New results on the combinatorial dynamics of the minimum entropy degree one circle maps depending on the rotation interval and its use as examples factory for graph maps

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Outline

Introduction and Motivation

- Transitivity,
- the existence of infinitely many periods and
- positive topological entropy

often characterize the complexity in dynamical systems.

Definition

A map $f\colon X \longrightarrow X$ is *transitive* if for every pair of open subsets $U, V \subset X$ there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.
A map f is called *totally transitive* if all iterates of f are transitive.

Introduction – Statement of the problem

A transitive map on a graph has positive topological entropy and dense set of periodic points (except for an irrational rotation on the circle).

A. M. Blokh.
On transitive mappings of one-dimensional branched manifolds.
In *Differential-difference equations and problems of mathematical physics (Russian)*, pages 3–9, 131. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984.

A. M. Blokh.
The connection between entropy and transitivity for one-dimensional mappings.
Uspekhi Mat. Nauk, 42(5(257)):209–210, 1987.

Ll. Alsedà, M. A. Del Río, and J. A. Rodríguez.
A survey on the relation between transitivity and dense periodicity for graph maps.
J. Difference Equ. Appl., 9(3-4):281–288, 2003.
Dedicated to Professor Alexander N. Sharkovsky on the occasion of his 65th birthday.

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
Transitivity and dense periodicity for graph maps.
J. Difference Equ. Appl., 9(6):577–598, 2003.

Thus, in view of

J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey.
On Devaney's definition of chaos.
Amer. Math. Monthly, 99(4):332–334, 1992.

every transitive map on a graph is chaotic in the sense of Devaney (except, again, for an irrational rotation on the circle).

Moreover, a totally transitive map on a graph which is not an irrational rotation on the circle has *cofinite set of periods* (meaning that the complement of the set of periods is finite or, equivalently, that it contains all positive integers larger than a given one).

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
A note on the totally transitive graph maps.
Internat. J. Bifur. Chaos Appl. Sci. Engrg., 11(3):841–843, 2001.

Summarizing

Totally transitive maps on graphs are complicate since they have *positive topological entropy* and *cofinite set of periods*.

However, for every graph that is not a tree and for every $\varepsilon > 0$, there exists a totally transitive map *with periodic points* such that its topological entropy is positive but smaller than ε.

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
A splitting theorem for transitive maps.
J. Math. Anal. Appl., 232(2):359–375, 1999.

Summarizing again

The complicate totally transitive maps on graphs may be relatively simple because they may have *arbitrarily small positive topological entropy*.

In this talk we consider the question whether the simplicity phenomenon that happens for the topological entropy can be extended to the set of periods. More precisely,

is it true that when a totally transitive graph map with periodic points has small positive topological entropy it also has small "cofinite part" of the set of periods?

This is, in fact, a study about the dynamics simplicity of complicate maps (totally transitive), and the consistency among the different ways of measuring it. The dynamics simplicity can be measured in different ways:

- small topological entropy,
- small set of periods (or small cofinite part of the set of periods),
- small cardinality of the set of periodic points of a given period,
- others ....

Introduction and Motivation

The aim of this study is to show the consistency of the measures of dynamical simplicity of the totally transitive graph maps. That is, when a sequence of totally transitive graph maps endeavours a path to simplification it does it with respect to all measures of dynamical complexity: entropy, size of the cofinite part of the set of periods, and others?

Currently we look at the consistency of the entropy and the size of the cofinite part of the set of periods.

Boundary of Cofiniteness definition

To measure the size of the "cofinite part" of the set of periods we introduce the notion of *boundary of cofiniteness*.

The *boundary of cofiniteness*

of a totally transitive map f, denoted by $\text{BdCof}(f)$, is defined as the largest positive integer $L \in \text{Per}(f)$, $L > 2$ such that $L - 1 \notin \text{Per}(f)$ but there exists $n \geq L$ such that $\text{Per}(f) \supset \{n, n+1, n+2, \dots\}$ and

$$\frac{\text{Card}\left(\{1, \dots, L-2\} \cap \text{Per}(f)\right)}{L-2} \leq \frac{2\log_2(L-2)}{L-2}.$$

That is, the cofinite part of the set of periods is beyond the boundary of cofiniteness and the density of the low periods is small.

Clearly, the larger it is the boundary of cofiniteness the simpler it is the set of periods.

A first attempt: proving theorems

Results for the circle and the σ graph.

Theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of totally transitive circle maps of degree one with periodic points such that $\lim_{n\to\infty} h(f_n) = 0$. For every n let $F_n \in \mathcal{L}_1$ be a lifting of f_n. Then,

- $\lim_{n\to\infty} \text{len}\left(Rot(F_n)\right) = 0$,
- *there exists $N \in \mathbb{N}$ such that $BdCof(f_n)$ exists for every $n \geq N$, and*
- $\lim_{n\to\infty} BdCof(f_n) = \infty$.

Results for the circle and the σ graph.

For *σ-maps* (continuous self maps of the space σ) we use the extension of *lifting*, *degree* and *rotation interval* $\text{Rot}_{\mathbb{R}}$ developed in

Ll. Alsedà and S. Ruette.
Rotation sets for graph maps of degree 1.
Ann. Inst. Fourier (Grenoble), 58(4):1233–1294, 2008.

to get the same result:

Theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of totally transitive σ-maps of degree one with periodic points such that $\lim_{n\to\infty} h(f_n) = 0$. For every n let $F_n \in \mathcal{L}_1$ be a lifting of f_n. Then,

- $\lim_{n\to\infty} \text{len}\left(Rot(F_n)\right) = 0$,
- *there exists $N \in \mathbb{N}$ such that $BdCof(f_n)$ exists for every $n \geq N$, and*
- $\lim_{n\to\infty} BdCof(f_n) = \infty$.

A first attempt: proving theorems

The problem is that these results deal with very particular (and simple) spaces, and require *a lot* of theory and dynamical knowledge of the spaces under study.

Or conversely: currently we do not have the tools to consider these problems out of the circle and the σ graph, unless we find a much simpler strategy for these results.

A second attempt: constructing examples for general graphs

When we show the three examples observe that they are constructed in two steps:

- Construct the desired example in the circle, with good control of the dynamics and verifying the desired properties (rotation number and set of periods).
- Extend the circle example to an arbitrary graph by keeping the basic desired properties.

Example I: with persistent fixed low periods

Theorem

For every $n \in \{4k+1, 4k-1 \colon k \in \mathbb{N}\}$ there exists a totally transitive continuous circle map of degree one such that: $Rot(f_n) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$, and

$$\operatorname{Per}(f_n) = \{2\} \cup \{p \text{ odd} \colon 2k+1 \le p \le n-2\} \cup \{n, n+1, n+2, \dots\}.$$

Moreover $\lim_{n\to\infty} h(f_n) = 0$. Furthermore, given any graph G with a circuit, the maps f_n can be extended to continuous totally transitive maps $g_n \colon G \longrightarrow G$ so that $\operatorname{Per}(g_n) = \operatorname{Per}(f_n)$ but still $\lim_{n\to\infty} h(g_n) = 0$.

Remark

- $2k+1 \le BdCof(g_n) = BdCof(f_n) < n$ *and, hence,* $\lim_{n\to\infty} BdCof(g_n) = \infty$.
- *The density of "lower" periods outside the cofinite part converges to $\frac{1}{4}$ and there is a very small period 2.*
- *Despite of the fact that still $\lim_{n\to\infty} h(g_n) = 0$, in general, $h(g_n)$ is slightly larger than $h(f_n)$.*

Example II: with non-constant low periods

Theorem

For every $n \in \mathbb{N}$ there exists a totally transitive continuous circle map of degree one such that: $Rot(f_n) = \left[\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right] = \left[\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}\right]$, and

$$\operatorname{Per}(f_n) = \{n\} \cup \{tn + k : t \in \{2,3,\dots,\nu-1\} \text{ and } -\tfrac{t}{2} < k \le \tfrac{t}{2},\ k \in \mathbb{Z}\} \cup \{L \in \mathbb{N} : L \ge n\nu + 1 - \tfrac{\nu}{2}\}$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd;} \end{cases}$$

and $n \le BdCof(f_n) \le n\nu - 1 - \frac{\nu}{2}$.

Moreover $\lim_{n\to\infty} h(f_n) = 0$ and $\lim_{n\to\infty} BdCof(f_n) = \infty$. Furthermore, given any graph G with a circuit, the maps f_n can be extended to continuous totally transitive maps $g_n \colon G \longrightarrow G$ so that $\operatorname{Per}(g_n) = \operatorname{Per}(f_n)$ but still $\lim_{n\to\infty} h(g_n) = 0$.

Example II: with non-constant low periods

Remark

- *The density of "lower" periods outside the cofinite part converges to* $\frac{1}{2}$ *but the smallest period is* n.
- *As before, despite of the fact that still* $\lim_{n\to\infty} h(g_n) = 0$, *in general,* $h(g_n)$ *is slightly larger than* $h(f_n)$.

Example III: the dream example

Theorem

For every $n \in \mathbb{N}$ *there exists a totally transitive continuous circle map of degree one such that:* $Rot(f_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$, *and*

$$\operatorname{Per}(f_n) = \{n, n+1, n+2, \dots\}.$$

Moreover $\lim_{n\to\infty} h(f_n) = 0$. *Furthermore, given any graph* G *with a circuit, the maps* f_n *can be extended to continuous totally transitive maps* $g_n \colon G \longrightarrow G$ *so that* $\operatorname{Per}(g_n) = \operatorname{Per}(f_n)$ *but still* $\lim_{n\to\infty} h(g_n) = 0$.

Remark

- $BdCof(g_n) = BdCof(f_n) = n$ *and, hence,* $\lim_{n\to\infty} BdCof(g_n) = \infty$.
- *There are no "lower" periods outside the cofinite part.*
- *As in the previous two cases, despite of the fact that still* $\lim_{n\to\infty} h(g_n) = 0$, *in general,* $h(g_n)$ *is slightly larger than* $h(f_n)$.

Circle dynamics of degree one — Liftings

Let $\mathrm{e} \colon \mathbb{R} \longrightarrow \mathbb{S}^1$ be the natural projection which is defined by $\mathrm{e}(x) := \exp(2\pi i x)$.

Given a continuous map $f \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we say that a continuous map $F \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a *lifting of* f if

$$\mathrm{e}(F(x)) = f(\mathrm{e}(x))$$

for every $x \in \mathbb{R}$.

For such F, there exists $d \in \mathbb{Z}$ such that

$$F(x+1) = F(x) + d \quad \text{for all} \quad x \in \mathbb{R},$$

and this integer is called both the *degree of* f and the *degree of* F.

If G and F are two liftings of f then $G = F + k$ for some integer k and so F and G have the same degree.

$\mathcal{L}_d$ denotes the set of all liftings of circle maps of degree d.

Circle dynamics of degree one — Rotation numbers

Let $F \in \mathcal{L}_1$ and let $x \in \mathbb{R}$. The number

$$\rho_F(x) := \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

will be called the *rotation number of* x. Moreover, the set

$$\operatorname{Rot}(F) := \{\rho_F(x) : x \in \mathbb{R}\} = \{\rho_F(x) : x \in [0,1]\}$$

will be called the *rotation interval of* F. It is well known that it is a closed interval of the real line (c.f. [Ito]).

Circle dynamics of degree one
More on rotation numbers

If $F \in \mathcal{L}_1$ is a non-decreasing map, then

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

for every $x \in \mathbb{R}$ and, moreover, it is independent on x.

Then this number ($\rho_F(x)$ for any $x \in \mathbb{R}$), will be called the *rotation number of F*.

Circle dynamics of degree one — Upper and lower maps

For every $F \in \mathcal{L}_1$ we define the *lower map* $F_l\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F_l(x) = \inf\{F(y) : y \geq x\}$$

and the *upper map* $F_u\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F_u(x) = \sup\{F(y) : y \leq x\}.$$

It is easy to see that F_l, F_u are non-decreasing maps from $\mathcal{L}_1$.

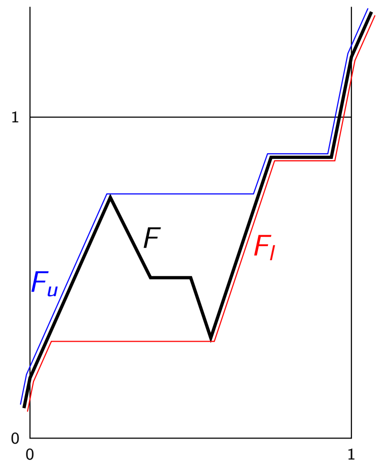


Figure: An example of a map $F \in \mathcal{L}_1$ with its lower map F_l in red and its upper map F_u in blue.

Circle dynamics of degree one
Upper and lower maps and rotation interval

The next theorem gives an effective way to compute the rotation interval from the rotation numbers of the upper and lower maps.

Theorem

For every $F \in \mathcal{L}_1$ it follows that $\operatorname{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

Circle dynamics of degree one — Lifted orbits

To study the periodic orbits of a circle map we introduce the notion of *lifted orbit*.

Let f be a continuous circle map of degree d and let $F \in \mathcal{L}_d$ be a lifting of f. A set $P \subset \mathbb{R}$ will be called a *lifted orbit of F* if there exists $z \in \mathbb{S}^1$ such that

$$P = \mathrm{e}^{-1}(\operatorname{Orb}_f(z)) \quad \text{and} \quad f(\mathrm{e}(x)) = \mathrm{e}(F(x))$$

for every $x \in P$.

Whenever z is a periodic point of f of period n, P will be called a *lifted periodic orbit of F of period n*. We will denote by $\operatorname{Per}(F)$ the set of periods of all lifted periodic orbits of F.

Remark

$\operatorname{Per}(F) = \operatorname{Per}(f)$.

Circle dynamics of degree one — A remark on lifted orbits

Let $F \in \mathcal{L}_1$ and let P be a lifted periodic orbit of F of period period n. Set

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

with $x_i < x_j$ if and only if $i < j$. The fact that $P = \mathrm{e}^{-1}(\mathrm{Orb}_f(z))$, in this case, gives

$$\mathrm{Card}\,(P \cap [r, r+1)) = n$$

for every $r \in \mathbb{R}$ and, hence,

$$x_{kn+i} = x_i + k$$

for every $i, k \in \mathbb{Z}$.

Moreover, there exists $m \in \mathbb{Z}$ such that $F^n(x_i) = x_i + m = x_{mn+i}$ for every $x_i \in P$. Consequently,

$$\rho_F(x_i) = \frac{m}{n}$$

for every $x_i \in P$.

Circle dynamics of degree one — More on lifted orbits

From the above remark it follows that if P is a lifted periodic orbit of $F \in \mathcal{L}_1$, then all the points of P have the same rotation number.

This number will be called the *rotation number of* P.

Circle dynamics of degree one — Twist lifted orbits

A lifted periodic orbit P of $F \in \mathcal{L}_1$ such that $F\big|_P$ is increasing will be called *twist*.

Remark

Let

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

be a twist lifted periodic orbit of $F \in \mathcal{L}_1$ of period n and rotation number m/n labelled so that $x_i < x_j$ if and only if $i < j$. Then, m and n are coprime and

$$F(x_i) = x_{i+m},$$

for all $i \in \mathbb{Z}$.

Circle dynamics of degree one — The set of periods

Recall that the *Sharkovskiĭ Ordering* ${}_{\mathrm{Sh}}\!\geq$ (the symbols ${}_{\mathrm{Sh}}\!>$, $<_{\mathrm{Sh}}$ and $\leq_{\mathrm{Sh}}$ will also be used in the natural way) is a linear ordering of $\mathbb{N}_{\mathrm{Sh}} := \mathbb{N} \cup \{2^\infty\}$ (we have to include the symbol $\{2^\infty\}$ in order to ensure the existence of supremum of every subset with respect to the ordering ${}_{\mathrm{Sh}}\!\geq$) defined as follows:

$$
\begin{array}{l}
3 \;{}_{\mathrm{Sh}}\!> 5 \;{}_{\mathrm{Sh}}\!> 7 \;{}_{\mathrm{Sh}}\!> 9 \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> \\
2\cdot 3 \;{}_{\mathrm{Sh}}\!> 2\cdot 5 \;{}_{\mathrm{Sh}}\!> 2\cdot 7 \;{}_{\mathrm{Sh}}\!> 2\cdot 9 \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> \\
4\cdot 3 \;{}_{\mathrm{Sh}}\!> 4\cdot 5 \;{}_{\mathrm{Sh}}\!> 4\cdot 7 \;{}_{\mathrm{Sh}}\!> 4\cdot 9 \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> \\
\qquad\qquad \vdots \\
2^n\cdot 3 \;{}_{\mathrm{Sh}}\!> 2^n\cdot 5 \;{}_{\mathrm{Sh}}\!> 2^n\cdot 7 \;{}_{\mathrm{Sh}}\!> 2^n\cdot 9 \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> \\
\qquad\qquad \vdots \\
2^\infty \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> 2^n \;{}_{\mathrm{Sh}}\!> \cdots \;{}_{\mathrm{Sh}}\!> 16 \;{}_{\mathrm{Sh}}\!> 8 \;{}_{\mathrm{Sh}}\!> 4 \;{}_{\mathrm{Sh}}\!> 2 \;{}_{\mathrm{Sh}}\!> 1.
\end{array}
$$

Circle dynamics of degree one — The set of periods
Notation

Given $c, d \in \mathbb{R}$, $c \le d$ we set

$$M(c,d) := \{n \in \mathbb{N} : c < k/n < d \text{ for some integer } k\}.$$

Let $F \in \mathcal{L}_1$ and let c be an endpoint of $\operatorname{Rot}(F)$. We define the set

$$Q_F(c) := \begin{cases} \emptyset & \text{if } c \notin \mathbb{Q} \\ \{sk : k \in \mathbb{N} \text{ and } k \le_{\text{Sh}} s_c\} & \text{if } c = r/s \text{ with } r,s \text{ coprime} \end{cases}$$

and $s_c \in \mathbb{N}_{\text{Sh}}$ is defined by the Sharkovskiĭ Theorem on the real line as follows: Indeed, since $c = r/s$ and r and s are coprime, the map $F^s - r$ is a continuous map on the real line with periodic points. By the Sharkovskiĭ Theorem there exists an $s_c \in \mathbb{N}_{\text{Sh}}$ such that the set of periods (not lifted periods) of $F^s - r$ is precisely $\{k \in \mathbb{N} : k \le_{\text{Sh}} s_c\}$.

Circle dynamics of degree one — The set of periods
Misiurewicz Theorem

Theorem

Let f be a continuous circle map of degree one having a lifting $F \in \mathcal{L}_1$. Assume that $\operatorname{Rot}(F) = [c,d]$. Then

$$\operatorname{Per}(f) = Q_F(c) \cup M(c,d) \cup Q_F(d).$$

Minimum dynamics in the circle — Definition

For $c, d \in \mathbb{R}$, $c < d$ and $z > 1$ we define

$$R_{c,d}(z) := \sum_{\{(p,q)\,:\, p\in\mathbb{Z},\ q\in\mathbb{N} \text{ and } c < \frac{p}{q} < d\}} z^{-q}.$$

$R_{c,d}(z) = \frac{1}{2}$ has a unique solution $\beta_{c,d} > 1$.

We define a lifting $G_{c,d} \in \mathcal{L}_1$ by:

$g_{c,d}$ denotes the circle map of degree 1 which has $G_{c,d}$ as a lifting.

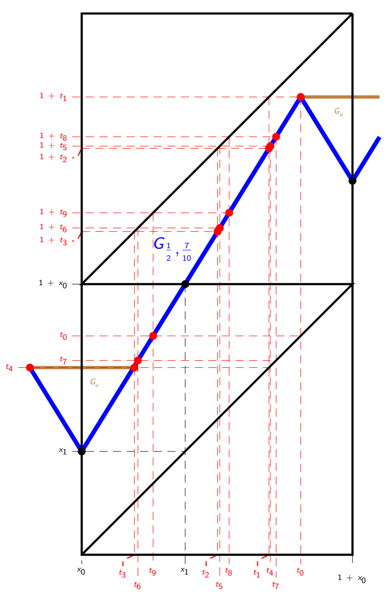
$$G_{c,d}(x) := \begin{cases} \beta_{c,d}x + b_{c,d} & \text{if } 0 \le x \le u_{c,d}, \\ \beta_{c,d}(1-x) + b_{c,d} + 1 & \text{if } u_{c,d} \le x \le 1, \\ G_{c,d}(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1], \end{cases}$$

with $u_{c,d} := \frac{\beta_{c,d}+1}{2\beta_{c,d}} \in (0,1)$ and

$$b_{c,d} := \frac{(\beta_{c,d}-1)^2}{\beta_{c,d}} \sum_{n=1}^{\infty} \lfloor nc \rfloor \beta_{c,d}^{-n}.$$

$\lfloor \cdot \rfloor$ denotes the integer part function.

Minimum dynamics in the circle — Definition



The graphs of the maps $G_{\frac{1}{2},\frac{7}{10}}$ in blue and $G_u = \left(G_{\frac{1}{2},\frac{7}{10}}\right)_u$ in brown (note that G_u coincides with $G_{\frac{1}{2},\frac{7}{10}}$ on the interval $[t_3, t_0]$ and hence it is left in blue color for clarity). The orbit of $0 = x_0$ is labelled so that $G_{\frac{1}{2},\frac{7}{10}}(x_0) = x_1$ and $G_{\frac{1}{2},\frac{7}{10}}(x_1) = x_0 + 1$ (in this case the (*temporal*) labelling leaves the orbit sorted in the real line) and the orbit of $u_{\frac{1}{2},\frac{7}{10}} = t_0$ is labelled so that $G_{\frac{1}{2},\frac{7}{10}}(t_i) = 1 + t_{i+1}$ for $i \in \{0,1,2,4,5,7,8\}$, $G_{\frac{1}{2},\frac{7}{10}}(t_i) = t_{i+1}$ for $i \in \{3,6\}$ and $G_{\frac{1}{2},\frac{7}{10}}(t_9) = t_0$. In this case the (*temporal*) labelling does not sort the orbit; the sorted version of this orbit is obtained by setting $y_9 := t_0$ and $y_{k+i} := t_{3i+3}$ for $k = 0,3,6$ and $i = 0,1,2$.

For $c \in \mathbb{R}$, $c > 0$, and $z > 1$ we define

$$T_c(z) := \sum_{n=0}^{\infty} z^{-\lfloor \frac{n}{c} \rfloor},$$

and, for definiteness, we set $T_0(z) \equiv 0$. Then, for $c, d \in \mathbb{R}$, $c < d$, $c \in [0, 1)$, and $z > 1$ we define

$$Q_{c,d}(z) := z + 1 + 2\left(\frac{z}{z-1} - T_{1-c}(z) - T_d(z)\right)$$

One can show that

$$Q_{c,d}(z) = (z-1)\left(1 - 2R_{c,d}(z)\right).$$

Hence, $\beta_{c,d}$ is the largest root of the equation $Q_{c,d}(z) = 0$.

This observation gives a much easier way of calculating the numbers $\beta_{c,d}$.

The next theorem justifies the importance of the the minimum dynamics maps:

Theorem ([A., Llibre, Mañosas, Misiurewicz])

Let f be a circle map of degree 1 with rotation interval $[c, d]$ with $c < d$. Then $h(f) \geq \log \beta_{c,d}$. Moreover, for every $c, d \in \mathbb{R}$ with $c < d$, $\text{Rot}\big(G_{c,d}\big) = [c, d]$ and $h(g_{c,d}) = \log \beta_{c,d}$.

Theorem A (entopy goes to zero with the rotation interval)
A nice natural property consistent with the minimality

Let $\{[c_n, d_n]\}_{n\in\mathbb{N}}$ be a sequence of non-degenerate intervals contained in the interval $(0, 1)$ such that $c_n, d_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \min M(c_n, d_n) = \infty$. Then,

$$\lim_{n\to\infty} h(g_{c_n,d_n}) = \log \beta_{c_n,d_n} = 0.$$

Remark

The inequality $\min M(c_n, d_n) > M$ *implies that* $M \notin M(c_n, d_n)$ *and this is equivalent to* $\frac{k}{M} \leq c_n < d_n \leq \frac{k+1}{M}$ *for some* $k \in \mathbb{Z}$*, which implies* $d_n - c_n \leq \frac{1}{M}$.
Hence, $\lim_{n\to\infty} \min M(c_n, d_n) = \infty$ *implies* $\lim_{n\to\infty} d_n - c_n = 0$.

Note: Theorem A is general and does not depend on a_n and b_n.

Notation

$\llbracket \cdot \rrbracket$ is a function similar to the integer part function, defined by:

$$\llbracket x \rrbracket := -1 - \lfloor -x \rfloor = \begin{cases} \lfloor x \rfloor & \text{if } x \notin \mathbb{Z}, \text{ and} \\ x - 1 & \text{otherwise.} \end{cases}$$

The fact that, for every $n \in \mathbb{N}$, $h(g_{c_n,d_n}) = \log \beta_{c_n,d_n}$ follows from Theorem 1. So, we need to show that $\lim_{n\to\infty} \beta_{c_n,d_n} = 1$.

For every $q \in \mathbb{N}$, $\llbracket qd_n \rrbracket - \lfloor qc_n \rfloor$ is equal to the number of integers contained in the interval (qc_n, qd_n) which, in turn, is the cardinality of the set

$$\{p \in \mathbb{Z} : \text{ such that } c_n < \tfrac{p}{q} < d_n\}.$$

Skipable proof of Theorem A (II)

For every $n \in \mathbb{N}$, let us denote $N_n := \min M(c_n, d_n) - 1$. So, for every $q \leq N_n$, $q \notin M(c_n, d_n)$ which is equivalent to

$$\{p \in \mathbb{Z} : \text{ such that } c_n < \tfrac{p}{q} < d_n\} = \emptyset \quad \Longleftrightarrow \quad \llbracket q d_n \rrbracket - \lfloor q c_n \rfloor = 0.$$

Moreover, if $\frac{1}{k} < d_n - c_n$, then $k \in M(c_n, d_n)$ and this implies that $k > N_n$. Hence, $k \leq N_n$ implies $d_n - c_n \leq \frac{1}{k}$, which is equivalent to $(d_n - c_n)k \leq 1$.

Skipable proof of Theorem A (III)

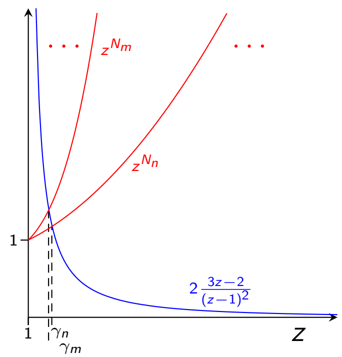
Consequently, when $z > 1$,

$$\begin{aligned} R_{c,d}(z) &:= \sum_{\{(p,q)\,:\, p\in\mathbb{Z},\ q\in\mathbb{N} \text{ and } c<\frac{p}{q}<d\}} z^{-q} = \sum_{k=N_n+1}^{\infty} \left(\llbracket k d_n \rrbracket - \lfloor k c_n \rfloor\right) z^{-k} \\ &\leq \sum_{k=N_n+1}^{\infty} \left((d_n - c_n)k + 1\right) z^{-k} \\ &= z^{-N_n} \sum_{k=1}^{\infty} \left((d_n - c_n)(N_n + k) + 1\right) z^{-k} \\ &\leq z^{-N_n} \sum_{k=1}^{\infty} \left((d_n - c_n)k + 2\right) z^{-k} \\ &< z^{-N_n} \sum_{k=1}^{\infty} (k+2) z^{-k} = z^{-N_n} \frac{3z-2}{(z-1)^2}. \end{aligned}$$

Skipable proof of Theorem A (IV)

Now let us consider the equation $z^{-N_n}\frac{3z-2}{(z-1)^2} = \frac{1}{2}$, which is equivalent to $z^{N_n} = 2\frac{3z-2}{(z-1)^2}$. We note that:

1. The map $z \mapsto 2\frac{3z-2}{(z-1)^2}$ is strictly decreasing on $(1, +\infty)$, $\lim_{z\to1^+} 2\frac{3z-2}{(z-1)^2} = +\infty$ and $\lim_{z\to\infty} 2\frac{3z-2}{(z-1)^2} = 0$.
2. The map $z \mapsto z^{N_n}$ is strictly increasing and $z^{N_n}\big|_{z=1} = 1$.
3. Clearly, $N_n < N_m$ implies $z^{N_n} < z^{N_m}$ for every $z > 1$.



Then, for each n, there exists a unique real number $\gamma_n > 1$ such that $\gamma_n^{N_n} = 2\frac{3\gamma_n-2}{(\gamma_n-1)^2}$, $\gamma_m < \gamma_n$ whenever $N_n < N_m$, and $\lim_{n\to\infty} \gamma_n = 1$ because $\lim_{n\to\infty} N_n = \infty$.

Skipable proof of Theorem A (and V)

Since, $R_{c_n,d_n}(\gamma_n) < \gamma_n^{-N_n}\frac{3\gamma_n-2}{(\gamma_n-1)^2} = \frac{1}{2}$, we have $1 < \beta_{c_n,d_n} < \gamma_n$ because R_{c_n,d_n} is continuous and decreasing on the interval $(1, \infty)$.

So, $\lim_{n\to\infty} \beta_{c_n,d_n}$ exists and satisfies

$$1 \leq \lim_{n\to\infty} \beta_{c_n,d_n} \leq \lim_{n\to\infty} \gamma_n = 1.$$

Theorem B

Let $[c, d]$ be a non-degenerate interval contained in the interval $(0, 1)$ such that $c = \frac{p}{q}$ and $d = \frac{r}{s}$ with $p, q, r, s \in \mathbb{N}$ and $(p, q) = (r, s) = 1$. Then, the following statements hold:

1. $g_{c,d}$ is totally transitive.
2. $Q = \mathrm{e}^{-1}(\mathrm{Orb}_{g_{c,d}}(\mathrm{e}(0)))$ is a twist periodic orbit of $G_{c,d}$ of period q and rotation number c with the property that $Q \subset \bigcup_{k\in\mathbb{Z}} \left[k, k + \frac{1}{\beta_{c,d}}\right)$. Moreover, Q is the unique lifted periodic orbit of $G_{c,d}$ with rotation number c.
3. $P = \mathrm{e}^{-1}(\mathrm{Orb}_{g_{c,d}}(\mathrm{e}(u_{c,d})))$ is a twist periodic orbit of $G_{c,d}$ of period s and rotation number d such that $P \subset \bigcup_{k\in\mathbb{Z}} \left(k + \frac{\beta_{c,d}-1}{2\beta_{c,d}}, k + u_{c,d}\right]$. Moreover, P is the unique lifted periodic orbit of $G_{c,d}$ with rotation number d.

Theorem B

4. Set

$$Q = \{\dots x_{-1}, x_0 = 0, x_1, x_2, \dots, x_{q-1}, x_q, x_{q+1}, \dots\}, \quad \text{and}$$
$$P = \{\dots y_{-1}, y_0, y_1, y_2, \dots, y_{s-1} = u_{c,d}, y_s, y_{s+1}, \dots\}$$

with $x_i < x_j$ and $y_i < y_j$ if and only if $i < j$. Then, for every $i, j \in \mathbb{Z}$,

$$x_{jq+i} = x_i + j, \quad y_{js+i} = y_i + j,$$
$$G_{c,d}(x_i) = x_{i+p} \quad \text{and} \quad G_{c,d}(y_i) = y_{i+r},$$

Moreover, we also have that $0 = x_0 < y_0$, $x_{q-1} < y_{s-1} = u_{c,d}$ and the points $y_{s-1} = u_{c,d} < 1 = x_q$ are consecutive in $Q \cup P$.

Theorem B

5. Assume that $\ell := \left\lfloor \frac{s}{r} \right\rfloor \neq \left\lfloor \frac{q}{p} \right\rfloor$. Then, $1 \le \ell < \left\lfloor \frac{q}{p} \right\rfloor$, $n := q - \ell p \ge p \ge 1$, $t := s - \ell r \in \{0, 1, 2, \dots, r-1\}$, and

$$\begin{array}{l}
x_0 = 0 < x_1 < \cdots < x_{n-1} < y_0 < y_1 < \cdots < y_{t-1} < \\
\begin{array}{cccccccccc}
 & x_n & < \cdots < & x_{n+p-1} & < & y_t & < \cdots < & y_{t+r-1} & < & \\
 & x_{n+p} & < \cdots < & x_{n+2p-1} & < & y_{t+r} & < \cdots < & y_{t+2r-1} & < & \\
 & \vdots & < \cdots < & \vdots & < & \vdots & < \cdots < & \vdots & < & \\
 & x_{n+(\ell-1)p} & < \cdots < & x_{n+\ell p-1} & < & y_{t+(\ell-1)r} & < \cdots < & y_{t+\ell r-1} & < & x_q = 1. \\
 & \parallel & & \parallel & & \parallel & & \parallel & & \\
 & x_{q-p} & & x_{q-1} & & y_{s-r} & & y_{s-1} & & \\
 & & & & & & & \parallel & & \\
 & & & & & & & u_{c,d} & &
\end{array}
\end{array}$$

Theorem B

6. $Q \cup P$ is a short Markov partition for $G_{c,d}$, and $g_{c,d}$ is Markov with respect to the Markov partition $\mathrm{Orb}_{g_{c,d}}(\mathrm{e}(0)) \cup \mathrm{Orb}_{g_{c,d}}(\mathrm{e}(u_{c,d}))$.

Short Markov Partition

Let X be a topological graph and let $f\colon X \longrightarrow X$ be a continuous map. A *Markov invariant set* is defined to be a finite f-invariant set $Q \supset V(X)$ ($f(Q) \subset Q$) such that the closure of each connected component of $X \setminus Q$, called a *Q-basic interval*, is an interval of X.

When f is monotone on each Q-basic interval Q is called a *Markov partition of X with respect to f* and f is called a *Markov map with respect to Q*.

Let $F \in \mathcal{L}_1$ be a lifting of f and let Q be a Markov partition with respect to F. A Q-basic interval will be called *F-short* if the length of the interval $F(I)$ is strictly smaller than 1. Then, Q will be called a *short Markov partition with respect to F* whenever every Q-basic interval is F-short.

Minimum dynamics on the circle (cont'd)

Corollary

Let $[c, d]$ be a non-degenerate interval contained in the interval $(0, 1)$ such that $c, d \in \mathbb{Q}$. Let q and s be the denominators of c and d when written in irreducible form, respectively. Then,

$$\mathrm{Per}(g_{c,d}) = \{q, s\} \cup M(c, d).$$

The examples factory: The minimalistic extension to graph maps

Let g be a Markov circle map of degree one with respect to a Markov partition Q. Assume that there exist pairwise disjoint basic intervals $I, J, K \in \mathcal{B}(Q)$ such that $f(I) = J$ and $f(J) = K$. Let G be an arbitrary graph with a circuit C. The *minimalistic extension of g to G with base at C*, is a continuous self map of G defined as follows.

Let $S \subset C$ be an interval such that $S \cap V(G) = \emptyset$ and let $\eta\colon \mathbb{S}^1 \longrightarrow C$ be a homeomorphism such that $C \setminus \mathrm{Int}(S) = \eta(J)$. Clearly, $X := G \setminus \mathrm{Int}(S) \supset \eta(J)$ is a subgraph of G and the two elements of $\partial S = \partial\eta(J)$ are endpoints (and thus vertices) of X but they are not vertices of G because $S \cap V(G) = \emptyset$. Then we have

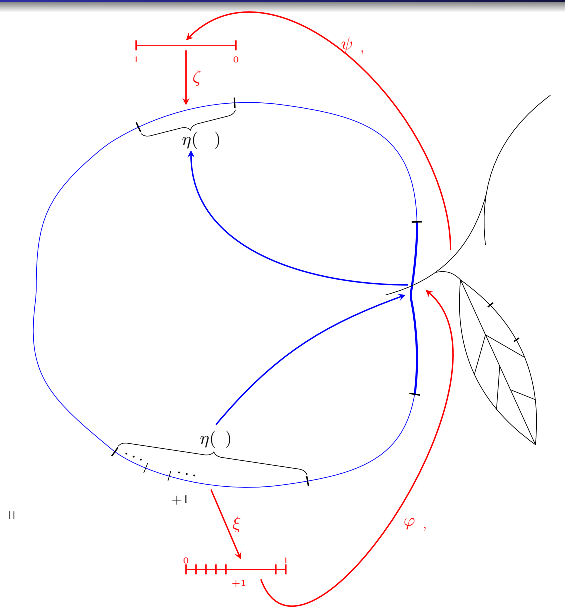
$$G = X \cup S = X \cup \mathrm{Int}(S) = (X \setminus \{a, b\}) \cup S,$$

and

$$V(X) = V(G) \cup \partial S = V(G) \cup \{a, b\}$$

where, for definiteness, we have set $\{a, b\} := \partial S = \partial\eta(J)$.

The minimalistic extension to graph maps — The figure



A topological graph G and the definition of $g^{G;C}$. The circuit C (with apple shape) is drawn in blue. Then, the interval S is the (thin) path in C from b to a counter-clockwise ($\partial S = \{a, b\}$) and the interval $C \setminus \mathrm{Int}(S) = \eta(J)$ is the **thick** path in C from b to a clockwise.

The minimalistic extension to graph maps

Now we define an auxiliary graph map $\widetilde{f_C} := \eta \circ g \circ \eta^{-1}\colon C \longrightarrow C$. Clearly, $\widetilde{f_C}$ is a Markov map with respect to $\eta(Q)$, conjugate to g.

The assumptions $g(I) = J$ and $g(J) = K$ imply that $\widetilde{f_C}$ sends homeomorphically $\partial\eta(I)$ to $\partial\eta(J)$ and the latter to $\partial\eta(K)$. Hence, we can consider a homeomorphism $\xi\colon I \longrightarrow [0, 1]$ with an orientation such that $\xi(\eta^{-1}(z_a^I)) = 0$ and $\xi(\eta^{-1}(z_b^I)) = 1$, where z_a^I (respectively z_b^I) denotes the endpoint (and unique element) of $\eta(I)$ such that $\widetilde{f_C}(z_a^I) = a$ (respectively $\widetilde{f_C}(z_b^I) = b$). Similarly, we consider a second homeomorphism $\zeta\colon [0, 1] \longrightarrow \eta(K)$ with an orientation such that $\zeta(0) = \widetilde{f_C}(a)$ and $\zeta(1) = \widetilde{f_C}(b)$.

The minimalistic extension to graph maps

When $G = C$ then we set $f^{G;C} := \widetilde{f_C}$. When $G \neq C$ we define the map $f^{G;C}$ by means of the Utility Lemma for the subgraph X. For every $x \in G$ we set:

$$f^{G;C}(x) := \begin{cases} \varphi_{a,b}(\xi(\eta^{-1}(x))) & \text{if } x \in \eta(I); \\ \zeta(\psi_{a,b}(x)) & \text{if } x \in X; \\ \widetilde{f_C}(x) & \text{if } x \in S \setminus \operatorname{Int}(\eta(I)). \end{cases}$$

Exporting circle dynamics to graphs — A utility lemma

Lemma

Let X be a topological graph which is not an interval and let $a, b \in V(X)$ be two different endpoints of X. Then, there exist a partition of the interval $[0,1]$, $0 = s_0 < s_1 < \dots < s_m = 1$, with $m = m(X, a, b) \geq 5$ odd, and two continuous surjective maps $\varphi_{a,b} \colon [0,1] \longrightarrow X$ and $\psi_{a,b} \colon X \longrightarrow [0,1]$ such that the following statements hold:

1. $\varphi_{a,b}^{-1}(W) = \{s_i : i \in \{0,1,\dots,m\}\}$, *where*

   $$W := \varphi_{a,b}(\{s_i : i \in \{0,1,\dots,m\}\}) \supset V(X),$$

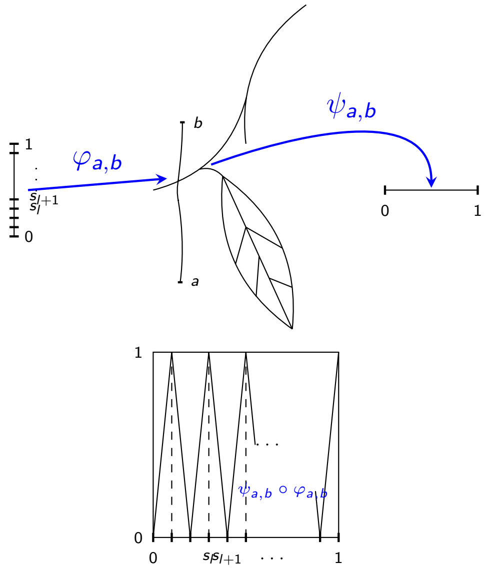
   and $\varphi_{a,b}(0) = a$ *and* $\varphi_{a,b}(1) = b$.
2. *For every* $i = 0,1,\dots,m-1$, $\varphi_{a,b}\big|_{[s_i,s_{i+1}]}$ *is injective and* $\varphi_{a,b}([s_i,s_{i+1}])$ *is an interval which is the closure of a connected component of the punctured graph* $X \setminus W$.

Exporting circle dynamics to graphs — A utility lemma

Lemma

3. *If* $\varphi_{a,b}(s_i) = \varphi_{a,b}(s_j)$ *then* $i \equiv j \pmod 2$.
4. $\psi_{a,b}(\varphi_{a,b}(s_i)) = 0$ *if i is even and* $\psi_{a,b}(\varphi_{a,b}(s_i)) = 1$ *if i is odd (in particular,* $\psi_{a,b}(a) = 0$ *and* $\psi_{a,b}(b) = 1$*).*
5. *The map* $\psi_{a,b}\big|_{\varphi_{a,b}([s_i,s_{i+1}])}$ *is injective and* $\psi_{a,b}(\varphi_{a,b}([s_i,s_{i+1}])) = [0,1]$ *for every* $i = 0,1,\dots,m-1$. *In particular, the map* $(\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i,s_{i+1}]}$ *is strictly monotone.*

Exporting circle dynamics to graphs — A utility lemma



A sketch of a topological graph X and the maps from the Lemma (top picture). The map $\psi_{a,b} \circ \varphi_{a,b}$ is shown in the bottom picture.

Theorem (The minimalistic extension factory)

Let g be a Markov circle map of degree one with respect to a Markov partition Q. Assume that there exist pairwise disjoint basic intervals $I, J, K \in \mathcal{B}(Q)$ such that $g(I) = J$, $g(J) = K$ and I is the unique basic interval that g-covers J. Let G be an arbitrary graph with a circuit C. Then, the minimalistic extension of g to G with base at C, f, is a continuous well defined self map of G which is Markov with respect to a Markov partition R. Moreover,

$$\text{Per}(g) \subset \text{Per}(f) \quad \text{and} \quad h(f) \le h(g) + \log \tfrac{L}{\rho}$$

where L is a constant depending solely on the graph G and not on the function g, and ρ is the minimal length of a loop from I to itself in the Markov graph of g.

Very Important Remark

The existence of the three pairwise disjoint basic intervals $I, J, K \in \mathcal{B}(Q)$ such that $g(I) = J$, $g(J) = K$ and I is the unique basic interval that g-covers J is assured by statement 5 of Theorem B, which is the key of the "factory" construction from the previous theorem.

Proposition

Let $H \in \mathcal{L}_1$ be a Markov map with respect to a Markov partition $P \cup Q$ such that $\text{Rot}(H)$ is a non-degenerate interval contained in $(0, 1)$. Let c be a rational endpoint of $\text{Rot}(H)$ and let q be the denominator of c when written in irreducible form. Assume also that P is a twist periodic orbit of H with period q and rotation number c and that P is the unique lifted orbit of H with rotation number c. Then, in the Markov graph of H with respect to $P \cup Q$ there exists a unique loop (modulo shifts) which is associated to P.

Definition

The above loop will be called the *P-fundamental loop of g*.

Theorem

In the assumptions of the minimalistic extension factory theorem suppose that g has a lifting H such that $Rot(H) = [c, d]$ is a non-degenerate interval contained in $(0, 1)$. Then $\text{Per}(g) = \text{Per}(f)$ provided that one of the following three contions is satisfied:

1. $\text{Per}(g) = M(c, d)$.
2. $\text{Per}(g) = \{q\} \cup M(c, d)$ *where q is the denominator of the rational endpoint c when written in irreducible form, $q \notin Q_H(d) \subset M(c, d)$, Q has a subset R such that $\exp^{-1}(R)$ is the unique lifted periodic orbit of H with period q and rotation number c, and the path $I \longrightarrow J \longrightarrow K$ is not a subpath of the R-fundamental loop of g.*
3. *Analogously for the other endpoint.*

Theorem

In the assumptions of minimalistic extension factory theorem suppose that g is Q-expanding, transitive, and $\text{Per}(g)$ is cofinite. Then, f is R-expanding and totally transitive.

Corollary

Let $\{[c_n, d_n]\}_{n\in\mathbb{N}}$ be a sequence of non-degenerate intervals contained in the interval $(0, 1)$ such that $c_n, d_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \min M(c_n, d_n) = \infty$. Assume also that the map g_{c_n,d_n} verifies the assumptions of the minimalistic extension factory theorem. Let G be an arbitrary graph with a circuit C. Then,

$$\lim_{n\to\infty} h\big(f_{c_n,d_n}\big) = 0,$$

where f_{c_n,d_n} denotes the minimalistic extension of g_{c_n,d_n} to G with base at the circuit C.