On dynamics of surface homeomorphisms with invariant continua

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1. Preliminaries

- A continuum is a compact and connected metric space containing at least two points.
- A continuum is decomposable if it is the union of two proper subcontinua.
- A continuum is hereditarily decomposable (HD for short) if every subcontinuum is decomposable.
- X is arc-like (or chainable) iff $X = \lim_{\leftarrow}\{[0,1], f_i\}$ (with each f_i continuous)
- X is circle-like iff $X = \lim_{\leftarrow}\{\mathbb{S}^1, f_i\}$ (with each f_i continuous)

2. Some known properties of circle-like HD continua

- All circle-like HD continua are planar (non-planar $\Rightarrow$ self-entwined $\Rightarrow$ indecomposable, Rogers 1970)
- There is a circle-like HD continuum that contains no arcs.
- There is a circle-like HD continuum (nonhomeomorphic to the circle) that admits arbitrarily small periodic homeomorphisms semiconjugate to arbitrarily small rigid rotations at the level of the tranche decomposition to the circle (Mouron 2003).
- Homeomorphisms of Souslinian circle-like HD continua have zero entropy (Ye 1994).

3. Related (and useful) results for chainable HD continua

- Sharkovskii Theorem holds for self-maps of chainable HD continua (Minc, Transue 1991).
- Homeomorphisms of chainable HD continua admit 2^n-periodic orbits for every n, but no other periods (Ingram 1989).
- Homeomorphisms of chainable HD continua have zero entropy (Mouron 2011).

4. Main Results

Let $h : X \to X$ be a self-homeomorphism of a hereditarily decomposable circle-like continuum X. We prove the following.

1. Any two periodic orbits of h with periods r and t must be such that $\frac{r}{t}$ is a power of 2.
2. If $r \lhd t$ and h has an r-periodic orbit then h also has a t-periodic orbit, where $\lhd$ is the Sharkovskii ordering of integers.
3. If h has a periodic orbit then the topological entropy of h is zero.
4. For every $q, j > 0$ there exists a hereditarily decomposable circle-like continuum $X_{q,j}$, and a planar homeomorphism $\bar{h}_{q,j} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $X_{q,j}$ is an attractor, and $\bar{h}_{q,j}$ has a $q2^i$-periodic orbit in $X_{q,j}$ for every $0 \le i \le j$.
5. There is a circle-like hereditarily decomposable continuum X_∞, a planar orientation preserving homeomorphism h_+, and a planar orientation reversing homeomorphism h_- with $h_+(X_\infty) = h_-(X_\infty) = X_\infty$, such that $h_+|X_\infty$ and $h_-|X_\infty$ are of type 2^∞. Analogous example exists with X_∞ being chainable.
6. Any finite-sheeted covering space of a circle-like hereditarily decomposable continuum is circle-like and hereditarily decomposable.
7. Any two-point compactification of the universal cover of such a continuum is chainable and hereditarily decomposable.

5. Hereditarily decomposable circle-like continua are planar

Suppose X is a hereditarily decomposable circle-like continuum. We can assume that X is embedded essentially into an annulus $\mathbb{A} = \{(r,\theta) \in \mathbb{R}^2 : 1 \le r \le 2, 0 \le \theta < 2\pi\}$; i.e. X separates the two boundary components of $\mathbb{A}$.

6. The two-point compactifications of the universal cover

By $\tilde{\mathbb{A}}$ we will denote the universal cover of $\mathbb{A}$ given by $\tilde{\mathbb{A}} = \{(r,\theta) \in \mathbb{R}^2 : 1 \le r \le 2\}$, with the covering map $\tau(r,\theta) = (r, \theta(\operatorname{mod} 2\pi))$. Consider a two-point compactification of $\tilde{\mathbb{A}}$ by the points c_1 and c_2. Set $\tilde{X} = \tau^{-1}(X) \cup \{c_1, c_2\}$ for the two-point compactification of $\tau^{-1}(X)$.

The following result is proved in [Bellamy, D. P.; Lewis, W. *An orientation reversing homeomorphism of the plane with invariant pseudo-arc.* Proc. Amer. Math. Soc. 114 (1992), no. 4, 1145–1149.]

Theorem 6.1 (Bellamy,Lewis 1992)**.** *The two-point compactification of the universal cover of the pseudo-circle is a pseudo-arc (i.e. hereditarily indecomposable and chainable).*

Theorem 6.2. *The two-point compactification $\tilde{X}$ of $\tau^{-1}(X)$ is a hereditarily decomposable chainable continuum.*

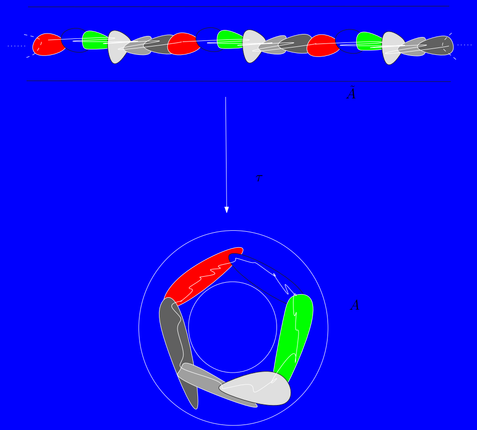

Figure 1: Lifting a circular chain to an infinite chain

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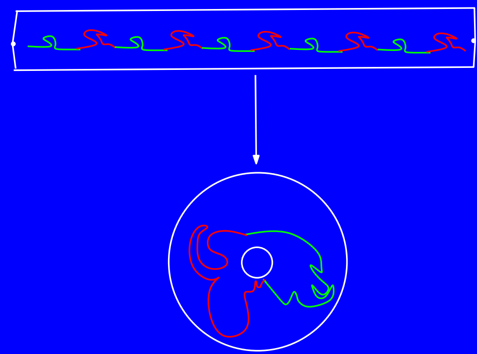

Figure 2: Decomposing the two-point compactification of the universal cover of X

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7. Finite-sheeted covering spaces of circle-like HD continua

Heath, J. W., *Weakly confluent, 2-to-1 maps on hereditarily indecomposable continua.* Proc. Amer. Math. Soc. 117 (1993), no. 2, 569-573.

Theorem 7.1 (Heath 1993)**.** *The n-fold covering space of the pseudo-circle is the pseudo-circle (i.e. circle-like and hereditarily indecomposable).*

One can show that.

Theorem 7.2. *The n-fold covering space of a hereditarily decomposable circle-like continuum is hereditarily decomposable and circle-like.*

Note that any homeomorphism $f : X \to X$ extends to a map $F : \mathbb{A} \to \mathbb{A}$ of degree ± 1. On the other hand X is the intersection of a nested sequence of annuli and thus $\check{H}_k(X, \mathbb{Z}) \cong \mathbb{Z}$ if $k = 1, 0$ and $\check{H}_k(X, \mathbb{Z}) \cong 0$ otherwise. Consequently f induces an automorphism of the Čech homology of X.

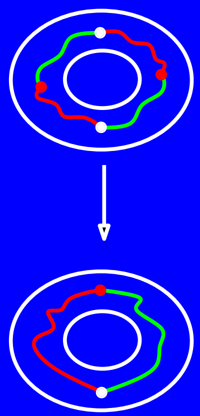


Figure 3: Decomposing the 2-fold cover of X.

- We shall call f homology preserving (reversing) if the degree of F is 1 (if the degree of F is -1).
- Any lift $\tilde{F}$ of a homology preserving (homology reversing) F extends to a map of the two-point compactification $\{c_1, c_2\} \cup \tilde{\mathbb{A}}$ by setting $\tilde{F}(c_1) = c_1, \tilde{F}(c_2) = c_2$ (by setting $\tilde{F}(c_1) = c_2, \tilde{F}(c_2) = c_1$).
- Since $\tilde{F}(\tau^{-1}(X)) = \tau^{-1}(X)$, one can consider a lift $\tilde{f}$ of f to $\tilde{X}$ by setting $\tilde{f} = \tilde{F}|\tilde{X}$.
- Clearly if f is a homeomorphism of X then $\tilde{f}$ is a homeomorphism of $\tilde{X}$.

Suppose $h : X \to X$ is a homology preserving homeomorphism. For a given $x \in X$ the rotation set $\rho(x, h)$ is the set of accumulation points of

$$\{\frac{1}{2\pi n}\pi[\tilde{h}^n(\tilde{x})](\operatorname{mod} 2\pi)\}_{n=1}^{\infty},$$

where $\tilde{h}$ is any lift of h, $\tilde{x} \in \tau^{-1}(x)$ and $\pi : \tilde{\mathbb{A}} \to \mathbb{R}$ is the projection onto the second coordinate.

Let p be a fixed point of h and y be a k-periodic point of h. Then

(HP1) There is $\tilde{h}$, a lift of h, such that $\tilde{h}(\tilde{p}) = \tilde{p}$ for any $\tilde{p} \in \tau^{-1}(p)$.

(HP2) If $\rho(y, h) = 0$ then $\tilde{h}^k(\tilde{y}) = \tilde{y}$ for any $\tilde{y} \in \tau^{-1}(y)$.

Suppose $h : X \to X$ is homology reversing. Let p be a fixed point of h. Then

(HR1) There is an integer $m[\tilde{h}, p]$ such that

$$\tilde{h}(\frac{\theta}{2\pi} + n, r) = (\frac{\theta}{2\pi} - n + m[\tilde{h}, p], r)$$

for every $(\frac{\theta}{2\pi} + n, r) \in \tau^{-1}(p)$,

(HR2) $\tilde{h}$ has a fixed point in $\tau^{-1}(p)$ iff $m[\tilde{h}, p]$ is even,

(HR3) if $m[\tilde{h}, p]$ is even then $\tilde{h}(x + 1, y)$ is a lift homeomorphism of h that does not have a fixed point in $\tau^{-1}(p)$.

Lemma 7.3. *(cf. Barge, Gillette) Let Y be a decomposable circle-like continuum. Suppose that $F : \mathbb{A} \to \mathbb{A}$ is a map of degree 1, $F(Y) = Y$ and $F|Y$ is a self-homeomorphism of Y. Then $\bigcup_{x \in X} \rho(x, F)$ is a singleton. In particular, if $F|Y$ has a fixed point then $\rho(y, F) = \{0\}$ for any $y \in Y$*

Barge, M.; Gillette, R. M., *Rotation and periodicity in plane separating continua.* Ergodic Theory Dynam. Systems 11 (1991), no. 4, 619–631.

8. Examples

Theorem 8.1. *For every $q, j > 0$ there exists a hereditarily decomposable circle-like continuum $X_{q,j}$, and a planar homeomorphism $\bar{h}_{q,j} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $X_{q,j}$ is an attractor, and $\bar{h}_{q,j}$ has a $q2^i$-periodic orbit in $X_{q,j}$ for every $0 \le i \le j$.*

First we consider the case when $q = 1$. It is known that for every j there is a piecewise monotone self-map of the circle f such that

- f has a fixed point,
- f has a 2^j-periodic point,
- f has zero topological entropy,
- f has degree 1.

Let $X_{1,j} = \lim_{\leftarrow}\{\mathbb{S}^1, f\}$ be the inverse limit space with f as the sole bonding map. Then $X_{1,j}$ is a hereditarily decomposable circle-like continuum by a result of Barge and Roe (1990). Let $h_{1,j} : X_{1,j} \to X_{1,j}$ be the shift map given by

$$h_{1,j}(a, b, c, \dots) = (f(a), f(b), f(c), \dots).$$

Notice that $h_{1,j}(x_o, x_o, x_o, \dots) = (x_o, x_o, x_o, \dots)$ where x_o is the fixed point of f. Additionally, the 2^j-periodic point of f gives rise to a 2^j-periodic point of $h_{1,j}$. Because f has degree 1, it is well known that $h_{1,j}$ extends to a homeomorphism $\bar{h}_{1,j}$ of the entire plane with $X_{1,j}$ as an attractor (e.g. Barge, Roe 1990).

To obtain the desired examples with $q > 1$ use the q-fold cover of $X_{1,j}$ and notice that the deck transformations commute with $h_{1,j}$.

Theorem 8.2. *There is a circle-like hereditarily decomposable continuum X_∞, a planar orientation preserving homeomorphism h_+, and a planar orientation reversing homeomorphism h_- with $h_+(X_\infty) = h_-(X_\infty) = X_\infty$, such that $h_+|X_\infty$ and $h_-|X_\infty$ are of type 2^∞ (and have zero entropy). Analogous example exists with X_∞ being chainable.*

Proof.

- For every n let X_n be a hereditarily decomposable circle-like continuum embedded into an annulus A_n, and h_n a homeomorphism of A_n with X_n invariant and of type 2^n on X_n.
- Let D_n be the disk that is the two-point compactification of the universal cover of A_n by the points a_n and b_n.
- Denote by $\tilde{X}_n$ and $\tilde{h}_n$ the chainable two-point compactification of the universal cover of X_n and lift of h_n to D_n respectively.
- Without loss of generality $\tilde{h}_n$ is orientation preserving and a_n and b_n are fixed points of $\tilde{h}_n$.

□

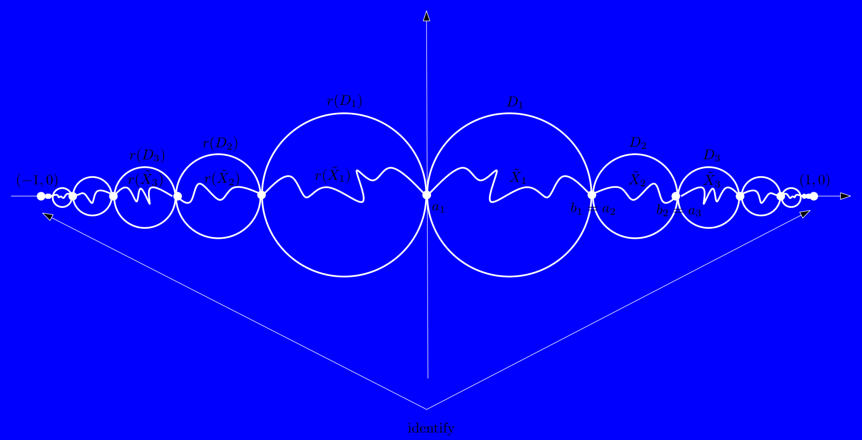
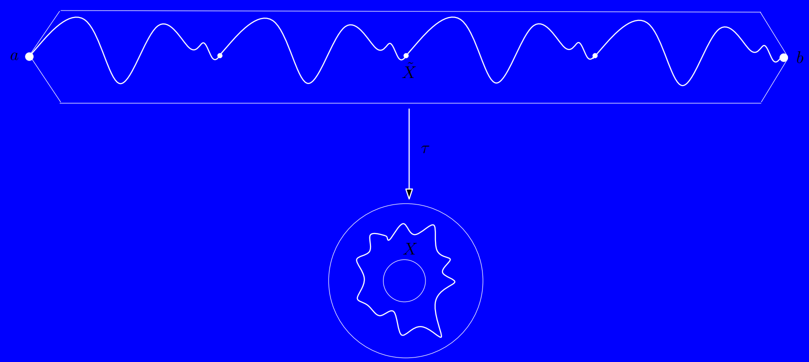


Figure 4: The continuum X_∞.

Theorem 8.3. *Suppose $h : X \to X$ is a self-homeomorphism of a hereditarily decomposable circle-like continuum X that has a q-periodic orbit, but no periodic orbit of period less than q. Then every periodic orbit of h must be of period $q2^k$, for some $k \in \mathbb{N}$. In particular, if h has a fixed point then every periodic point is of period 2^k for some positive integer k.*

Theorem 8.4. *Suppose $h : X \to X$ is a self-homeomorphism of a hereditarily decomposable circle-like continuum X that has a q-periodic orbit, but no periodic orbit of period less than q. If h has a $q2^n$-periodic orbit then it also has a $q2^m$-periodic orbit, for every $0 \le m \le n$.*

Theorem 8.5. *Suppose $h : X \to X$ is a self-homeomorphism of a hereditarily decomposable circle-like continuum X that has a fixed point. Then the topological entropy of h is zero.*



9. More on circle-like continua: $\frac{1}{2}$-homogeneity.

- For a continuum Y by Homeo(Y) we shall denote its homeomorphism group.
- Y is $\frac{1}{2}$-homogeneous if the action of Homeo(Y) on Y has exactly two orbits.
- In 2005 Víctor Neumann-Lara, Patricia Pellicer-Covarrubias and Isabel Puga-Espinosa described a decomposable $\frac{1}{2}$-homogeneous circle-like continuum: in an arc of pseudoarcs pinch the two end-pseudoarcs and then identify them to a unique local separating point.

Question.(Neumann-Lara & Pellicer-Covarrubias & Puga-Espinosa) *Does there exist an indecomposable $\frac{1}{2}$-homogeneous circle-like continuum?*

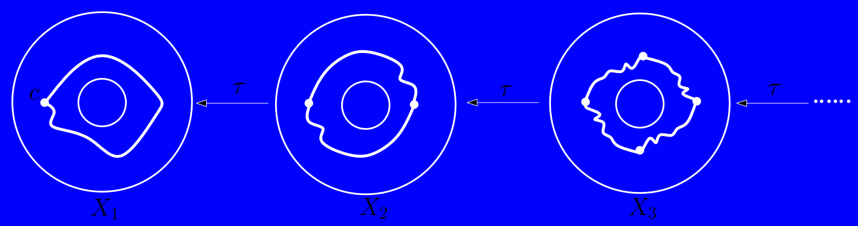


Figure 5: The inverse system of finite-sheeted covering spaces.

Theorem 9.1. *There is an uncountable family $\mathcal{S}$ of topologically distinct indecomposable $\frac{1}{2}$-homogeneous circle-like continua. Each member of $\mathcal{S}$ is a solenoidal continuum obtained from the $\frac{1}{2}$-homogeneous circle-like continuum as an inverse limit of its finite-sheeted covering spaces.*

$$\begin{tikzcd}
\tilde{X} \subseteq \mathbb{A} \arrow[r, "\tilde{H}", "\tilde{h}=\tilde{H}|\tilde{X}"'] \arrow[d, "\tau"'] & \tilde{X} \subseteq \mathbb{A} \arrow[d, "\tau"'] \\
X \subseteq \mathbb{A} \arrow[r, "H", "h=H|X"'] & X \subseteq \mathbb{A}
\end{tikzcd}$$

Theorem 9.2. *Let X be a $\frac{1}{2}$-homogeneous continuum. If X is indecomposable then the two orbits of* $\mathrm{Homeo}(X)$ *are uncountable.*

Proof.

- Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the two orbits of $\mathrm{Homeo}(X)$.
- First, suppose that there is a composant C of X with the property $\mathcal{O}_1 \cap C \neq \emptyset \neq \mathcal{O}_2 \cap C$.
- Since X is indecomposable it has uncountably many pairwise disjoint composants.
- Therefore it is enough to show that each composant must have a point in common with the two orbits.
- Let $x_1 \in \mathcal{O}_1 \cap C$ and $x_2 \in \mathcal{O}_2 \cap C$. Let also C' be a composant. Fix $x_3 \in C'$.
- Since X is $\frac{1}{2}$-homogeneous there is a homeomorphism h such that either $h(x_1) = x_3$ or $h(x_2) = x_3$. Without loss of generality assume $h(x_1) = x_3$. Since $h(C) = C'$ it follows that $h(x_2) \in C'$.
- Therefore $\mathcal{O}_1 \cap C' \neq \emptyset \neq \mathcal{O}_2 \cap C'$.
- Second, suppose that for every composant K we have $\mathcal{O}_1 \cap K = \emptyset$ or $\mathcal{O}_2 \cap K = \emptyset$.
- Since both $\mathcal{O}_1$ and $\mathcal{O}_2$ are nonempty there must be two composant K_1 and K_2 such that $K_1 \subseteq \mathcal{O}_1$ and $K_2 \subseteq \mathcal{O}_2$.
- But a composant contains a subcontinuum of X so $\mathcal{O}_1$ and $\mathcal{O}_2$ are uncountable.

$\square$

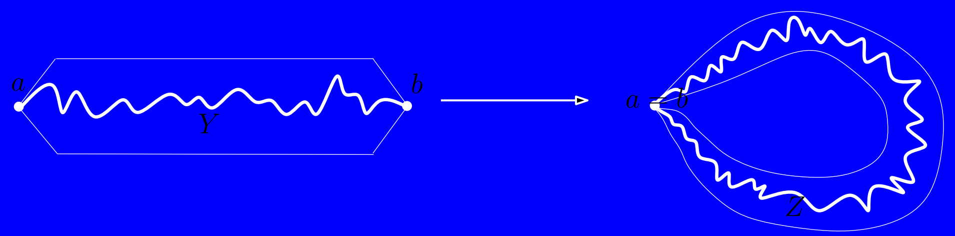


Figure 6: The continuum Z is not $\frac{1}{2}$-homogeneous.

Example. Let Y be the pseudoarc and let a and b be two points that are in two different composants of Y. Identify a and b to obtain a circle-like, plane separating continuum Z. Z is not $\frac{1}{2}$-homogeneous.