Periods of periodic orbits for vertex maps on graphs

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Introduction

One of the basic starting points for one-dimension combinatorial dynamics is Sharkovsky's Theorem.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of least period v then f also has a periodic point of least period m for any $m \lhd v$, where

$$1 \lhd 2 \lhd 4 \lhd \dots \dots 28 \lhd 20 \lhd 12 \lhd \dots 14 \lhd 10 \lhd 6 \dots 7 \lhd 5 \lhd 3.$$

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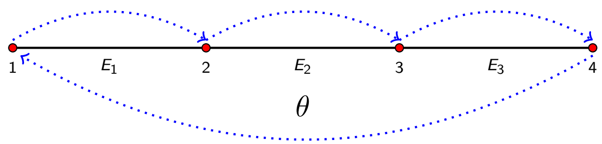
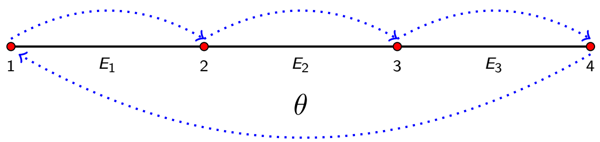
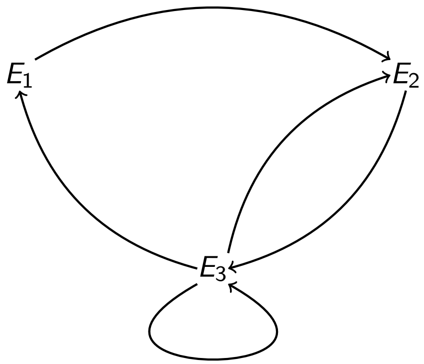
an example



an example





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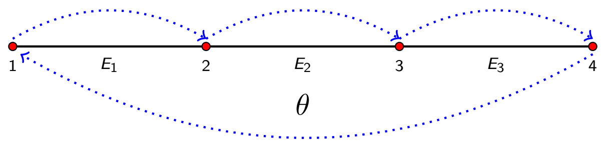

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

a basic result

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Theorem

Let M be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the ijth entry of M^k gives the number of walks of length k from E_j to E_i.

a basic result

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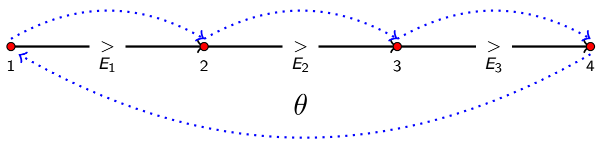
Theorem

Let M be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the ijth entry of M^k gives the number of walks of length k from E_j to E_i.

Corollary

The trace of M^k gives the total number of closed walks of length k.

an example – with orientation



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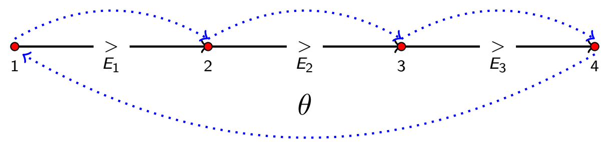

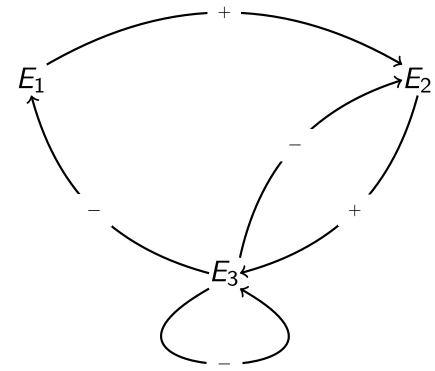

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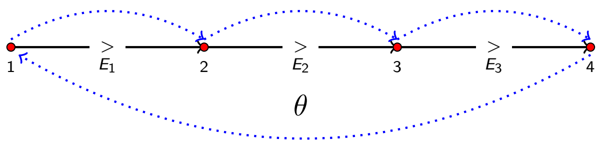

$$M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

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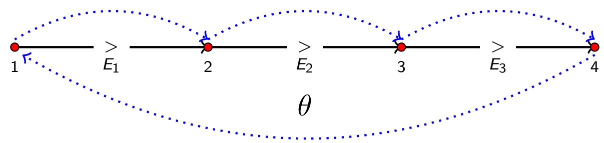

$$M_0(\theta) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

basic properties

Theorem

The ijth entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length k from E_j to E_i minus the number negatively oriented walks from E_j to E_i.

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Theorem

The ijth entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length k from E_j to E_i minus the number negatively oriented walks from E_j to E_i.

Corollary

The trace of $(M_1(\theta))^k$ gives the number of positively oriented closed walks of length k minus the number of negatively oriented closed walks of length k.

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Theorem

1. $(M_0(\theta))^k = M_0(\theta^k)$
2. $(M_1(\theta))^k = M_1(\theta^k)$

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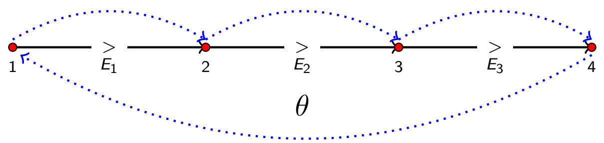

$$M_0(\theta) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

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Theorem

Trace $(M_0(\theta)) -$ *Trace* $(M_1(\theta)) = 1$.

first lemma

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Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that f has a periodic point of period 17. Then f has a periodic point of period 2^k for any non-negative integer k.

first lemma

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Proof.

Since 17 is not a divisor of 2^k we know that θ^{2^k} does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So Trace $(M_0(\theta^{2^k})) = 0$.

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Proof.

Since 17 is not a divisor of 2^k we know that θ^{2^k} does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So Trace $(M_0(\theta^{2^k})) = 0$. So Trace $(M_1(\theta^{2^k})) = -1$.

first lemma

Proof.

Since 17 is not a divisor of 2^k we know that θ^{2^k} does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So Trace $(M_0(\theta^{2^k})) = 0$. So Trace $(M_1(\theta^{2^k})) = -1$. So the oriented Markov graph has a vertex E_j with a closed walk from E_j to itself of length 2^k with negative orientation.

first lemma

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Proof.

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first lemma

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Proof.

Since 17 is not a divisor of 2^k we know that θ^{2^k} does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So Trace $(M_0(\theta^{2^k})) = 0$. So Trace $(M_1(\theta^{2^k})) = -1$. So the oriented Markov graph has a vertex E_j with a closed walk from E_j to itself of length 2^k with negative orientation. Since the orientation is negative it cannot be the repetition of a shorter closed walk, as any shorter closed walk would have to be repeated an even number of times. So there is a periodic point in E_j with minimum period 2^k. □

second lemma

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Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that f has a periodic point of period 17. Then f has a periodic point of period m for any non-negative integer $m > 17$.

second lemma

Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation.

second lemma

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Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation.

second lemma

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Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1.

second lemma

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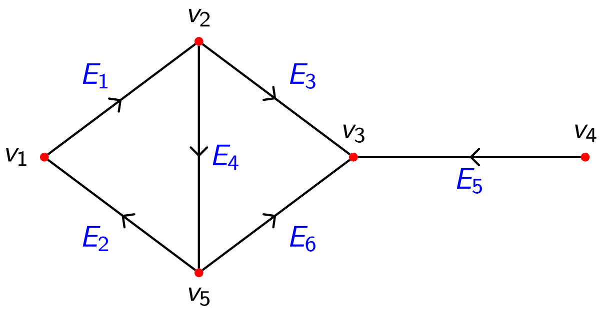
Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1. We can construct a non-repetitive closed walk of length m by going once around the walk of length 17 and then $m - 17$ times around the walk of length 1. $\square$

vertex maps on graphs



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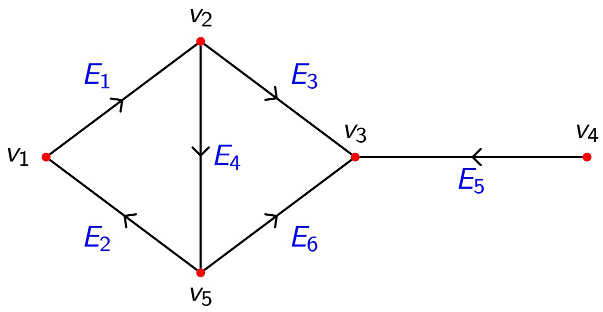

$$M_0(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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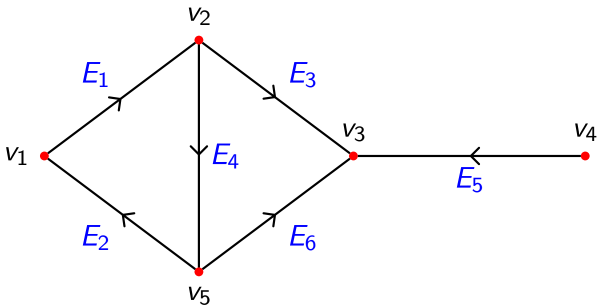

$$M_1(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

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Theorem

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Theorem

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3. $Trace\,(M_0(\theta)) - Trace\,(M_1(\theta)) = L_f$

Theorem

1. $(M_0(\theta))^k = M_0(\theta^k)$
2. $(M_1(\theta))^k = M_1(\theta^k)$
3. $Trace\,(M_0(\theta)) - Trace\,(M_1(\theta)) = L_f$

Corollary

If the underlying map is homotopic to the identity, then $Trace\,(M_0(\theta)) - Trace\,(M_1(\theta)) = v - e$

Lemma

Let G be a graph and f a vertex map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If v is not a divisor of 2^k, then f has a periodic point with period 2^k.

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.
Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.
Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of f has a non-repetitive loop of length 2 with negative orientation.

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.
Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of f has a non-repetitive loop of length 2 with negative orientation.
etc – use induction □

Lemma

Let G be a graph and f a map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.

Lemma

Let G be a graph and f a map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.

Proof.

Similar trace argument. □

The Sharkovsky ordering can be defined as follows:
(what positive integers does v force?)

1. $2^l \triangleleft 2^k = v$ if $l \leq k$.

Sharkovsky ordering

The Sharkovsky ordering can be defined as follows:
(what positive integers does v force?)

1. $2^l \lhd 2^k = v$ if $l \leq k$.
2. If $v = 2^k s$, where $s > 1$ is odd, then

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1. $2^l \lhd 2^k = v$ if $l \leq k$.
2. If $v = 2^k s$, where $s > 1$ is odd, then
   1. $2^l \lhd v$, for all positive integers l.

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   1. $2^l \lhd v$, for all positive integers l.
   2. $2^k r \lhd v$, where $r \geq s$.
   3. $2^l r \lhd v$, where $l > k$ and such that $2^l r < v$.

final remarks

1. This is a way of generalizing from maps of the interval and circle to maps on graphs.

final remarks

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1. This is a way of generalizing from maps of the interval and circle to maps on graphs.
2. This is not the most general method of generalizing, but it leads to interesting results, and is very accessible.

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2. This is not the most general method of generalizing, but it leads to interesting results, and is very accessible.
3. More info at: *Sharkovsky's theorem and one-dimensional combinatorial dynamics* arxiv.org/abs/1201.3583

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Thank you!