$ and $y_0 = \bar{y} - \mathbf{i}y\mathbf{i} = 2 + 2\mathbf{j}$. Notice that $|x| = |y|$ but $|x_0| \neq |y_0|$.
- $\widehat{X}^* = UDV^* = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+\mathbf{j}) \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{2} \end{pmatrix} \mathrm{I}.$
This is an adapted SVD.
- $\Sigma(h_D^G) \cong \Sigma(1) \times \Sigma(1)$, that is, four points. Explicitly
$$\Sigma(h_D^G) = \{A \in Sp(2)\colon DA^* = AD\} = \{\pm I, \pm P\}.$$
- Now, $\Sigma(h^G_{\sigma(\widehat{X})}) = V\Sigma(h_D^G)U^* = \{\pm U^*, \pm PU^*\}$, and these four points are in M, then it follows that h_X^M is a Morse function.

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