On the definition of Strange Nonchaotic Attractor

Lluís Alsedà

Departament de Matemàtiques
Universitat Autònoma de Barcelona

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Outline

Introduction — the setting

In the last two decades a lot of works have been devoted to find and study Strange Non-chaotic Attractors (SNA).
Many of these objects are found and studied for non autonomous quasiperiodically forced dynamical systems of the type:

$$(1) \qquad \begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$.
Similar models are studied also in higher dimensions and for systems that are both discrete and continuous.
Other important studies are developed in the framework of cocycles and spectral theory.

The start of the story

The term *Strange Non-chaotic attractor (SNA)* was introduced and coined in

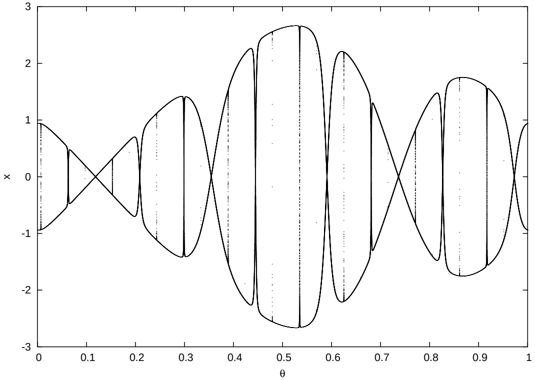
[GOPY] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke.
Strange attractors that are not chaotic.
Phys. D, 13(1-2):261–268, 1984.

After this paper the study of these objects became rapidly popular and a number a papers studying different related models appeared.

The [GOPY] model (ω equals the golden mean)

(2) $$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= 2\sigma \tanh(x_n)\cos(2\pi\theta_n) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1, \omega \in \mathbb{R} \setminus \mathbb{Q}$ and $\sigma > 1$.



The authors called this object an SNA since:

- the orbit of the point (θ, x) for almost every $\theta \in \mathbb{S}^1$ and every $x > 0$ *converges* to the SNA.
- it is *strange* because *it is not piecewise differentiable*: The SNA cuts the line $x = 0$ (and then it does so at the orbit of a point which is dense in $x = 0$) and it is different from zero in a set whose projection to $\mathbb{S}^1$ is dense.

  Remark: The line $x = 0$ is invariant because $x_{n+1} = \sigma \tanh(x_n) \cos(2\pi\theta_n)$. Moreover this invariant line turns to be a repellor.
- it is *non-chaotic* because *the Lyapunov exponents are non positive* (computed numerically).

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Early results

Constructions of flows containing SNA's can be found in

- V.M. Millionščikov.
  Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients.
  Differ. Uravn., 4 (3): 391–396, 1968.
- V.M. Millionščikov.
  Proof of the existence of irregular systems of linear differential equations with quasi periodic coefficients.
  Differ. Uravn., 5 (11): 1979–1983, 1969.
- R.E. Vinograd.
  A problem suggested by N.P. Erugin.
  Differ. Uravn.,11 (4): 632–638, 1975.

Notice that these results were obtained much before than the notion and term SNA was coined.

The current situation

Looking at the relevant literature one sees that

- The notion of SNA is neither unified nor precisely formulated
- The existence of SNA, usually, is not proved rigorously. Some authors just give very rough/rude numerical evidences of their existence that easily can turn out to be wrong.
- The theoretical tools to study these objects and derive these consequences, are often used in a wrong way.

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On the positive side there are few works where the existence of an SNA is rigorously proved. Mainly:

[BO] Z. I. Bezhaeva and V. I. Oseledets.
On an example of a "strange nonchaotic attractor".
Funktsional. Anal. i Prilozhen., 30(4):1–9, 95, 1996.

[Kel] G. Keller.
A note on strange nonchaotic attractors.
Fund. Math., 151(2):139–148, 1996.

Aims

- propose a rigorous definition for the notion of SNA in the topological setting.
- discuss some methodological aspects relative to the non-chaoticity part of the definition.

To do this we will discuss a paradigmatic example that, in some sense, generalises the [GOPY] model.

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A paradigmatic example: [Kel]

(3) $$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= f(x_n)g(\theta_n) \end{cases}$$

where $x \in \mathbb{R}^+, \theta \in \mathbb{S}^1$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and

1. $f \colon [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, strictly increasing, strictly concave and verifies $f(0) = 0$ (to fix ideas take $f(x) = \tanh(x)$ as in the [GOPY] model).
   Thus, $x = 0$ will be invariant.
2. $g \colon \mathbb{S}^1 \longrightarrow [0, \infty)$ is bounded and continuous (to fix ideas take $g(\theta) = B|\cos(2\pi\theta)|$ with $B > 2$ in a similar way to the [GOPY] model – except for the absolute value).

We also define

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g(\theta) d\theta\right) < \infty.$$

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The vertical Lyapunov exponent of Keller's model

It is given by

$$\lambda_v(\theta_0, x_0) := \lim_{n\to\infty} \frac{1}{n} \log \left\| \begin{pmatrix} \frac{\partial \theta_n}{\partial \theta} & \frac{\partial \theta_n}{\partial x} \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|$$

whenever this limit exists.
By Oseledec's theorem, the existence of this limit is assured for almost every point in the support of any invariant measure. Moreover, it can be easily proved that the other Lyapunov exponent is zero.

Keller's Theorem

There exists an upper semicontinuous map $\phi\colon \mathbb{S}^1 \longrightarrow [0,\infty)$ whose graph is invariant under the Model (3). Moreover,

1. $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |x_k - \phi(\theta_k)| = 0$ for any *generic point* (by a "generic point" we mean *Lebesgue almost every $\theta \in \mathbb{S}^1$ and every $x > 0$*). In particular, the lifting of the Lebesgue measure on $\mathbb{S}^1$ to the graph of ϕ is a Bowen-Ruelle-Sinai measure.
2. If $\sigma > 1$, then $\lambda(\theta, x) < 0$ for generic points. Thus, both Lyapunov exponents are nonpositive for generic points.
3. The set $\{\theta : \phi(\theta) > 0\}$ has full Lebesgue measure. On the other hand, if there exists $\hat{\theta} \in \mathbb{S}^1$ such that $g(\hat{\theta}) = 0$, then the set $\{\theta : \phi(\theta) = 0\}$ is meager and ϕ is discontinuous for Lebesgue almost all $\theta \in \mathbb{S}^1$.
4. Whenever $\sigma \neq 0$, $|x_n - \phi(\theta_n)|$ converges to zero exponentially fast for generic points, as n tends to infinity.

Consequences of Keller's Theorem

We have rigorously seen that:

- The graph of ϕ *attracts* generic points by 4.
- The graph of ϕ is *strange* by 3.
- When $\sigma > 1$, Model (3) is *non-chaotic* by 2.

Our next step is to formulate a rigorous definition that captures this situation and the other related ones such as the [GOPY] model.

However, some remarks have to be done first.

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Remarks

A– The word *strange* in this theory is used in a different way that it is *often* used in the “world” of chaotic attractors. It refers to *strange geometry*.

B– Since the SNA intersects the line $x = 0$ at a dense pinched set and and this line is also invariant (it is a repellor), it can be seen that the basin of attraction of the attractor does not contain an open set.

C– The map ϕ is upper semicontinuous. Hence, its graph is not closed. So, the choice is either to take a non-closed attractor or to take the closure of the graph of ϕ as the attractor that will contain a repellor (the line $x = 0$). This is the alternative we have chosen.

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The notion of attractor

We use the definition of attractor proposed by Milnor in

J. Milnor.
On the concept of attractor.
Comm. Math. Phys., 99(2):177–195, 1985.
(Erratum: Comm. Math. Phys, **102(3)** (1985), 517–519).

The notion of attractor–II

Let (M, f) be a dynamical system where M is a smooth compact manifold endowed with a measure μ equivalent to the Lebesgue one when it is restricted to any coordinate neighbourhood. A closed subset $\mathcal{A} \subset M$ will be called an *attractor* if it satisfies the following two conditions:

(I) The set $\rho(\mathcal{A}) := \{x : \omega(x) \subset A\}$ has positive Lebesgue measure;

(II) there is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ so that $\rho(\mathcal{A}')$ coincides with $\rho(\mathcal{A})$ (up to sets of measure zero).

The notion of attractor–III

The set $\rho(\mathcal{A})$ is called the *realm of attraction* of $\mathcal{A}$, and it can be defined for every subset of M. When it is open, it is called the *basin of attraction of $\mathcal{A}$*.

If the space M contains a compact set N with positive measure such that $f(N) \subset N$ then there exists at least an attractor in $N \subset M$.

The notion of attractor–IV

Notice that an attractor $\mathcal{A}$ may contain a smaller attractor $\mathcal{A}' \subsetneq \mathcal{A}$ as long as $\rho(\mathcal{A})$ and $\rho(\mathcal{A}')$ differ in a set of positive measure. The attractors for which this condition is not satisfied, i.e. do not have a proper smaller attractor inside them, are called *minimal attractors*. These are the attractors that we are going to consider. A subset $\mathcal{A}$ of *M* is a *minimal attractor* if it satisfies the condition (I) from the definition of attractor and

(II') There is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ such that $\mu(\rho(\mathcal{A}'))$ is positive.

Towards the definition of strangeness

It has been proved by

J. Stark.
Invariant graphs for forced systems.
Phys. D, 109(1-2):163–179, 1997.
Physics and dynamics between chaos, order, and noise (Berlin, 1996).

that the invariant curves of models of type

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases}$$

are the graph of a correspondence from $\mathbb{S}^1$ to $\mathbb{R}$.

The different definitions of strangeness used in the literature

An attractor which is the graph of a correspondence is called strange when

(A) it is not a finite set of points neither piecewise differentiable.

(B) it has fractal geometry. That is its Hausdorff dimension is greater than its topological one.

(C) its Hausdorff dimension is greater than one.

The three definitions above are used in articles where two-dimensional systems are studied, while for higher-dimensional systems only the first definition is used.

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A remark in the two dimensional case

Using elemental dimension theory one can prove that the Hausdorff dimension of the graph of a one-dimensional piecewise differentiable map from $\mathbb{S}^1$ to $\mathbb{R}$ is one.
Therefore, in the two dimensional case the definition (A) above is the most general one.
This justifies the choice of the following

The notion of strangeness

An attractor is called *strange* when it is not a finite set of points neither a piecewise differentiable manifold.
A manifold M is *piecewise differentiable* if there exists a finite set of disjoint differentiable submanifolds $A_1, \ldots, A_k$ such that

$$M \subset \mathrm{Cl}(\cup_{i=1}^{k} A_i).$$

If M has boundary, then it must be piecewise differentiable too.

On the notion of non-chaoticity
A usual argument in the literature

Recall that

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases}$$

The usual procedure is to compute numerically the vertical Lyapunov exponent:

$$\lambda_v(\theta_0, x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|$$

and call the system *non-chaotic* if $\lambda_v(\theta_0, x_0) < 0$.

Theoretical explanation (in dimension 2 to simplify) — Oseledec's Theorem

Assume that μ is an ergodic measure of the system. Then, according to the Oseledec's Theorem, μ almost every point (θ_0, x_0) is *regular*. A regular point verifies the following properties:

(R1) the above limit exists and takes a value $\lambda_v = \lambda_v(\theta_0, x_0)$; which it is independent on the choice of the point.

(R2) for all $v \in T_{(\theta_0,x_0)}\mathbb{S}^1 \times \mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|M(\theta_n, x_n)v\| \quad \text{where} \quad M(\theta_n, x_n) = \begin{pmatrix} \frac{\partial \theta_n}{\partial \theta} & \frac{\partial \theta_n}{\partial x} \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix}$$

takes at most two different values λ_v and another one (which may coincide with λ_v), that we will denote by $\widehat{\lambda}$.

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(R3)

$$\begin{aligned}
\lambda_v + \widehat{\lambda} &= \lim_{n\to\infty} \frac{1}{n} \log \det |M(\theta_n, x_n)| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \det \left| \begin{pmatrix} 1 & 0 \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| = \lambda_v
\end{aligned}$$

Consequently, $\widehat{\lambda} = 0$ and no Lyapunov exponent is positive μ-a.e. if and only if the vertical Lyapunov exponent is not positive μ-a.e..

Some problems — Some complains

- Usually the vertical Lyapunov exponent is computed numerically in a very rude way and with a serious lack of precision. Thus the nonpositivity of λ_v is not guaranteed.
- Almost never the convergence of the limit is justified and the invariant ergodic measure that is used is not specified. Thus, if the considered initial point is not regular the following problems should be dealt with:

(P1) the limit may not exist (see the next very simple example)

(P2) λ_v need not be independent on the initial point, and

(P3) the formula in (R3) need not hold. Consequently, it well may happen that $\widehat{\lambda}(\theta, x) > 0$.

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(P1) the limit may not exist (see the next very simple example)

(P2) λ_v need not be independent on the initial point, and

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A very simple example on the non existence of lim in Lyapunov exponents

Consider

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = \tau(x_n) + \varepsilon \cos(2\pi\theta_n); \end{cases}$$

where $\tau(x)$ is a tent-like map:

$$\tau(x) = \begin{cases} \alpha(x - \frac{1}{2}) + 1 & \text{if } 0 \le x \le \frac{1}{2} \\ -\beta(x - \frac{1}{2}) + 1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

with $0 \le \alpha, \beta \le 2$.

Computing the vertical Lyapunov exponent

Then,

$$\frac{1}{n}\log\left|\frac{\partial x_n}{\partial x}\right| = \frac{1}{n}\log\left|\frac{\partial\tau^n(x_0)}{\partial x}\right| = \frac{1}{n}\log\left(\alpha^{n_1}\cdot\beta^{n_2}\right)$$

where $n_1 + n_2 = n$ and n_1 (respectively n_2) is the number of times that the orbit $x_0, x_1, \ldots, x_{n-1}$ visits the the interval $[0, \frac{1}{2})$ (respectively $(\frac{1}{2}, 1]$).
Clearly,

$$\frac{1}{n}\log\left(\alpha^{n_1}\cdot\beta^{n_2}\right) = \frac{n_1\log(\alpha) + n_2\log(\beta)}{n_1 + n_2}.$$

By using elementary symbolic dynamics, we see that there exits an infinite set of points (with Lebesgue measure zero) so that the above sequence has no limit (even it can have the interval formed with endpoints $\log(\alpha)$ and $\log(\beta)$ as the set of accumulation points).

Jager's approach to the definition of non-chaoticity

Another approach to the definition of non-chaoticity is to consider the dynamical system in dimension one restricted to the invariant (non-closed) attracting graph.

Then the original system is called *non-chaotic* if the unique Lyapunov exponent of this reduced system is non positive.

This argument can be made rigorous by means of the Birkhoff Ergodic Theorem since the dynamics on the attractor is driven by $\theta_{n+1} = \theta_n + \omega \pmod 1$, which is uniquely ergodic with the unique ergodic measure being the Lebesgue measure.

A first approach to the notion of non-chaoticity – I

- In our opinion, none of the previous approaches is satisfactory because the non-chaoticity condition should be observable (positive Lebesgue measure).
- In the first approach, the “non-chaotic” points are the regular ones which “live” in the support of invariant measures which are not absolutely continuous with respect to the Lebesgue one. Thus, being not observable in the above sense.
- In the second approach the “non-chaotic” points are almost all points in the invariant (non-closed) attracting graph. The situation is similar.

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A first approach to the notion of non-chaoticity – II

- Finally, if we forget about regularity then we have to deal correctly with the problems (P1), (P2), (P3) pointed out before. The solution to the problem:

(P1) is to consider $\limsup$ instead of $\lim$ (see the previous example).

(P2) is to estimate the Lyapunov exponents for all relevant points; since now no point can be chosen as a representative.

(P3) is to compute the maximal Lyapunov exponent (see the next example).

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And finally: the notion of non-chaoticity

An attractor $\mathcal{A}$ is *non-chaotic* if the set of points in its realm of attraction $\rho(\mathcal{A})$, whose maximal upper Lyapunov exponent

$$\lambda_{\max}(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\|$$

is positive, has zero Lebesgue measure.

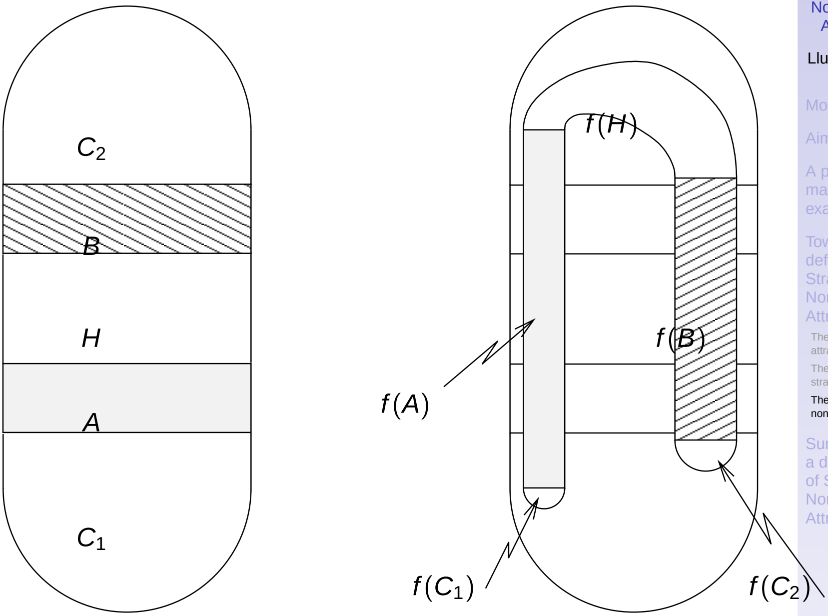
An example from de la Llave

We have seen before that the problems (P1), (P2) pointed out before cannot be avoided. The main question is whether the argument presented in (R3) works for non regular points. Although this is not known to us in the case of quasiperiodically forced skew products there is the following nice example of de la Llave that suggests that the definition that we gave, in general, cannot be simplified.
Consider an asymmetric horseshoe $\mathcal{C}^\infty$ diffeomorphism f as the one shown in the next picture ($N = C_1 \cup C_2 \cup A \cup B \cup H$ denotes the whole disc). To fix ideas we assume that

$$Df\big|_A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 24 \end{pmatrix} \quad \text{and} \quad Df\big|_B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -12 \end{pmatrix}.$$

Observe that both matrices have determinant 6.

The asymmetric horseshoe



The set of bad points

It is well known that the invariant set for f is given by

$$\Lambda := \{x \in A \cup B \ : \ x_i := f^i(x) \in A \cup B \ \forall i \in \mathbb{Z}\}.$$

Denote by $\widetilde{\Lambda}$ the subset of Λ consisting of all points whose itinerary has the form

$$(4) \qquad \cdots A^{n_1} B^{n_3-n_2} A^{n_4-n_3} B^{n_5-n_4} \cdots A^{n_{2k+1}-n_{2k}} B^{n_{2k+2}-n_{2k+1}} \cdots$$

where $n_k = \lfloor \rho^k \rfloor$ with $\rho > 1$. That is, if we set $n_0 = 0$ and we take $n \in [n_k, n_{k+1} - 1]$ for some k then

$$f^n(x) \in \begin{cases} A & \text{if } k \text{ is even,} \\ B & \text{if } k \text{ is odd.} \end{cases}$$

Clearly, $f(\tilde{\Lambda}) \subset \tilde{\Lambda}$.

Computing Lyapunov exponents

Simple Lemma

For every point $x \in \tilde{\Lambda}$ it follows that

1. $\limsup_{n\to\infty} \frac{1}{n} \log \left\| Df^n x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \log(\frac{1}{2})$,
2. $\limsup_{n\to\infty} \frac{1}{n} \log \left\| Df^n x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \log(24)$,
3. $\limsup_{n\to\infty} \frac{1}{n} \log |\det(Df^n x)| = \lim_{n\to\infty} \frac{1}{n} \log |\det(Df^n x)| = \log(6)$.

In particular, the "determinant formula" (R3) does not hold in this case.

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In particular, the "determinant formula" (R3) does not hold in this case.

Summarising: a definition of Strange Nonchaotic Attractor

Summarising, a Strange Nonchaotic Attractor is a closed set $\mathcal{A}$ such that

1. is an *attractor* in the sense of Milnor: its *realm of attraction* $\rho(\mathcal{A}) := \{x : \omega(x) \subset A\}$ has positive Lebesgue measure and there is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ such that $\mu(\rho(\mathcal{A}'))$ is positive.
2. is *strange*: it is not a finite set of points neither a piecewise differentiable manifold.
3. is *non-chaotic*: the set of points in its realm of attraction $\rho(\mathcal{A})$ whose maximal upper Lyapunov exponent $\lambda_{\max}(x)$ is positive, has zero Lebesgue measure.

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Is the definition useful?

With the above definition the model [GOPY] and the class of models studied by Keller [Kel] are SNA.

Some open problems in SNA

- Fractalization as a route to SNA
- Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems.
- May SNA coexist in models with more complicate base maps?

Fractalization as a route to SNA

There exists a so called fractalization route to SNA described for instance for the model (see Heagy and Hammel for a similar model and Prasad, Negi and Ramaswamy for a description of the different routes to SNA described)

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = \alpha x_n(1 - x_n) + \varepsilon \cos(2\pi\theta_n) \end{cases}$$

[NK] T. Nishikawa, K. Kaneko.
Fractalization of torus as a strange nonchaotic attractor.
Phys. Rev. E, **56(6)** (1997), 6114–6124.

Fractalization as a route to SNA

Recently, careful numerical studies have shown that there are serious flaws on the computations of the original papers describing the fractalization route to SNA. What seemed to be a fractal curve now seems to be a $\mathcal{C}^{\infty}$ curve!!

[HS] A. Haro, C. Simó.
To be or not to be a SNA: That is the question.
preprint, 2005.

Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems

In [FJJK] it is studied the coexistence of periodic *pinched strips* as a generalisation of the Sharkovskii Theorem that studies the coexistence of periodic orbits for interval maps.

[FJJK] R. Fabbri, T. Jäger, R. Johnson, and G. Keller.
A Sharkovskii-type theorem for minimally forced interval maps.
Topological Methods in Nonlinear Analysis, **26** (2005) 163– 188.

Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems

The aim of this problem is to look deeply to the dynamical structure of these objects. Instead of just looking at the period we want to look at the combinatorial structure ("permutation") of the whole orbit of the periodic pinched strip and derive dynamical consequences from it. Since we are using more information than just the period we will definitely obtain more information on the *forced* dynamics. Namely we aim at

- Study the forcing relation of the orbit of pinched strips,
- Perhaps, construct models with minimal dynamics (fixed a given combinatorial data of an orbit of pinched strips),
- Obtain lower bounds of the topological entropy of the system.

May SNA coexist in models with more complicate base maps?

Consider a model of the type

$$\begin{cases} \theta_{n+1} & = \varphi(\theta_n), \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases} \tag{5}$$

where φ is a continuous circle map of degree one with *nondegenerate* rotation interval.
The question we are interested in is whether may coexist (of course simultaneously) different SNA's associated to the Birkhoff orbits of φ with different irrational rotational number.

Remark

Each of these orbits is semiconjugate — and plays the same role of — a single orbit of the rigid rotation $\theta_{n+1} = \theta_n + \omega$ (mod 1) in the usual models.