Continuum-wise Expansive Homoclinic classes

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Notations

- M : a compact C^∞ manifold
- Diff(M) : the space of C^1 diffeomorphisms on M endowed with the C^1 topology.
- For any $x \in M$ and $f \in \text{Diff}(M)$, $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$: the orbit of f through x.

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Goal

- In this talk, we will study the hyperbolicity of continuum-wise expansive homoclinic classes.

Hyperbolicity

- A closed invariant set $\Lambda \subset M$ called (uniformly) hyperbolic for f if $T_\Lambda M$ has a splitting $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$ such that
  - E^s_Λ and E^u_Λ are Df- invariant,
  - Df is contractive on E^s_Λ and Df is expansive on E^u_Λ.

  There are several notions extending (uniform) hyperbolicity; partial hyperbolicity, dominated splitting, etc.

Dominated splitting

- A closed invariant set $\Lambda \subset M$ admits a dominated splitting if $T_\Lambda M$ has a splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ such that
  - E_Λ and F_Λ are $Df-$ invariant;
  - there are constans $C > 0$ and $0 < \lambda < 1$ such that

  for any $x \in \Lambda$, any unit vectors $u \in E_x, v \in F_x$,

$$\frac{\|D_x f^n(u)\|}{\|D_x f^n(v)\|} \leq C\lambda^n, \quad \forall n \geq 0$$

$$\Longleftrightarrow \frac{\|D_x f^n(E_x)\|}{m(D_x f^n(F_x))} \leq C\lambda^n, \quad \forall n \geq 0$$

  where, m denotes the minimum norm.

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Homoclinic class

- $P_h(f)$ = the set of hyperbolic periodic points of f.
- For any $p, q \in P_h(f)$, we say that p and q are homoclinically related ($p \sim q$) if $W^s(p) \pitchfork W^u(q) \neq \varnothing$ and $W^u(p) \pitchfork W^s(q) \neq \varnothing$

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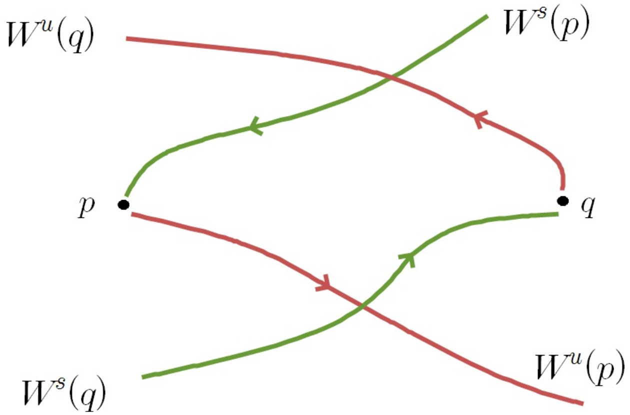

Homoclinic class

- "$\sim$" is an equivalence relation on $P_h(f)$ by the λ-lemma.
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$$H_f(p) = \overline{\{q \in P_h(f) : q \sim p\}}$$
$$= \overline{\{x \in W^s(p) \pitchfork W^u(p)\}}$$

= the homoclinic class of f associated to p.

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Why homoclinic class?

- Every basic set is a homoclinic class;
  More precisely, if $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_n$
  is a spectral decomposition, then for each $i = 1, 2, \cdots, n$,
  there is a hyperbolic periodic point $p_i \in \Lambda_i$ such that
  $\Lambda_i = H_f(p_i)$
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Hyperbolic-like properties

- When does the homoclinic class $H_f(p)$ have the hyperbolicity or hyperbolic-like properties such as

  - partial hyperbolicity

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Continuum-wise Expansivity

- We say that $\Lambda \subset M$ is continuum-wise expansive for f if there is a constant $\alpha > 0$ such that for any subsontinuum $A \subset \Lambda$, $\mathrm{diam} f^n(A) > \alpha$ for some $n \in \mathbb{Z}$. (H. Kato ('93))
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CW-expansive Diffeomorphisms

- From differential view point, we can see that the class of continuum-wise expansive diffeomorphism is strictly larger than the class of expansive diffeomorpshisms.

- For example, we denote $\mathbb{T}^2$ by the 2-dimensional torus. Let us consider the quotient space $\mathbb{P}^2 = \mathbb{T}^2 / \backsim$ obtained from the torus $\mathbb{T}^2$ by identifying each point $x \in \mathbb{T}^2$ with its antipodal point $-x$.

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CW-expansive Diffeomorphisms

- Let $\pi : \mathbb{T}^2 \to \mathbb{P}^2$ be the projection, and take a linear hyperbolic diffeomorphsim

$$f : \mathbb{T}^2 \to \mathbb{T}^2, \qquad \text{e.g. } f = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

  Then we can see that the map $g = \pi \circ f \circ \pi^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ is continuum-wise expansive, but g is not expansive.
  Note that every hyperbolic diffeomorphism is expansive.

- In fact, if f is expansive with an expansive constant $\alpha > 0$, then g is CW-expansive with an CW-expansive constant $\frac{1}{2}\alpha$.

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$$\mathbb{T}^2 \xrightarrow{\quad f \quad} \mathbb{T}^2$$

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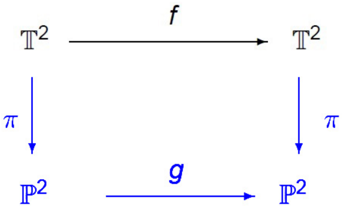

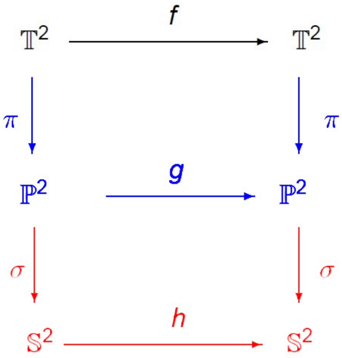
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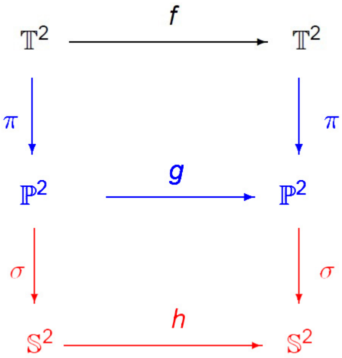

In this way, we can construct many diffeomorphisms on $\mathbb{S}^2$ which are CW-expansive.

Theorem 1

- We say that $CR(f)$ is C^1-persistently CW-expansive if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $CR(g)$ is CW-expansive.
- $CR(f)$ is C^1-persistently CW-expansive if and only if f satisfies Axiom A (i.e., $\Omega(f) = \overline{P(f)}$ is hyperbolic) and no-cycle condition (2012, Das-L-Lee).

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Theorem 2

- C^1-generically, every CW-expansive homoclinic class is hyperbolic (2012, Das-L-Lee).

- More precisely, there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}(M)$ such that for any $f \in \mathcal{R}$ and for any $p \in P_h(f)$, if $H_f(p)$ is CW-expansive, then $H_f(p)$ is hyperbolic.

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- We say that $H_f(p)$ is C^1-stably CW-expansive if there are a neighborhood U of $H_f(p)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that
  · $H_f(p) = \bigcap_{n\in\mathbb{Z}} f^n(U)$
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- If $H_f(p)$ is C^1-stably CW-expansive, then it is hyperbolic (2012, Das-L-Lee).

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Open

- If $H_f(p)$ is C^1-persistently expansive (or, CW-expansive) then is it hyperbolic?

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