

Euclidean tilings

Invariant measures

Asymptotic Thurston norm

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Tilings of \mathbb{R}^2

Prototiles: $\mathcal{P} = \{p_1, \dots, p_n\}$ is a finite set of polygons with colored edges.

Definition

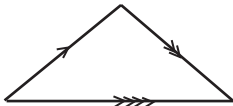
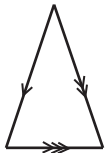
A **\mathcal{P} -tiling of \mathbb{R}^2** is a collection of polygons with colored edges (*tiles*) such that:

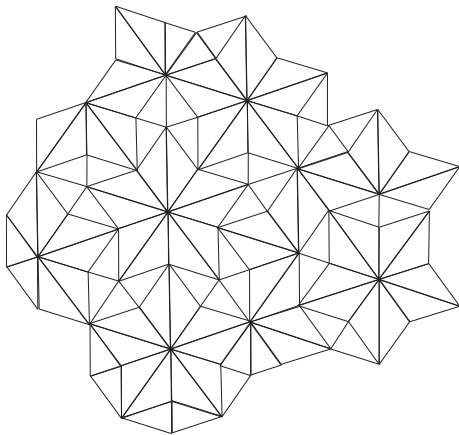
1. $\mathbb{R}^2 = \bigcup_i t_i$.
2. The tiles t_i have disjoint interiors.
3. If two tiles t_i, t_j meet, they meet along edges whose colors match.
4. Each tile t_i is a translate of some prototile $p_j \in \mathcal{P}$.

$\Omega_{\mathcal{P}}$ is the set of all \mathcal{P} -tilings.

Remark

$\Omega_{\mathcal{P}}$ might be empty: this is an undecidable problem.





The Anderson-Putnam complex $\mathcal{A}_{\mathcal{P}}$

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Definition

1. *The 2-cells are the prototiles p_j .*
2. *Two 2-cells are glued along the edges e_i, e_j if and only if there is a translation which carries e_i to e_j and the colors match.*

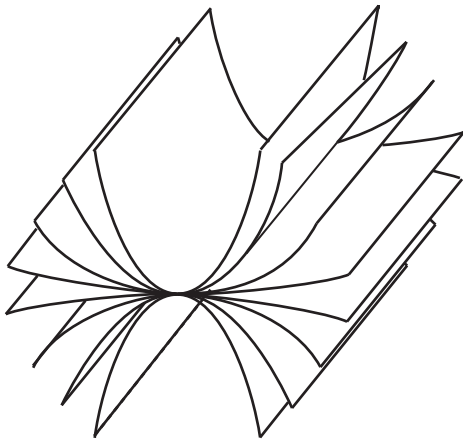
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2. *Two 2-cells are glued along the edges e_i, e_j if and only if there is a translation which carries e_i to e_j and the colors match. Orient the 2-cells with the orientation of the plane and choose an orientation for the edges.*
3. *Each edge has two sides: the collection of 2-cells where it appears with a $+$ sign in the boundary and the collection of 2-cells where it appears with a $-$ sign.*

\Rightarrow Structure of Branched Surface

The Anderson-Putnam complex $\mathcal{A}_{\mathcal{P}}$



Homology and surfaces

$$H_2(\mathcal{A}_{\mathcal{P}}; \mathbb{R}) = \text{Ker}(\partial: C_2(\mathcal{A}_{\mathcal{P}}; \mathbb{R}) \rightarrow C_1(\mathcal{A}_{\mathcal{P}}; \mathbb{R})).$$

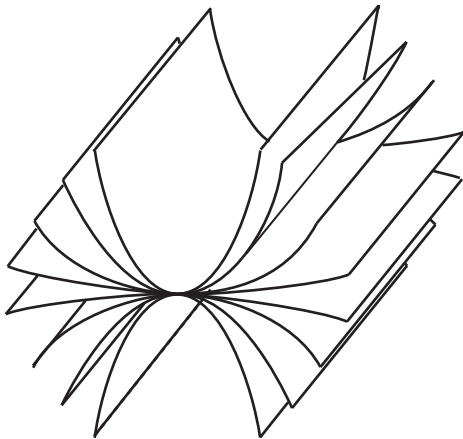
Equivalent to look at the **switch equations**.

Lemma

Any non-negative integer 2-cycle $c \in H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{Z})$ is represented by a closed (i.e. with no boundary) compact surface S , denoted by $[S] = c$.

This surface S is not necessarily unique up to homeomorphism.

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Lemma

1. $\|c\| = 0$ if and only if there is a torus representing c .
2. $\|c_1 + c_2\| \leq \|c_1\| + \|c_2\|$.
3. $\|nc\| \leq |n|\|c\|$.

It might happen $\|nc\| < |n|\|c\|$.

Asymptotic Thurston norm

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The asymptotic Thurston norm is uniformly continuous.

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1. *The asymptotic Thurston norm is well-defined on $H_2^+(\mathcal{A}_P; \mathbb{R})$.*
2. $|||c_1 + c_2||| \leq |||c_1||| + |||c_2|||$.
3. $|||nc||| = |n| |||c|||$.
4. $|||c||| = 0$ does not imply that there is a torus representing c .

A geometric interpretation of the tiling problem

Theorem (Chazottes-Gambaudo-G)

$\Omega_{\mathcal{P}}$ is non-empty (which is equivalent to \mathcal{P} tiles the plane) if and only if the asymptotic Thurston norm vanishes on some non-trivial class $c \in H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{Z})$.

Metrizable topology on $\Omega_{\mathcal{P}}$

$T, T' \in \Omega_{\mathcal{P}}$. $B_{\epsilon}(0)$: open ball of radius ϵ around the origin.

$A = \{\epsilon \in (0, 1) \text{ s.t. there exists } u \in \mathbb{R}^2 \text{ with } \|u\| < \epsilon \text{ and}$

$$(T + u) \cap B_{1/\epsilon}(0) = T' \cap B_{1/\epsilon}(0)\}$$

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$\delta(T, T') = \inf(A)$ if A is non-empty and 1 otherwise.

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$(\Omega_{\mathcal{P}}, \delta)$ is a compact metric space, together with a continuous action of \mathbb{R}^2 .

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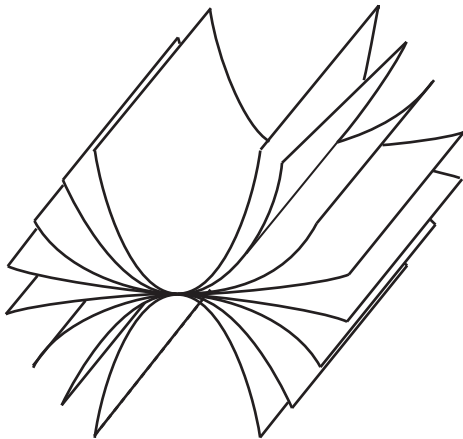
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$(\Omega_{\mathcal{P}}, \delta)$ is a compact metric space, together with a continuous action of \mathbb{R}^2 .

\mathbb{R}^2 amenable \Rightarrow Existence of an invariant measure \Rightarrow Existence of a non-negative real 2-cycle in $H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$.

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Asymptotic Thurston norm and invariant measures

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Theorem (Chazottes-Gambaudo-G)

Let $c \in H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$. There exists $\mu \in \mathcal{M}(\Omega_{\mathcal{P}})$ such that $c = \pi(\mu)$ if and only if the asymptotic Thurston norm of c vanishes.

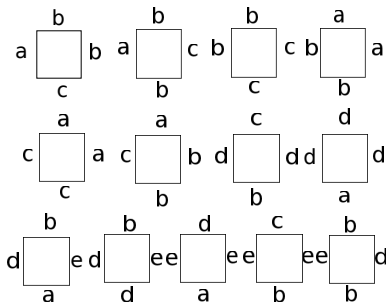
Wang tilings

A **Wang tiling** is (a tiling made from) a finite collection of unit squares with sides parallel to the axis of \mathbb{R}^2 and colored edges.

Theorem (Sadun-Williams)

For any finite collection of polygons \mathcal{P} there is a Wang tiling \mathcal{W} such that $(\Omega_{\mathcal{P}}, \mathbb{R}^2)$ and $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ are topologically equivalent.

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Proposition

It is sufficient to prove our theorem for Wang tilings.

Hint of proof for Wang tilings

$c = \pi(\widehat{\mu}) \Rightarrow |||c||| = 0$: Forget the colors to obtain a new Wang tiling $\widehat{\mathcal{W}}$ and a new Anderson-Putnam complex $\mathcal{A}_{\widehat{\mathcal{W}}}$. The system $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ is a sub-system of $(\Omega_{\widehat{\mathcal{W}}}, \mathbb{R}^2)$.

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Periodic orbits of \mathbb{R}^2 (tori) are dense in $(\Omega_{\widehat{\mathcal{W}}}, \mathbb{R}^2)$.

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$c = \pi(\mu)$ in $H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$ is approximated by a sequence of 2-cycles (c_i) in $H_2^+(\mathcal{A}_{\widehat{\mathcal{W}}}; \mathbb{R})$ such that $|||c_i||| = 0$.

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By continuity $|||c||| = 0$.