Motivation

Our goal is to better understand the topological structure of inverse limit spaces.

We use combinatoric tools, including Hofbauer towers, to study the collection of endpoints of the inverse limit space $(I, f)$ where $f$ is a unimodal map with $\lim_{k \to \infty} Q(k) = \infty$.

Unimodal Maps

A unimodal map is a continuous map $f : [0, 1] \to [0, 1]$ for which there exists a point $c \in (0, 1)$ such that $f|_{[0,c)}$ is strictly increasing and $f|_{(c, 1]}$ is strictly decreasing.

The point $c$ is called the turning point and we set $c_i$ to be the $i$th iterate of $c$; i.e., $c_i = f^i(c)$.

Symmetric Tent Maps

The symmetric tent map $T_a : [0, 1] \to [0, 1]$ with $a \in [0, 2]$ is given by

$$T_a(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{2}, \\ a(1-x) & \text{if } x \geq \frac{1}{2}. \end{cases}$$
Logistic Maps

The logistic map $g_a : [0, 1] \rightarrow [0, 1]$ with $a \in [0, 4]$ is defined by $g_a(x) = ax(1 - x)$.

Kneading Sequences

For a unimodal map $f$ and a point $x \in [0, 1]$, the itinerary of $x$ under $f$ is given by $I(x) = I_0I_1I_2 \cdots$, where

$$l_j = \begin{cases} 
0 & \text{if } f^j(x) < c, \\
* & \text{if } f^j(x) = c, \\
1 & \text{if } f^j(x) > c.
\end{cases}$$

The kneading sequence of a map $f$, denoted $K(f)$, is the sequence $I(c_1) = e_1e_2e_3 \cdots$.

Cutting Times and Kneading Maps

An iterate $n$ is called a cutting time if the image of the central branch of $f^n$ contains $c$. The cutting times are denoted $S_0, S_1, S_2, \ldots$, where $S_0 = 1$ and $S_1 = 2$.

An integer function $Q : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$, called the kneading map, may be defined by $S_k - S_{k-1} = S_{Q(k)}$.

The kneading sequence, kneading map, and cutting times each completely determine the combinatorics of the map $f$. 
Hofbauer Towers

Given a unimodal map $f$, the associated **Hofbauer tower** is the disjoint union of intervals $\{D_n\}_{n \geq 1}$ where $D_1 = [0, c_1]$ and, for $n \geq 1$,

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin D_n, \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

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Inverse Limit Spaces

Here a **continuum** is a compact connected metrizable space. Given a continuum $I$ and a continuous map $f : I \to I$, the associated **inverse limit space** $(I, f)$ is defined by

$$(I, f) = \{ x = (x_0, x_1, \ldots) | x_n \in I \text{ and } f(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N} \}$$

and has metric

$$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$
Endpoints and $E$

In our case, a point $x \in (I, f)$ is an endpoint of $(I, f)$ provided for every pair $A$ and $B$ of subcontinua of $(I, f)$ with $x \in A \cap B$, either $A \subset B$ or $B \subset A$.

Given a unimodal map $f$, define
$$E_f := \{(x_0, x_1, \ldots) \in (I, f) \mid x_i \in \omega(c, f) \text{ for all } i \in \mathbb{N}\}$$

Lemma (2010, Alvin and Brucks, Fund. Math.)

Let $f$ be a unimodal map with $K(f) \neq 10^\infty$ and suppose $x = (x_0, x_1, \ldots) \in (I, f) \setminus E$. Then $x$ is not an endpoint of $(I, f)$.

Backward Itineraries

For each $x \in (I, f)$ such that $x_i \neq c$ for all $i > 0$, set
$$\tau_R(x) = \sup\{n \geq 1 \mid I_{n-1}(x)I_{n-2}(x)\cdots I_1(x) = e_1e_2\cdots e_{n-1} \text{ and } 
\#\{1 \leq i \leq n-1 \mid e_i = 1\} \text{ is even}\}, \text{ and}$$

$$\tau_L(x) = \sup\{n \geq 1 \mid I_{n-1}(x)I_{n-2}(x)\cdots I_1(x) = e_1e_2\cdots e_{n-1} \text{ and } 
\#\{1 \leq i \leq n-1 \mid e_i = 1\} \text{ is odd}\}.$$ 

Known Results About Endpoints

Bruin provides a characterization with both a combinatoric and analytic component when $f$ is unimodal and the turning point is not periodic.

Proposition (1999, Bruin, Topology Appl.)

Let $f$ be a unimodal map and $x \in (I, f)$ be such that $x_i \neq c$ for all $i \geq 0$. Then $x$ is an endpoint of $(I, f)$ if and only if $\tau_R(x) = \infty$ and $x_0 = \sup \pi_0(\Gamma(x))$ (or $\tau_L(x) = \infty$ and $x_0 = \inf \pi_0(\Gamma(x))$).
The Adding Machine Map

Let $\alpha = \langle q_1, q_2, \ldots \rangle$ be a sequence of integers where each $q_i \geq 2$. Denote by $\Delta_\alpha$ the set of all sequences $(a_1, a_2, \ldots)$ such that $0 \leq a_i \leq q_i - 1$ for each $i$.

The map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$, defined by

$$f_\alpha((x_1, x_2, \ldots)) = (x_1, x_2, x_3, \ldots) + (1, 0, 0, \ldots),$$

is called the $\alpha$-adic adding machine map.

Kneading Maps, Adding Machines, and Endpoints

Theorem (2011, Alvin and Brucks, Topology Appl.)

Let $f \in A$ be such that $\lim_{k \to \infty} Q(k) = \infty$. Then $E$ is precisely the collection of endpoints of $(I, f)$.

Further, if $f \in A$ and $\lim_{k \to \infty} Q(k) \neq \infty$, then it may be that $E$ is exactly the collection of endpoints of $(I, f)$, or it may be that $E$ properly contains the collection of endpoints of $(I, f)$.

Kneading Maps and Endpoints

Is it possible that every unimodal map $f$ with $\lim_{k \to \infty} Q(k) = \infty$ is such that $E$ is the collection of endpoints of $(I, f)$?

Recall that if $f|_{\omega(c)}$ is topologically conjugate to an adding machine, then $f|_{\omega(c)}$ is one-to-one.
Proof of Main Result

Let \( x = (x_0, x_1, x_2, \ldots) \in \mathcal{E} \) be such that \( x_i \neq c \) for all \( i \geq 0 \).
Recall that \( x_0 \in \omega(c) \).

We can find an increasing sequence of \( D_{n_k} \) such that \( x_0 \in D_{n_k} \) for all \( k \in \mathbb{N} \).

As \( Q(k) \to \infty \) and \( f|_{\omega(c)} \) is one-to-one, there exists some level \( D_N \) of the Hofbauer tower where if \( x_0 \in D_n \) for some \( n \geq N \), then the unique preimage \( x_1 \in \omega(c) \) lies in \( D_{n-1} \).

WLOG take \( \{n_k\} \) such that \( n_1 > S_l > N \).

Hence \( \mathcal{I}_{\beta(n_k)-1}(x) \cdots \mathcal{I}_1(x) = e_1 e_2 \cdots e_{\beta(n_k)-1} \).

Note that \( \beta(n_k) \to \infty \).

\( \tau_R(x) = \infty \) or \( \tau_L(x) = \infty \).

In both cases we show \( x \) must be an endpoint of \((I, f)\), using Bruin’s characterization.
Is it the case that for all unimodal maps \( f \) with \( \lim_{k \to \infty} Q(k) = \infty \) the collection \( E \) is precisely the collection of endpoints for \( (I, f) \)?

How will this better understanding of the collection of endpoints help us to understand the topological structure of the inverse limit space?

Can we use the behavior of the endpoints to distinguish between two inverse limit spaces?

Thank you for your attention.