THE CENTRAL LIMIT THEOREM FOR INNER FUNCTIONS II

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Abstract. A sharp version of the Central Limit Theorem for linear combinations of iterates of an inner function is proved. The authors previously showed this result assuming a suboptimal condition on the coefficients of the linear combination. Here we explain a variation of the argument which leads to the sharp result. We also review the steps of the proof as well as the main technical tool which is Aleksandrov Disintegration Theorem of Aleksandrov-Clark measures.

1. Introduction and main results

Let $\mathbb{D}$ be the unit disc in the complex plane and let m be the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$. An analytic mapping from $\mathbb{D}$ into $\mathbb{D}$ is called inner if its radial limits have modulus one at almost every point of the unit circle. Hence any inner function f induces a map defined at almost every point $z \in \partial\mathbb{D}$ as $f(z) = \lim_{r\to 1^-} f(rz)$. Let $f^n = f \circ \ldots \circ f : \partial\mathbb{D} \to \partial\mathbb{D}$ denote the n-th iterate of the inner function f. It has been recently shown that the iterates $\{f^n\}$ behave as a sequence of independent random variables in the sense that they obey appropriate versions of classical results on sequences of independent random variables (see [Nic22] and [NS22]). There are of course other situations in Complex Analysis where one encounters probabilistic behaviors in settings where the notion of independence is not directly present. A classical example of such setting is the assymptotic behaviour of lacunary series. In a series of classical papers by Paley and Zygmund, Salem and Zygmund and Weiss, the authors consider versions of the Khintchine-Kolmogorov Theorem for pointwise convergence (see [Zyg88, Section V.6]), versions of the Central Limit Theorem ([SZ47] and [SZ48]) and of the Law of the Iterated Logarithm ([Wei59]), for lacunary series. Our work is inspired by the Central Limit Theorem for lacunary series proved by Salem and Zygmund.

Recall that a sequence of measurable functions $\{f_N\}$ defined at almost every point in the unit circle converges in distribution to a (circularly symmetric) standard complex normal variable if for any Borel set $K \subset \mathbb{C}$ such that its boundary ∂K has zero area, one has

$$\lim_{N\to\infty} m\left(\{z \in \partial\mathbb{D}\colon f_N(z) \in K\}\right) = \frac{1}{2\pi}\int_K e^{-|w|^2/2}\, dA(w).$$

Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers. A version of the Central Limit Theorem for linear combinations $\sum a_n f^n$ of iterates has been given in [NS22] under certain conditions on the size of the coefficients $\{a_n\}$. The main purpose of this paper is to present the following version of this result which holds under the minimal assumption on the size of the coefficients $\{a_n\}$.

Theorem 1. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers with*

$$\sum_{n=1}^{\infty} |a_n|^2 = \infty. \tag{1.1}$$

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Consider

$$\sigma_N^2 = \sum_{n=1}^{N} |a_n|^2 + 2\,\mathrm{Re} \sum_{k=1}^{N-1} f'(0)^k \sum_{n=1}^{N-k} \overline{a_n} a_{n+k}, \quad N = 1, 2, \dots \tag{1.2}$$

Assume that the coefficients $\{a_n\}$ *satisfy that*

$$\lim_{N\to\infty} \frac{|a_N|^2}{\sum_{n=1}^{N} |a_n|^2} = 0. \tag{1.3}$$

Then

$$\frac{\sqrt{2}}{\sigma_N} \sum_{n=1}^{N} a_n f^n$$

converges in distribution to a standard complex normal variable.

Other versions of the Central Limit Theorem in this context have been given in [IU23] and [AN23].

We will use an argument due to Salem and Zygmund to show that condition (1.3) is optimal in the sense that for any sequence $\{a_n\}$ with $\sum |a_n|^2 = \infty$ and

$$\limsup_{N\to\infty} \frac{|a_N|^2}{\sum_{n=1}^{N} |a_n|^2} > 0,$$

the conclusion cannot hold. For $1 \leqslant p < \infty$ let $\|g\|_p$ denote the L^p norm of the function g on the unit circle defined by

$$\|g\|_p^p = \int_{\partial\mathbb{D}} |g|^p dm.$$

Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers. It was proved in [NS22] that

$$\frac{1 - |f'(0)|}{1 + |f'(0)|} \sum_{n=1}^{N} |a_n|^2 \leqslant \sigma_N^2 = \left\| \sum_{n=1}^{N} a_n f^n \right\|_2^2 \leqslant \frac{1 + |f'(0)|}{1 - |f'(0)|} \sum_{n=1}^{N} |a_n|^2, \quad N = 1, 2, \dots \tag{1.4}$$

Hence the series $\sum a_n f^n$ converges in $\mathrm{L}^2(\partial\mathbb{D})$ if and only if $\sum |a_n|^2 < \infty$. Moreover if this last condition holds, then the series $\sum a_n f^n(z)$ converges at almost every point $z \in \partial\mathbb{D}$ (see [Nic22]). In this context, repeating the proof of Theorem 1, one can show that when one has pointwise convergence, the tails obey a Central Limit Theorem.

Theorem 2. *Let* f *be an inner function with* $f(0) = 0$ *which is not a rotation. Let* $\{a_n\}$ *be a sequence of complex numbers with* $\sum |a_n|^2 < \infty$. *Consider*

$$\sigma^2(N) = \sum_{n\geqslant N} |a_n|^2 + 2\,\mathrm{Re} \sum_{k\geqslant 1} f'(0)^k \sum_{n\geqslant N} \overline{a_n} a_{n+k}, \quad N = 1, 2, \dots$$

Assume that the coefficients $\{a_n\}$ *satisfy that*

$$\lim_{N\to\infty} \frac{|a_N|^2}{\sum_{n\geqslant N} |a_n|^2} = 0. \tag{1.5}$$

Then

$$\frac{\sqrt{2}}{\sigma(N)} \sum_{n=N}^{\infty} a_n f^n$$

converges in distribution to a standard complex normal variable.

Given a set $\mathcal{A}$ of positive integers, consider the corresponding partial sum

$$\xi(\mathcal{A}) = \sum_{n\in\mathcal{A}} a_n f^n.$$

The proof of Theorem 1 contains two main ideas. The first one is a convenient splitting of the partial sums

$$\sum_{n=1}^{N} a_n f^n = \sum_k (\xi(\mathcal{A}_k) + \xi(\mathcal{B}_k))$$

into certain alternating blocks of consecutive terms $\xi(\mathcal{A}_k)$ and $\xi(\mathcal{B}_k)$ which depend on N, that is, satisfying $\max \mathcal{A}_k + 1 = \min \mathcal{B}_k$ and $\max \mathcal{B}_k + 1 = \min \mathcal{A}_{k+1}$, which obey two counteracting properties. On the one hand $\| \sum \xi(\mathcal{B}_k)\|_2/\sigma_N$ must be small so that $\sum \xi(\mathcal{B}_k)$ becomes irrelevant. On the other hand the number of terms of each block $\xi(\mathcal{B}_k)$ must be large so that the correlations between different blocks $\xi(\mathcal{A}_k)$ decay sufficiently fast. The construction of these blocks is inspired by a similar construction of Weiss in [Wei59]. The second main idea in the proof was already present in [NS22] and concerns the decay of certain correlations which naturally appear when proving a Central Limit Theorem. We mention two main properties. If $\mathcal{A}$ and $\mathcal{B}$ are two sets of positive integers such that $a < b$ for any $a \in \mathcal{A}$ and $b \in B$, it turns out that $|\xi(\mathcal{A})|^2$ and $|\xi(\mathcal{B})|^2$ are uncorrelated, that is,

$$\int_{\partial\mathbb{D}} |\xi(\mathcal{A})|^2 |\xi(\mathcal{B})|^2 \, dm = \left(\int_{\partial\mathbb{D}} |\xi(\mathcal{A})|^2 \, dm \right) \left(\int_{\partial\mathbb{D}} |\xi(\mathcal{B})|^2 \, dm \right) \tag{1.6}$$

(see Theorem 6 in Section 3). The second property provides an exponential decay of the higher order correlations of the iterates. More concretely, let $\varepsilon_i = 1$ or $\varepsilon_i = -1$ for $i = 1, 2, \ldots, k$ and $n_1 < \ldots < n_k$ be positive integers satisfying $n_j - n_{j-1} \geqslant q \geqslant 1$, $j = 2, \ldots, k$. Denote $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ and $\boldsymbol{n} = (n_1, \ldots, n_k)$. For a positive integer n, denote by f^{-n} the function defined by $f^{-n}(z) = \overline{f^n(z)}$, $z \in \partial\mathbb{D}$. It was shown in [NS22] that

$$\left| \int_{\partial\mathbb{D}} \prod_{j=1}^{k} f^{\varepsilon_j n_j} \, dm \right| \leqslant C^k k! |f'(0)|^{\Phi(\boldsymbol{\varepsilon}, \boldsymbol{n})}, \quad k = 1, 2, \ldots, \tag{1.7}$$

if q is sufficiently large and where Φ is a certain function depending on the choice of indices that satisfies $\Phi(\boldsymbol{\varepsilon}, \boldsymbol{n}) \geqslant kq/4$ and that is well suited for summing over the indices afterwards (see Theorem 10 in Section 4). The main technical tool in the proof of both properties (1.6) and (1.7) is the theory of Aleksandrov-Clark measures and more concretely, the Aleksandrov Disintegration Theorem.

We have tried to make this paper self-contained. We use some auxiliary results from [NS22] but, when possible, we have provided simple proofs. This paper is structured as follows. In Section 2 we show that the growth condition (1.3) on the coefficients cannot be improved. Section 3 contains a brief exposition on Aleksandrov-Clark measures and their application in the proof of Theorem 1. In Section 4 we state some auxiliary results. Finally, we prove Theorem 1 in Section 5

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2. Optimality of the assumption on the coefficients

We now prove the optimality of condition (1.3) using an argument due to Salem and Zygmund [SZ47]. The main idea is contained in the following result, from which we will deduce the optimal nature of condition (1.3).

Proposition 3. *Le $\{g_n\}$ be a sequence of functions in the unit circle with $|g_n(z)| \leqslant 1$ for every $n \geqslant 1$ and for almost every $z \in \partial\mathbb{D}$. Let $\{a_n\}$ be a sequence of complex numbers that is not square summable and define*

$$S_N^2 = \sum_{n=1}^{N} |a_n|^2.$$

Denote

$$G_N(z) = \sum_{n=1}^{N} a_n g_n(z), \qquad z \in \partial\mathbb{D}.$$

Assume that the sequence $\{G_N/S_N\}$ *satisfies the uniform decay*

$$m\left(\left\{z \in \partial\mathbb{D}\colon \left|\frac{G_N(z)}{S_N}\right| > r\right\}\right) \leqslant \varphi(r), \qquad \text{for all } r > 0, N \geqslant 1, \tag{2.1}$$

where $\varphi(r)$ *is a positive decreasing function with* $\lim_{r\to\infty} \varphi(r) = 0$. *Assume that*

$$\limsup_{N\to\infty} \frac{|a_N|^2}{S_N^2} > 0. \tag{2.2}$$

Then, there exists $r_0 > 0$ *such that*

$$\lim_{N\to\infty} m\left(\left\{z \in \partial\mathbb{D}\colon \left|\frac{G_N(z)}{S_N}\right| \leqslant r\right\}\right) = 1 \tag{2.3}$$

for all $r \geqslant r_0$.

Proof. Given $r > 0$ and $N \geqslant 1$, denote

$$A_N(r) = \left\{z \in \partial\mathbb{D}\colon \left|\frac{G_N(z)}{S_N}\right| \leqslant r\right\}$$

and define the functions

$$P_N(r) = m\left(\{A_N(r)\}\right), \qquad P(r) = \liminf_{N\to\infty} P_N(r).$$

From the definitions, it is clear that P_N are nondecreasing functions and, hence, so is function P. Moreover, for each $N \geqslant 1$ we have that $\lim_{r\to\infty} P_N(r) = 1$. Furthermore, condition (2.1) implies that also $\lim_{r\to\infty} P(r) = 1$. Indeed, the bound (2.1) for a given $r > 0$ is equivalent to

$$m\left(\{A_N(r)\}\right) \geqslant 1 - \varphi(r)$$

for all $N \geqslant 1$ and, in particular, $P(r) \geqslant 1 - \varphi(r)$. Therefore, the decay of $\varphi(r)$ ensures that $\lim_{r\to\infty} P(r) = 1$.

Note that (2.3) is the same as to say that $P(r) = 1$ for all $r \geqslant r_0$. Thus, assume that this is not the case, so $P(r) < 1$ for any finite r. Because of (2.2), there exists $\varepsilon > 0$ such that $|a_N|^2 > \varepsilon S_N^2$ for infinitely many values of N. For such values of N we get that

$$\frac{S_{N-1}^2}{S_N^2} = \frac{S_N^2 - |a_N|^2}{S_N^2} < 1 - \varepsilon.$$

Now write

$$\frac{G_N}{S_N} = \frac{G_{N-1}}{S_{N-1}}\frac{S_{N-1}}{S_N} + \frac{a_N g_N}{S_N}.$$

For $r > 0$, observe that if $z \in A_{N-1}(r)$, by the previous identity we have that $z \in A_N(r\sqrt{1-\varepsilon} + 1)$. In particular, by the definition of P, taking the limit in N we find that $P(r) \leqslant P(r\sqrt{1-\varepsilon} + 1)$. For $r \geqslant 1/(1-\sqrt{1-\varepsilon})$, it holds that $r\sqrt{1-\varepsilon} + 1 \leqslant r$ and, since P is nondecreasing, we actually have that $P(r) = P(r\sqrt{1-\varepsilon} + 1)$.

Pick $r_0 > 1/(1-\sqrt{1-\varepsilon})$ large enough so that $s_0 = r_0\sqrt{1-\varepsilon} + 1 < (1 - \varepsilon/4)r_0$. By the previous argument and because P is nondecreasing, we find that $P(r) = P(r_0) < 1$ for all $s_0 \leqslant r \leqslant r_0$, that is P is constant in the interval $[s_0, r_0]$ (see Figure 1). Next, pick $s_1 \in [r_0(1-\varepsilon/4), r_0]$ and let

$$r_1 = \frac{s_1 - 1}{\sqrt{1-\varepsilon}}.$$

Note that by our choice of r_0 and s_1, there exists a constant $C > 1$ such that $r_1 \geqslant Cr_0$. Repeating the previous argument and using that $P(s_1) = P(r_0)$, we get that $P(r) = P(r_0) < 1$ for all $s_0 \leqslant r \leqslant r_1$. Iterating this procedure, we see that for any $r \geqslant s_0$ it holds that $P(r) = P(r_0) < 1$. However, this contradicts the fact that $\lim_{r\to\infty} P(r) = 1$, which must hold because of (2.1), and thus concludes the proof. □

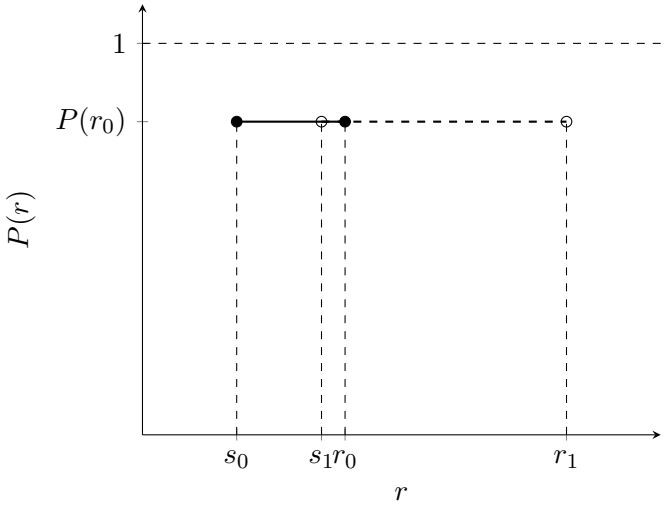

FIGURE 1. First, we choose points $r_0 > 1/(1-\sqrt{1-\varepsilon})$ and $s_0 = r_0\sqrt{1-\varepsilon}+1$. Thus, P is constant and equal to $P(r_0)$ in the interval $[s_0, r_0]$. Then, by choosing s_1 in $[s_0, r_0]$ we see that P is also equal to $P(r_0)$ in $[s_0, r_1]$, where $r_1 = (s_1 - 1)/\sqrt{1-\varepsilon}$.

We will now deduce that condition (1.3) is optimal.

Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers whose modulus is not square summable. Let σ_N be defined by (1.2). Assume that

$$\lim_{N\to\infty} \frac{|a_N|^2}{\sum_{n=1}^{N} |a_n|^2} > 0. \tag{2.4}$$

We will now show that

$$\frac{\sqrt{2}}{\sigma_N} \sum_{n=1}^{N} a_n f^n$$

does not converge in distribution to a standard complex normal variable.

Define S_N as before and denote

$$G_N(z) = \sum_{n=1}^{N} a_n f^n(z), \qquad z \in \partial\mathbb{D}.$$

Assume now that G_N/σ_N converges in distribution to a normal random variable. Estimate (1.4) asserts that there exists $C = C(f) \geqslant 1$ such that $C^{-1} \leqslant \sigma_N^2/S_N^2 \leqslant C$ (see also Theorem 8). Therefore, even though $\{G_N/S_N\}$ might not converge in distribution, the properties of the normal distribution and the boundedness of the ratio σ_N/S_N ensure that for every $r > 0$ exists $N_0 = N_0(r)$ such that

$$m\left(\left\{z \in \partial\mathbb{D}\colon \left|\frac{G_N(z)}{S_N}\right| \leqslant r\right\}\right) < 1$$

for all $N \geqslant N_0$. In particular,

$$\liminf_{N\to\infty} m\left(\left\{z \in \partial\mathbb{D}\colon \left|\frac{G_N(z)}{S_N}\right| \leqslant r\right\}\right) < 1 \tag{2.5}$$

for all finite r. Next, note that since $\sigma_N = \|G_N\|_2$, we have the uniform bound

$$\left\|\frac{G_N}{S_N}\right\|_2 \leqslant C^{1/2}.$$

Hence, by Chebyshev's inequality, the functions $\{G_N/S_N\}$ satisfy the uniform decay (2.1) with $\varphi(r) = C/r^2$. Thus, (2.5) and Proposition 3, provide a contradiction.

3. Aleksandrov-Clark measures

Given an analytic mapping f from the unit disc into itself (not necessarily inner) and a point $\alpha \in \partial\mathbb{D}$, the function $(\alpha + f)/(\alpha - f)$ has positive real part and hence there exists a positive measure $\mu_\alpha = \mu_\alpha(f)$ in the unit circle and a constant $C_\alpha \in \mathbb{R}$ such that

$$\frac{\alpha + f(w)}{\alpha - f(w)} = \int_{\partial\mathbb{D}} \frac{z+w}{z-w}\, d\mu_\alpha(z) + iC_\alpha, \quad w \in \mathbb{D}. \tag{3.1}$$

The measures $\{\mu_\alpha \colon \alpha \in \partial\mathbb{D}\}$ are called the Aleksandrov-Clark measures of the function f. Clark introduced them in his paper [Cla72] and many of their deepest properties were found by Aleksandrov in [Ale86], [Ale87] and [Ale89]. The two surveys [PS06] and [Sak07] as well as [CMR06, Chapter IX] contain their main properties and a wide range of applications. Observe that if $f(0) = 0$ then μ_α are probability measures. Moreover, f is inner if and only if μ_α is a singular measure for some (all) $\alpha \in \partial\mathbb{D}$. From the definition it is clear that, if f is an inner function, the mass of μ_α is carried by the set $f^{-1}(\{\alpha\}) \subset \partial\mathbb{D}$.

Assume $f(0) = 0$. Computing the first two derivatives in formula (3.1) and evaluating at the origin, we obtain

$$\int_{\partial\mathbb{D}} \overline{z}\, d\mu_\alpha(z) = f'(0)\overline{\alpha}, \quad \alpha \in \partial\mathbb{D}, \tag{3.2}$$

and

$$\int_{\partial\mathbb{D}} \overline{z}^2\, d\mu_\alpha(z) = \frac{f''(0)}{2}\overline{\alpha} + f'(0)^2\overline{\alpha}^2, \quad \alpha \in \partial\mathbb{D}.$$

More generally, if we expand both terms of identity (3.1) in power series, we get that for every positive integer l it holds that

$$\int_{\partial\mathbb{D}} \overline{z}^l\, d\mu_\alpha(z) = \sum_{k=1}^{l} \overline{\alpha}^k \int_{\partial\mathbb{D}} f(z)^k \overline{z}^l\, dm(z), \quad \alpha \in \partial\mathbb{D}.$$

Hence for any integer l, the l-th moment of μ_α is a trigonometric polynomial in the variable α of degree less than or equal to $|l|$. Moreover, the coefficients of this polynomials are given by derivatives of powers of f due to Cauchy's formula.

Our main technical tool is Aleksandrov Disintegration Theorem which asserts that

$$m = \int_{\partial\mathbb{D}} \mu_\alpha\, dm(\alpha) \tag{3.3}$$

holds true in the sense that

$$\int_{\partial\mathbb{D}} G\, dm = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} G(z)\, d\mu_\alpha(z)\, dm(\alpha)$$

for any integrable function G on the unit circle. In other words, for any given analytic self-mapping f of the unit disk, the Lebesgue measure is the average of the Aleksandrov-Clark measures $\{\mu_\alpha\}$ of f.

Before showing the main application of the Aleksandrov Disintegration Theorem in our context, we mention a basic fact of inner functions, which is just a restatement of Löwner's Lemma. Recall that Löwner's Lemma claims that if f is an inner function such that $f(0) = 0$, then $m\left(f^{-1}(E)\right) = m(E)$ for any measurable set $E \subseteq \partial\mathbb{D}$ (see for instance [DM91, Corollary 1.5]).

Lemma 4. *Let f be an inner function with $f(0) = 0$.*
(a) Let G be an integrable function on $\partial\mathbb{D}$. Then

$$\int_{\partial\mathbb{D}} G(f(z))\, dm(z) = \int_{\partial\mathbb{D}} G(z)\, dm(z).$$

(b) Let $k < j$ be positive integers. Then

$$\int_{\partial\mathbb{D}} \overline{f^k} f^j\, dm = f'(0)^{j-k}.$$

Proof of Lemma 4. Assume that G is the characteristic function of a measurable set $E \subset \partial\mathbb{D}$. Since $m(f^{-1}(E)) = m(E)$, the identity (a) follows. The result for general integrable functions holds because of density of linear combinations of characteristic functions. Using (a) and Cauchy formula, we have

$$\int_{\partial\mathbb{D}} \overline{f^k} f^j \, dm = \int_{\partial\mathbb{D}} \overline{z} f^{j-k}(z) \, dm(z) = f'(0)^{j-k},$$

where the last equality follows from the chain rule. □

When proving Theorem 1, we will need to estimate integrals of products of iterates of an inner function f. We will use Aleksandrov Disintegration Theorem to integrate factors of these products separately. The following lemma is a particular example of this procedure. However, the reader will easily see how to apply the same technique to more general products of iterates. The reason to choose this particular example is not only for clarity, but also because it will have a direct application in studying the L^2 norms of partial sums in Section 4.

Lemma 5. *Let f be an inner function with $f(0) = 0$. For $k = 1, 2, \dots, p$, let n_k, j_k, be positive integers such that*

$$\max\{n_k, j_k\} < \min\{n_{k+1}, j_{k+1}\}, \quad k = 1, \dots, p-1. \tag{3.4}$$

Then

$$\int_{\partial\mathbb{D}} \prod_{k=1}^{p} f^{n_k} \overline{f^{j_k}} dm = \prod_{k=1}^{p} \int_{\partial\mathbb{D}} f^{n_k} \overline{f^{j_k}} dm. \tag{3.5}$$

Proof of Lemma 5. We argue by induction on p. Assume (3.5) holds for $p-1$ products. We can assume $n_1 < j_1$. By part (a) of Lemma 4 we have

$$\int_{\partial\mathbb{D}} \prod_{k=1}^{p} f^{n_k} \overline{f^{j_k}} \, dm = \int_{\partial\mathbb{D}} z \overline{f^{j_1 - n_1}(z)} \prod_{k=2}^{p} f^{n_k - n_1}(z) \overline{f^{j_k - n_1}(z)} \, dm(z).$$

Let $\{\mu_\alpha \colon \alpha \in \partial\mathbb{D}\}$ be the Aleksandrov-Clark measures of the inner function $f^{j_1 - n_1}$. The Aleksandrov Disintegration Theorem (3.3) and the fact that the mass of μ_α is carried by the set $\left\{z \in \partial\mathbb{D} \colon f^{j_1 - n_1}(z) = \alpha\right\}$ gives that last integral can be written as

$$\int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} z\overline{\alpha} \prod_{k=2}^{p} f^{n_k - j_1}(\alpha) \overline{f^{j_k - j_1}(\alpha)} \, d\mu_\alpha(z) \, dm(\alpha).$$

By (3.2) and part (b) of Lemma 4, we have

$$\int_{\partial\mathbb{D}} z \, d\mu_\alpha(z) = \overline{f'(0)}^{j_1 - n_1} \alpha = \alpha \int_{\partial\mathbb{D}} f^{n_1} \overline{f^{j_1}} \, dm.$$

Hence

$$\int_{\partial\mathbb{D}} \prod_{k=1}^{p} f^{n_k} \overline{f^{j_k}} \, dm = \left(\int_{\partial\mathbb{D}} f^{n_1} \overline{f^{j_1}} \, dm \right) \int_{\partial\mathbb{D}} \prod_{k=2}^{p} f^{n_k - j_1} \overline{f^{j_k - j_1}} \, dm$$

and we can apply the inductive assumption. One more use of the invariance property of part (a) of Lemma 4 finishes the proof. □

Linear combinations of iterates of inner functions are not independent, but under suitable conditions their correlations decay fast enough, which is sufficient for many applications. The following theorem states a simple condition under which the moduli squared of linear combinations of iterates of an inner function are uncorrelated.

Theorem 6. *Let f be an inner function with $f(0) = 0$. Let $\mathcal{A}_k$, $k = 1, 2, \dots, p$, be finite collections of positive integers such that*

$$\max\{n \colon n \in \mathcal{A}_k\} < \min\{n \colon n \in \mathcal{A}_{k+1}\}, \quad k = 1, \dots, p-1. \tag{3.6}$$

Consider

$$\xi_k = \sum_{n\in\mathcal{A}_k} a_n f^n.$$

Then

$$\int_{\partial\mathbb{D}} \prod_{k=1}^{p} |\xi_k|^2 \, dm = \prod_{k=1}^{p} \int_{\partial\mathbb{D}} |\xi_k|^2 \, dm.$$

Proof. At almost every point of the unit circle we have

$$|\xi_k|^2 = \sum_{n\in\mathcal{A}_k} |a_n|^2 + \sum (\overline{a_n} a_j \overline{f^n} f^j + \overline{a_j} a_n \overline{f^j} f^n),$$

where the last sum is taken over all indices $n, j \in \mathcal{A}_k$ with $j > n$. Hence $\prod |\xi_k|^2$ can be written as a linear combination of terms of the form

$$\prod f^{n_k} \overline{f^{j_k}},$$

where $n_k, j_k \in \mathcal{A}_k$. Observe that (3.6) gives the assumption (3.4) in Lemma 5. Now Lemma 5 finishes the proof. □

4. Auxiliary results

This section is devoted to collect some results which will be used in the proof of Theorem 1.

Given an inner function f on the unit disk and a positive integer n, we denote $f^{-n} = \overline{f^n}$. By applying Lemma 5 several times we can estimate expressions of the form

$$\int_{\partial\mathbb{D}} f^{\varepsilon_1 n_1} f^{\varepsilon_2 n_2} \dots f^{\varepsilon_k n_k} \, dm,$$

where $n_1, \dots, n_k$ are positive integers and $\varepsilon_1, \dots, \varepsilon_k \in \{+1, -1\}$. The following lemma states the corresponding results for some particular configurations with 4 factors that will turn out to be useful later on.

Lemma 7. *Let f be an inner function with $f(0) = 0$ which is not a rotation. For $k = 1, 2, 3, 4$, let n_k be positive integers and $\varepsilon_k \in \{+1, -1\}$. Consider*

$$I = I(\varepsilon_1 n_1, \varepsilon_2 n_2, \varepsilon_3 n_3, \varepsilon_4 n_4) = \int_{\partial\mathbb{D}} f^{\varepsilon_1 n_1} f^{\varepsilon_2 n_2} f^{\varepsilon_3 n_3} f^{\varepsilon_4 n_4} \, dm.$$

(a) Assume $\varepsilon_1 = -\varepsilon_2$, $\varepsilon_3 = \varepsilon_4$ and $\max\{n_1, n_2\} < \min\{n_3, n_4\}$. Then $I = 0$.

(b) Assume $n_1 < n_2 < n_3 < n_4$. If $\varepsilon_1\varepsilon_2 = \varepsilon_3\varepsilon_4 = -1$ then we have $|I| = |f'(0)|^{n_2 - n_1 + n_4 - n_3}$. If $\varepsilon_1 = \varepsilon_2$ and $n_3 - n_2 \geqslant 3$, then there exists a constant $C = C(f) > 0$, independent of the indices n_1, n_2, n_3, n_4, such that $|I| \leqslant C|f'(0)|^{n_4 - n_1}$.

(c) Let $n_1 < n_2 < n_3$ be positive integers with $n_2 - n_1 \geqslant 3$. Then there exists a constant $C = C(f) > 0$, independent of the indices n_1, n_2, n_3, such that

$$\left| \int_{\partial\mathbb{D}} (f^{\varepsilon_1 n_1})^2 f^{\varepsilon_2 n_2} f^{\varepsilon_3 n_3} \, dm \right| \leqslant C|f'(0)|^{n_3 - n_1}.$$

Proof of Lemma 7. We can assume $n_1 < n_2$ and $\varepsilon_1 = 1$. Since Lebesgue measure is invariant under f we have

$$I = \int_{\partial\mathbb{D}} z f^{\varepsilon_2(n_2 - n_1)}(z) f^{\varepsilon_3(n_3 - n_1)}(z) f^{\varepsilon_4(n_4 - n_1)}(z) dm(z).$$

Let $\{\mu_\alpha \colon \alpha \in \partial\mathbb{D}\}$ be the Aleksandrov-Clark measures of the inner function $f^{n_2 - n_1}$. The Aleksandrov Disintegration Theorem (3.3) and the fact that the mass of μ_α is carried by the set $\{z \in \partial\mathbb{D} \colon f^{n_2 - n_1}(z) = \alpha\}$ give that

$$I = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} z \alpha^{\varepsilon_2} f^{\varepsilon_3(n_3 - n_2)}(\alpha) f^{\varepsilon_4(n_4 - n_2)}(\alpha) \, d\mu_\alpha(z) \, dm(\alpha).$$

By (3.2) and part (b) of Lemma 4, we have

$$\int_{\partial\mathbb{D}} z\, d\mu_\alpha(z) = \overline{f'(0)}^{n_2-n_1}\alpha.$$

We deduce that

$$I = \overline{f'(0)}^{n_2-n_1}\int_{\partial\mathbb{D}} \alpha^{1+\varepsilon_2} f^{\varepsilon_3(n_3-n_2)}(\alpha) f^{\varepsilon_4(n_4-n_2)}(\alpha)\, dm(\alpha).$$

If $\varepsilon_2 = -1$ and $\varepsilon_3 = \varepsilon_4$ last integral vanishes and we obtain the statement in (a). If $\varepsilon_2 = -1$ and $\varepsilon_3 = -\varepsilon_4$, the modulus of the last integral is $|f'(0)|^{n_4-n_3}$ and we deduce first part of (b). Assume now $\varepsilon_2 = 1$. Then

$$I = \overline{f'(0)}^{n_2-n_1}\int_{\partial\mathbb{D}} z^2 f^{\varepsilon_3(n_3-n_2)}(z) f^{\varepsilon_4(n_4-n_2)}(z)\, dm(z).$$

Let $\{\sigma_\alpha\colon \alpha \in \partial\mathbb{D}\}$ be the Aleksandrov-Clark measures of the inner function $f^{n_3-n_2}$. The Aleksandrov Disintegration Theorem (3.3) and the fact that the mass of σ_α is carried by the set $\{z \in \partial\mathbb{D}\colon f^{n_3-n_2}(z) = \alpha\}$ gives that

$$I = \overline{f'(0)}^{n_2-n_1}\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} z^2\alpha^{\varepsilon_3} f^{\varepsilon_4(n_4-n_3)}(\alpha)\, d\mu_\alpha(z)\, dm(\alpha).$$

By (3.2)

$$\int_{\partial\mathbb{D}} z^2 d\mu_\alpha(z) = \alpha^2\overline{f'(0)}^{n_3-n_2} + \alpha b,$$

and $|b| \leqslant C|f'(0)|^{n_3-n_2}$ because $n_3 - n_2 \geqslant 3$. We deduce that

$$|I| \leqslant C|f'(0)|^{n_3-n_1}\max\left\{\left|\int_{\partial\mathbb{D}} \alpha^l f^{\varepsilon_4(n_4-n_3)}(\alpha)\, dm(\alpha)\right|\right\},$$

where the maximum is taken over all positive integers l with $|l| \leqslant 3$. This gives the second part of statement (b). Similar arguments give part (c). □

The next statement collects convenient estimates of L^2 and L^4 norms of linear combinations of iterates of an inner function (see [NS22] for details). Here, given $u, v \in \mathbb{C}$, we denote by $\langle u, v\rangle = \mathrm{Re}(\overline{u}v)$ the standard scalar product of u and v when considered as elements of $\mathbb{R}^2$.

Theorem 8. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers with $\sum_n |a_n|^2 < \infty$. Consider*

$$\xi = \sum_{n=1}^{\infty} a_n f^n$$

and

$$\sigma^2 = \sum_{n=1}^{\infty}|a_n|^2 + 2\,\mathrm{Re}\sum_{k=1}^{\infty} f'(0)^k \sum_{n=1}^{\infty}\overline{a_n}a_{n+k}.$$

(a) We have $\|\xi\|_2^2 = \sigma^2$ and

$$\kappa^{-1}\sum_{n=1}^{\infty}|a_n|^2 \leqslant \sigma^2 \leqslant \kappa\sum_{n=1}^{\infty}|a_n|^2,$$

where $\kappa = (1 + |f'(0)|)(1 - |f'(0)|)^{-1}$.

(b) For any $t \in \mathbb{C}$ we have

$$\int_{\partial\mathbb{D}}\langle t, \xi\rangle^2\, dm = \frac{1}{2}|t|^2\sigma^2.$$

(c) There exists a constant $C = C(f) > 0$ independent of the sequence $\{a_n\}$, such that $\|\xi\|_4 \leqslant C\,\|\xi\|_2$.

In the proof of Theorem 1 we will need to consider partial sums of the series of iterates of an inner function and also partial sums restricted to certain subsets of indices $\{\mathcal{A}(k)\}$. This auxiliary result, proved in [NS22], will allow us to compare the L^2 norms of these partial sums.

Lemma 9. *Let $\{a_n\}$ be a sequence of complex numbers and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Consider the sequence*

$$\sigma_N^2 = \sum_{n=1}^{N} |a_n|^2 + 2\,\mathrm{Re} \sum_{k=1}^{N-1} \lambda^k \sum_{n=1}^{N-k} \overline{a_n} a_{n+k}, \quad N = 1, 2 \dots$$

For $N > 1$, let $\mathcal{A}(j) = \mathcal{A}(j, N)$, for $j = 1, \dots, M = M(N)$, be pairwise disjoint sets of consecutive positive integers smaller than N. Consider

$$\sigma^2(\mathcal{A}(j)) = \sum_{n \in \mathcal{A}(j)} |a_n|^2 + 2\,\mathrm{Re} \sum_{k \geqslant 1} \lambda^k \sum_{n \in \mathcal{A}(j)\colon\, n+k \in \mathcal{A}(j)} \overline{a_n} a_{n+k}, \quad j = 1, 2 \dots, M.$$

Let $\mathcal{A} = \cup \mathcal{A}(j)$. Assume

$$\lim_{N \to \infty} \frac{\sum_{n \in \mathcal{A}} |a_n|^2}{\sum_{n=1}^{N} |a_n|^2} = 1 \tag{4.1}$$

and

$$\lim_{j \to \infty} \frac{\max\{|a_n|^2 \colon n \in \mathcal{A}(j)\}}{\sum_{n \in \mathcal{A}(j)} |a_n|^2} = 0. \tag{4.2}$$

Then

$$\lim_{N \to \infty} \frac{\sum_{j=1}^{M} \sigma^2(\mathcal{A}(j))}{\sigma_N^2} = 1.$$

Another technical tool for the proof of Theorem 1 is the following general version of Lemma 7 (see [NS22]). It quantifies the correlation of iterates of an inner function when the number of iterations differ by a large amount. More concretely, if the minimum difference between the number of iterations on each iterate is at least q, large enough, then we have certain control over correlations of up to q different such iterates. Here we use the notation $f^{-n} = \overline{f^n}$.

Theorem 10. *Let f be an inner function with $f(0) = 0$ and $a = |f'(0)| < 1$. Let $1 \leqslant k \leqslant q$ be positive integers. Let $\boldsymbol{\varepsilon} = \{\varepsilon_j\}_{j=1}^k$ where $\varepsilon_j = 1$ or $\varepsilon_j = -1$, and let $\boldsymbol{n} = \{n_j\}_{j=1}^k$ where $n_1 < n_2 < \ldots < n_k$ are positive integers with $n_{j+1} - n_j > q$ for any $j = 1, 2, \dots, k-1$. Consider*

$$I(\boldsymbol{\varepsilon}, \boldsymbol{n}) = \left| \int_{\partial \mathbb{D}} \prod_{j=1}^{k} f^{\varepsilon_j n_j}\, dm \right|$$

Then there exist constants $C = C(f) > 0$, $q_0 = q_0(f) > 0$ independent of $\boldsymbol{\varepsilon}$ and of $\boldsymbol{n}$, such that if $q \geqslant q_0$ we have

$$I(\boldsymbol{\varepsilon}, \boldsymbol{n}) \leqslant C^k k! a^{\Phi(\boldsymbol{\varepsilon}, \boldsymbol{n})}, k = 1, 2, \dots,$$

where $\Phi(\boldsymbol{\varepsilon}, \boldsymbol{n}) = \sum_{j=1}^{k-1} \delta_j (n_{j+1} - n_j)$, with $\delta_j \in \{0, 1/2, 1\}$ for any $j = 1, \dots, k-1$, and with $\delta_1 = 1$ and $\delta_{k-1} \geqslant 1/2$. In addition, for $j = 2, \dots, k-1$ the coefficient $\delta_j = 1$ if and only if $\delta_{j-1} = 0$. Furthermore, if $\delta_{j-1} > 0$, the coefficient δ_j depends on $\varepsilon_{j+1}, \dots, \varepsilon_k$ and $n_j, \dots, n_k$ for $j = 2, \dots, k-1$.

The last auxiliary result of this section will be used to simplify the proof of Theorem 1. Its proof is an elementary application of Cauchy-Schwarz's inequality (see [NS22]).

Lemma 11. *Let $\{f_n\}$, $\{g_n\}$ be two sequences of measurable functions defined at almost every point of the unit circle. Assume that there exists a constant $C > 0$ such that the following conditions hold*

(a) $\sup_n \|f_n\|_2 \leqslant C$ *and*

$$\lim_{n \to \infty} \int_{\partial \mathbb{D}} f_n\, dm = 1$$

(b) $g_n(z) > -C$ *for almost every* $z \in \partial \mathbb{D}$ *and* $\lim_{n \to \infty} \|g_n\|_2 = 0$.

Then

$$\lim_{n\to\infty}\int_{\partial\mathbb{D}} f_n e^{-g_n}\,dm = 1.$$

5. Proof of the Central Limit Theorem

The proof of Theorem 1 relies on splitting the partial sum

$$\sum_{n=1}^{N} a_n f^n = \sum_k (\xi_k + \eta_k)$$

into certain blocks of consecutive terms which depend on N and that are of two alternating types: ξ_k and η_k. In other words, we will have a block η_k between blocks ξ_k and ξ_{k+1}. The main idea in this construction relies on balancing two counteracting properties. On the one hand $\|\sum \eta_k\|_2/\sigma_N$ must be small so that $\sum \eta_k$ becomes irrelevant. On the other hand the number of terms of each block η_k must be large so that the correlations between different blocks ξ_k decay sufficiently fast. This procedure was used by M. Weiss in order to prove the Law of Iterated Logarithm for lacunary series [Wei59] and is detailed in our context in next lemma. Let $|\mathcal{A}|$ denote the number of elements of a set $\mathcal{A}$ of integers.

Lemma 12. *Consider an inner function f with $f(0) = 0$ which is not a rotation and a sequence $\{a_n\}$ of complex numbers. Denote*

$$\sigma_N^2 = \sum_{n=1}^{N} |a_n|^2 + 2\,\mathrm{Re}\sum_{k=1}^{N-1} f'(0)^k \sum_{n=1}^{N-k} \overline{a_n} a_{n+k},$$

$$S_N^2 = \sum_{n=1}^{N} |a_n|^2$$

and assume that there exists a nonincreasing function φ with $\lim_{N\to\infty}\varphi(N) = 0$ such that

$$\sup\{|a_n|^2 \colon 1 \leqslant n \leqslant N\} \leqslant \varphi(N) S_N^2. \tag{5.1}$$

If N is large enough, then we can choose indices $0 \leqslant M_k < N_{k+1} < M_{k+1} \leqslant N$, for $0 \leqslant k \leqslant Q_N - 1$, such that if we define the blocks of consecutive integers

$$\mathcal{A}(k) = \{n \in \mathbb{N} \colon M_{k-1} < n \leqslant N_k\}, \quad \mathcal{B}(k) = \{n \in \mathbb{N} \colon N_k < n \leqslant M_k\}, \quad 1 \leqslant k \leqslant Q_N \tag{5.2}$$

and the block sums

$$\xi_k = \sum_{n\in\mathcal{A}(k)} a_n f^n, \quad \eta_k = \sum_{n\in\mathcal{B}(k)} a_n f^n, \quad 1 \leqslant k \leqslant Q_N, \tag{5.3}$$

it holds that,

$$\lim_{N\to\infty} Q_N \varphi(N)^{1/8} = \frac{1}{2}, \tag{5.4}$$

$$\|\xi_k\|_2^2 \lesssim \varphi(N)^{1/8}\sigma_N^2, \tag{5.5}$$

$$\frac{1}{\sigma_N}\left\|\sum_{n=1}^{N} a_n f^n - \sum_{k=1}^{Q_N}\xi_k\right\|_2 \lesssim \varphi(N)^{1/16} \tag{5.6}$$

and

$$|\mathcal{A}(k)| \geqslant \varphi(N)^{-7/8}, \quad |\mathcal{B}(k)| \geqslant \frac{1}{2}\varphi(N)^{-1/2}, \quad k = 1, 2, \ldots, Q_N. \tag{5.7}$$

In particular, observe that (5.4) and (5.7) imply that

$$Q_N \leqslant \varphi(N)^{-1/8} \leqslant \varphi(N)^{-1/2} \leqslant |\mathcal{B}(k)|, \quad 1 \leqslant k \leqslant Q_N,$$

if N is sufficiently large. Hence the number of blocks ξ_k is much smaller than the number of (possibly null) terms in the blocks η_k.

$\mathcal{J}(1)$					$\cdots$	$\mathcal{J}(P_N)$			coefficients in T_N		
1	2	3	$\cdots$	J_1	$\cdots$	$J_{P_N-1}+1$	$\cdots$	J_{P_N}	$J_{P_N}+1$	$\cdots$	N

FIGURE 2. The blocks of indices $\mathcal{J}(k)$ are pairwise disjoint blocks of consecutive indices that contain all positive integers up to J_{P_N}.

Proof. We may assume that N is large enough for $\varphi(N)$ to be already small. Recall that, by Theorem 8, we have that $S_N^2 \simeq \sigma_N^2$. First we define an auxiliary sequence of indices $0 \leqslant J_k \leqslant J_{k+1} \leqslant N$ as follows. Pick $J_0 = 0$ and let J_1 be the smallest positive integer such that

$$\sum_{n=1}^{J_1} |a_n|^2 \geqslant \varphi(N)^{1/8} S_N^2.$$

Now define the auxiliary block of consecutive integers

$$\mathcal{J}(1) = \{n \in \mathbb{N} \colon J_0 < n \leqslant J_1\}.$$

It is clear that the minimality of J_1 and the bound (5.1) on the coefficients imply that

$$\sum_{n \in \mathcal{J}(1)} |a_n|^2 \leqslant \varphi(N)^{1/8} S_N^2 + |a_{J_1}|^2 \leqslant (\varphi(N)^{1/8} + \varphi(N)) S_N^2.$$

Assume that we have chosen $J_0, J_1, \ldots, J_{k-1}$ and defined the blocks of consecutive integers $\mathcal{J}(1), \ldots, \mathcal{J}(k-1)$. Then pick J_k to be the smallest positive integer such that

$$\sum_{n=J_{k-1}+1}^{J_k} |a_n|^2 \geqslant \varphi(N)^{1/8} S_N^2,$$

and let

$$\mathcal{J}(k) = \{n \in \mathbb{N} \colon J_{k-1} < n \leqslant J_k\}.$$

As before, the minimality of J_k and (5.1) give that

$$\varphi(N)^{1/8} S_N^2 \leqslant \sum_{n \in \mathcal{J}(k)} |a_n|^2 \leqslant (\varphi(N)^{1/8} + \varphi(N)) S_N^2. \tag{5.8}$$

We continue this process until we reach J_{P_N} such that

$$\sum_{n=J_{P_N}+1}^{N} |a_n|^2 < \varphi(N)^{1/8} S_N^2 \tag{5.9}$$

and denote $T_N^2 = \sum_{n=J_{P_N}+1}^{N} |a_n|^2$ (see Figure 2). Observe that by estimate (5.8), summing over k, we have that

$$\varphi(N)^{1/8} S_N^2 P_N \leqslant S_N^2 - T_N^2 \leqslant (\varphi(N)^{1/8} + \varphi(N)) S_N^2 P_N.$$

Thus, since $\varphi(N) \to 0$, by (5.9) we get that the number P_N of auxiliary blocks satisfies

$$\lim_{N \to \infty} \varphi(N)^{1/8} P_N = 1. \tag{5.10}$$

Also note that, since $|a_n|^2 \leqslant \varphi(N) S_N^2$ for $n \leqslant N$, we deduce from the lower bound in (5.8) that $\varphi(N) S_N^2 |\mathcal{J}(k)| \geqslant \varphi(N)^{1/8} S_N^2$, $1 \leqslant k \leqslant P_N$. Hence, the number of indices in $\mathcal{J}(k)$ satisfies

$$|\mathcal{J}(k)| \geqslant \varphi(N)^{-7/8}, \quad 1 \leqslant k \leqslant P_N. \tag{5.11}$$

Next, we modify the auxiliary blocks $\mathcal{J}(k)$ to obtain the blocks $\mathcal{A}(k)$ and $\mathcal{B}(k)$ (see Figure 3). Fix $1 \leqslant k \leqslant \lfloor P_N/2 \rfloor$. By (5.11), each block $\mathcal{J}(2k)$ has length larger than $\varphi(N)^{-7/8}$, which is larger than $\varphi(N)^{-1/2}$. We pick p_k smaller pairwise disjoint blocks of consecutive integers $\mathcal{B}(k, j)$

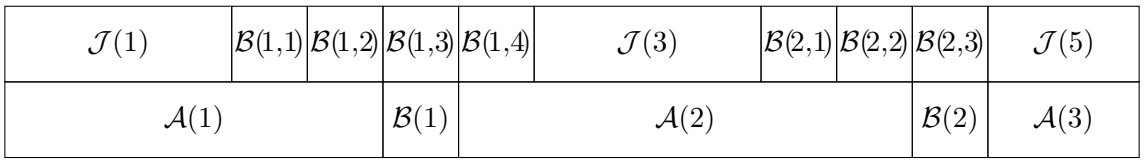

FIGURE 3. In each block $\mathcal{J}(2k)$ consider smaller blocks $\mathcal{B}(k,j)$. Then, we choose $\mathcal{B}(k)$ to be one of these blocks for which $\sum_{n\in\mathcal{B}(k,j)}|a_n|^2$ is minimal. Finally, the blocks $\mathcal{A}(k)$ are the sets of indices between blocks $\mathcal{B}(k-1)$ and $\mathcal{B}(k)$.

of lengths $\lfloor\varphi(N)^{-1/2}\rfloor$. Note that the estimate (5.11) on the number of indices in $\mathcal{J}(k)$, gives that we can get at least

$$p_k \geqslant \frac{|\mathcal{J}(2k)|}{\varphi(N)^{-1/2}} \geqslant \frac{1}{2}\varphi(N)^{-3/8} \tag{5.12}$$

such shorter blocks. Now we pick $\mathcal{B}(k)$ to be one of the blocks $\mathcal{B}(k,j)$ such that the sum

$$\sum_{n\in\mathcal{B}(k,j)}|a_n|^2$$

is minimal. It turns out that

$$\sum_{n\in\mathcal{B}(k)}|a_n|^2 \leqslant \frac{1}{p_k}\sum_{n\in\mathcal{J}(k)}|a_n|^2 \leqslant 2C\varphi(N)^{1/2}S_N^2, \tag{5.13}$$

where the last inequality follows from (5.8) and (5.12).

We define now the sequences M_k and N_k. For clarity, we will assume that P_N is even, since the minor modifications when it is odd will be obvious. Set $Q_N = P_N/2$, $M_0 = 0$ and, $N_k = \min\mathcal{B}(k) - 1$, $M_k = \max\mathcal{B}(k)$ for $1 \leqslant k \leqslant Q_N$, and

$$\mathcal{A}(k) = \{n\in\mathbb{N}\colon M_{k-1} < n \leqslant N_k\}, \qquad 1 \leqslant k \leqslant Q_N.$$

Note that (5.2) is clear by the definition of M_k and N_k. In addition, by the construction of $\mathcal{B}(k)$ for $1 \leqslant k \leqslant Q_N$ we have that

$$|\mathcal{B}(k)| \geqslant \frac{1}{2}\varphi(N)^{-1/2}.$$

Also, taking into account that each block $\mathcal{A}(k)$, $1 \leqslant k \leqslant Q_N$, contains some block $\mathcal{J}(i)$ with $1 \leqslant i \leqslant P_N$ odd, estimate (5.11) gives that

$$|\mathcal{A}(k)| \geqslant \varphi(N)^{-7/8}$$

and this proves (5.7). After defining the sets of indices $\mathcal{A}(k)$ and $\mathcal{B}(k)$, the block sums ξ_k and η_k are given by (5.3) for $1 \leqslant k \leqslant Q_N$. Also note that for any $1 \leqslant k \leqslant Q_N$ we have $\mathcal{A}(k) \subset \mathcal{J}(i-1)\cup\mathcal{J}(i)\cup\mathcal{J}(i+1)$ for some odd $1 \leqslant i \leqslant P_N$ (where we consider $\mathcal{J}(0) = \varnothing$). Hence applying Theorem 8 to ξ_k we find

$$\|\xi_k\|_2^2 \lesssim \sum_{n\in\mathcal{J}(i-1)\cup\mathcal{J}(i)\cup\mathcal{J}(i+1)}|a_n|^2 \lesssim \varphi(N)^{1/8}\sigma_N^2, \quad 1 \leqslant k \leqslant Q_N,$$

which is property (5.5).

When P_N is even, the construction already gives that $Q_N = P_N/2$, while if P_N is odd, then $Q_N = (P_N+1)/2$. In any of these two cases, using (5.10) we get (5.4). The other differences that we would have when P_N is odd are that we would have one more block $\mathcal{A}(k)$ than $\mathcal{B}(k)$ and that the last index in the sequences $\{M_k\}$ and $\{N_k\}$ would be N_{Q_N} instead of M_{Q_N}.

We are only left with checking (5.6). First denote the sums over indices that are not in any of the blocks ξ_k and η_k, by

$$R_N^2 = \sum_{n=M_{Q_N}+1}^{N}|a_n|^2, \qquad \rho_N = \sum_{n=M_{Q_N}+1}^{N}a_n f^n.$$

In particular, note that

$$R_N^2 = \sum_{n=M_{Q_N}+1}^{J_{P_N}} |a_n|^2 + \sum_{n=J_{P_N}+1}^{N} |a_n|^2 \leqslant (2\varphi(N)^{1/8} + \varphi(N))S_N^2.$$

Thus, using Theorem 8 twice we see that

$$\|\rho_N\|_2^2 = \left\| \sum_{n=1}^{N} a_n f^n - \sum_{k=1}^{Q_N} (\xi_k + \eta_k) \right\|_2^2 \lesssim R_N^2 \lesssim \varphi(N)^{1/8} \sigma_N^2. \tag{5.14}$$

Note that

$$\sum_{n=1}^{N} a_n f^n - \sum_{k=1}^{Q_N} \xi_k = \sum_{k=1}^{Q_N} \eta_k + \rho_N.$$

Then using (5.13), (5.14) and (5.4), we deduce

$$\begin{aligned} \left\| \sum_{n=1}^{N} a_n f^n - \sum_{k=1}^{Q_N} \xi_k \right\|_2 &\leqslant \sum_{k=1}^{Q_N} \|\eta_k\|_2 + \|\rho_N\|_2 \lesssim \\ &\lesssim Q_N \varphi(N)^{1/4} \sigma_N + \varphi(N)^{1/16} \sigma_N \lesssim \varphi(N)^{1/16} \sigma_N. \end{aligned} \tag{5.15}$$

This concludes the proof of (5.6). □

We use Lemma 12 to prove Theorem 1. The idea is that we will discard the blocks η_k and the tail ρ_N as their total L^2 norm is irrelevant compared to that of the full partial sum. Then, the correlation between the blocks ξ_k will be controlled due to the length of the blocks η_k that separate them. In practice, this will imply that the blocks ξ_k will behave almost as if they were independent.

Proof of Theorem 1. For a given value of N split the partial sum

$$\sum_{n=1}^{N} a_n f^n$$

into the blocks ξ_k and η_k, with $1 \leqslant k \leqslant Q_N$, given by Lemma 12. Observe that, because of property (5.6) and Chebyshev's inequality, we only need to see that

$$T_N = \frac{\sqrt{2}}{\sigma_N} \sum_{k=1}^{Q_N} \xi_k$$

tends in distribution to a standard complex normal random variable as N tends to infinity. We split the proof into several parts.

1. The characteristic function of T_N. By the Levi Continuity Theorem, it is sufficient to show that for any complex number t we have

$$\varphi_N(t) = \int_{\partial\mathbb{D}} e^{i\langle t, T_N\rangle}\, dm \to e^{-|t|^2/2}, \quad \text{as} \quad N \to \infty. \tag{5.16}$$

As before $\langle t, w\rangle = \mathrm{Re}(\bar{t}w)$ is the standard scalar product in the plane. Fix $t \in \mathbb{C}$. Our first step is to show the approximation

$$\lim_{N\to\infty} \varphi_N(t) - \int_{\partial\mathbb{D}} \prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right) \exp\left(-\frac{\langle t, \xi_k\rangle^2}{{\sigma_N}^2}\right) dm = 0. \tag{5.17}$$

We start with some apriori estimates. Given $\delta > 0$, consider the sets $E_k = E_k(\delta, N) = \{z \in \partial\mathbb{D} \colon |\xi_k(z)| > \delta\sigma_N\}$, $k = 1, 2, \ldots, Q_N$. By part (c) of Theorem 8 and estimate (5.5) we have $\|\xi_k\|_4^4 \lesssim \varphi(N)^{1/4}\sigma_N^4$. Chebyshev's inequality and estimate (5.4) give

$$\sum_{k=1}^{Q_N} m(E_k) \lesssim \frac{\varphi(N)^{1/4}\sigma_N^4 Q_N}{\delta^4 \sigma_N^4} \lesssim \frac{\varphi(N)^{1/8}}{\delta^4}$$

if N is sufficiently large. Given $\mu > 1$, consider the set

$$E_0 = E_0(\mu, t, N) = \left\{ z \in \partial\mathbb{D} \colon \sum_{k=1}^{Q_N} \langle t, \xi_k(z)\rangle^2 > \mu\sigma_N^2 \right\}.$$

Part (b) of Theorem 8, estimates (5.4), (5.5) and Chebyshev's inequality yield

$$m(E_0) \lesssim \frac{|t|^2\varphi(N)^{1/8}\sigma_N^2 Q_N}{\mu\sigma_N^2} \lesssim \frac{|t|^2}{\mu},$$

if N is sufficiently large. Hence the set

$$E = E(\delta, \mu, t, N) = \bigcup_{k=0}^{Q_N} E_k$$

satisfies

$$m(E) \lesssim \left(\frac{\varphi(N)^{1/8}}{\delta^4} + \frac{|t|^2}{\mu} \right). \tag{5.18}$$

Now we will choose appropriate constants $\delta > 0$ and $\mu > 1$. Using the elementary identity

$$\exp(z) = (1+z)\exp\left(\frac{z^2}{2} + o(|z^2|)\right),$$

where $o(|z|^2)/|z|^2 \to 0$ as $z \to 0$, we deduce

$$\exp(i\langle t, T_N\rangle) = \left(\prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right) \exp\left(-\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right)\right) \exp\left(\sum_{k=1}^{Q_N} o\left(\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right)\right).$$

Fix $\varepsilon > 0$ and let $\mu > 1$ be a constant to be fixed later. Note that $\langle t, \xi_k(z)\rangle^2/\sigma_N^2 \leqslant \delta^2|t|^2$ if $z \in \partial\mathbb{D}\backslash E_k$, $k = 1, \ldots, Q_N$. So picking $\delta = \delta(\varepsilon, t) > 0$ sufficiently small we have

$$\sum_{k=1}^{Q_N} o\left(\frac{\langle t, \xi_k(z)\rangle^2}{\sigma_N^2}\right) \leqslant \varepsilon \sum_{k=1}^{Q_N} \frac{\langle t, \xi_k(z)\rangle^2}{\sigma_N^2} \leqslant \varepsilon\mu, \quad z \in \partial\mathbb{D}\backslash E.$$

Note that the last inequality holds because $z \in \partial\mathbb{D}\backslash E_0$. Hence

$$\left| \int_{\partial\mathbb{D}\backslash E} \exp(i\langle t, T_N\rangle)\, dm - \int_{\partial\mathbb{D}\backslash E} \prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right) \exp\left(-\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right) dm \right| \leqslant$$

$$\leqslant (e^{\varepsilon\mu} - 1) \int_{\partial\mathbb{D}\backslash E} \prod_{k=1}^{Q_N} \left(1 + \frac{2\langle t, \xi_k\rangle^2}{\sigma_N^2}\right)^{1/2} \exp\left(-\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right) dm \leqslant e^{\varepsilon\mu} - 1.$$

Last inequality follows from the elementary estimate $(1+x)^{1/2}e^{-x/2} \leqslant 1$ if $x \geqslant 0$. Hence

$$\left| \varphi_N(t) - \int_{\partial\mathbb{D}} \prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right) \exp\left(-\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right) dm \right| \leqslant 2m(E) + e^{\varepsilon\mu} - 1.$$

Taking $\mu = 1/\sqrt{\varepsilon} > 1$ we get that both $|t|^2/\mu$ and $\varepsilon\mu$ are small. Then estimate (5.18) yields the approximation (5.17) as $N \to \infty$. Therefore to prove (5.16) it is sufficient to show that for any $t \in \mathbb{C}$ one has

$$\lim_{N\to\infty} \int_{\partial\mathbb{D}} \prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right) \exp\left(-\frac{\langle t, \xi_k\rangle^2}{\sigma_N^2}\right) dm = \exp(-|t|^2/2).$$

This will follow from Lemma 11 applied to the functions

$$f_N = \prod_{k=1}^{Q_N} \left(1 + \frac{i\sqrt{2}\langle t, \xi_k\rangle}{\sigma_N}\right),$$

$$g_N = \frac{1}{\sigma_N^2}\sum_{k=1}^{Q_N}\langle t,\xi_k\rangle^2 - \frac{|t|^2}{2}.$$

According to Lemma 11 it is sufficient to show

$$\sup_N \|f_N\|_2 < \infty, \tag{5.19}$$

$$\lim_{N\to\infty}\|g_N\|_2 = 0, \tag{5.20}$$

$$\lim_{N\to\infty}\int_{\partial\mathbb{D}} f_N dm = 1. \tag{5.21}$$

2. The norm $\|f_N\|_2$. Observe that

$$\prod_{k=1}^{Q_N}\left(1+\frac{2\langle t,\xi_k\rangle^2}{\sigma_N^2}\right) = 1+\sum_{k=1}^{Q_N}\frac{2^k}{\sigma_N^{2k}}\sum\langle t,\xi_{j_1}\rangle^2\ldots\langle t,\xi_{j_k}\rangle^2,$$

where the last sum is taken over all collections of indices $1\leqslant j_1<\ldots<j_k\leqslant Q_N$. Since $\langle t,\xi_n\rangle^2\leqslant|t|^2|\xi_n|^2$, Theorem 6, part (a) of Theorem 8 and estimate (5.5) give that

$$\int_{\partial\mathbb{D}}\langle t,\xi_{j_1}\rangle^2\ldots\langle t,\xi_{j_k}\rangle^2\,dm\leqslant C(f)^k|t|^{2k}\varphi(N)^{k/8}\sigma_N^{2k},\quad 1\leqslant k\leqslant Q_N.$$

Since the total number of distinct collections of indices $j_1,\ldots,j_k$ verifying $1\leqslant j_1<\ldots<j_k\leqslant Q_N$ is $\binom{Q_N}{k}$, we deduce

$$\int_{\partial\mathbb{D}}\prod_{k=1}^{Q_N}\left(1+\frac{2\langle t,\xi_k\rangle^2}{{\sigma_N}^2}\right)dm\leqslant 1+\sum_{k=1}^{Q_N}\binom{Q_N}{k}\frac{C(f)^k2^k|t|^{2k}\varphi(N)^{k/8}\sigma_N^{2k}}{\sigma_N^{2k}}.$$

Simplifying last expression we deduce

$$\int_{\partial\mathbb{D}}\prod_{k=1}^{Q_N}\left(1+\frac{2\langle t,\xi_k\rangle^2}{\sigma_N^2}\right)dm\leqslant\left(1+2C(f)|t|^2\varphi(N)^{1/8}\right)^{Q_N}.$$

Now (5.4) implies that $Q_N\lesssim\varphi(N)^{-1/8}$, if N is sufficiently large, from which we get that $\|f_N\|_2^2\leqslant\exp\left(3C(f)|t|^2\right)$. This gives (5.19).

3. The norm $\|g_N\|_2$. Recall that

$$\xi_k = \sum_{n\in\mathcal{A}(k)} a_n f^n,\qquad k=1,\ldots,Q_N. \tag{5.22}$$

Let $\mathcal{A}=\bigcup_{k=1}^{Q_N}\mathcal{A}(k)$. Observe that (5.6) together with Theorem 8 gives

$$\lim_{N\to\infty}\frac{\sum_{n\in\mathcal{A}}|a_n|^2}{S_N^2}=1.$$

This is assumption (4.1) of Lemma 9. Assumption (4.2) is an immediate consequence of (5.1) and (5.5). Thus, Lemma 9 gives

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{Q_N}\|\xi_k\|_2^2}{\sigma_N^2}=1. \tag{5.23}$$

Denote $\lambda=t/|t|$. We have

$$g_N=\frac{|t|^2}{4\sigma_N^2}\sum_{k=1}^{Q_N}\left(2|\xi_k|^2+\overline{\lambda^2}\xi_k^2+\lambda^2\overline{\xi_k^2}-2\frac{\sigma_N^2}{Q_N}\right).$$

Applying (5.23), the proof of (5.20) reduces to show

$$\lim_{N\to\infty}\left\|\frac{1}{\sigma_N^2}\sum_{k=1}^{Q_N}\psi_k\right\|_2=0,$$

where $\psi_k = 2(|\xi_k|^2 - \|\xi_k\|_2^2) + \overline{\lambda^2}\xi_k^2 + \lambda^2\overline{\xi_k^2}$. Now

$$\left\|\sum_{k=1}^{Q_N}\psi_k\right\|_2^2 = \sum_{k=1}^{Q_N}\|\psi_k\|_2^2 + 2\,\mathrm{Re}\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\overline{\psi_k}\psi_j\,dm. \tag{5.24}$$

Since $|\psi_k| \leqslant 4|\xi_k|^2 + 2\,\|\xi_k\|_2^2$, parts (a) and (c) of Theorem 8 and estimate (5.5) give that $\|\psi_k\|_2^2 \leqslant C(f)\varphi(N)^{1/4}\sigma_N^4$. Hence, using that $Q_N \simeq \varphi(N)^{-1/8}$ by expression (5.4), we get

$$\sum_{k=1}^{Q_N}\|\psi_k\|_2^2 \lesssim \varphi(N)^{1/8}\sigma_N^4$$

and we deduce

$$\lim_{N\to\infty}\frac{1}{\sigma_N^4}\sum_{k=1}^{Q_N}\|\psi_k\|_2^2 = 0.$$

The second term in (5.24) splits as

$$\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\overline{\psi_k}\psi_j\,dm = A + B + C + D,$$

where

$$A = 4\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\left(|\xi_k|^2 - \|\xi_k\|_2^2\right)\left(|\xi_j|^2 - \|\xi_j\|_2^2\right)\,dm,$$

$$B = 2\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\left(|\xi_k|^2 - \|\xi_k\|_2^2\right)\left(\overline{\lambda^2}\xi_j^2 + \lambda^2\overline{\xi_j^2}\right)\,dm,$$

$$C = 2\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\left(\overline{\lambda^2}\xi_k^2 + \lambda^2\overline{\xi_k^2}\right)\left(|\xi_j|^2 - \|\xi_j\|_2^2\right)\,dm,$$

$$D = \sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\left(\overline{\lambda^2}\xi_k^2 + \lambda^2\overline{\xi_k^2}\right)\left(\overline{\lambda^2}\xi_j^2 + \lambda^2\overline{\xi_j^2}\right)\,dm.$$

By Theorem 6, $\|\xi_k\xi_j\|_2 = \|\xi_k\|_2\,\|\xi_j\|_2$ if $k \neq j$ and we deduce $A = 0$. Since the mean of ξ_j^2 over the unit circle vanishes and at almost every point in the unit circle one has

$$|\xi_k|^2 = \sum_{n\in\mathcal{A}(k)}|a_n|^2 + 2\,\mathrm{Re}\sum_{n\in\mathcal{A}(k)}\sum_{j\in\mathcal{A}(k),j>n}\overline{a_n}a_j\overline{f^n}f^j, \tag{5.25}$$

the integrals in B can be written as a linear combination of

$$\int_{\partial\mathbb{D}} f^{n_1}\overline{f^{j_1}}\left(\overline{\lambda^2}\xi_j^2 + \lambda^2\overline{\xi_j^2}\right)\,dm,$$

where $n_1, j_1 \in \mathcal{A}(k)$ and hence $\max\{n_1, j_1\} < \min\{n\colon n \in \mathcal{A}(j)\}$. According to part (a) of Lemma 7,

$$\int_{\partial\mathbb{D}} f^{n_1}\overline{f^{j_1}}\xi_j^2\,dm = 0$$

and we deduce $B = 0$. Since the mean of ξ_k^2 over the unit circle vanishes, we have

$$C = 4\,\mathrm{Re}\,\overline{\lambda^2}\sum_{k=1}^{Q_N-1}\sum_{j>k}^{Q_N}\int_{\partial\mathbb{D}}\xi_k^2|\xi_j|^2\,dm.$$

Fix $j > k$. Using again that ξ_k vanishes at the origin and formula (5.25), we have

$$\int_{\partial\mathbb{D}}\xi_k^2|\xi_j|^2\,dm = \int_{\partial\mathbb{D}}\xi_k^2\,\mathrm{Re}\,h_j\,dm,$$

where

$$h_j = 2 \sum_{r,l\in\mathcal{A}(j),l>r} \overline{a_r} a_l \overline{f^r} f^l.$$

Using formula (5.22) to expand ξ_k^2, we obtain

$$\int_{\partial\mathbb{D}} \xi_k^2 |\xi_j|^2 \, dm = E + F,$$

where

$$E = \sum_{n\in\mathcal{A}(k)} \sum_{r,l\in\mathcal{A}(j),l>r} a_n^2 \int_{\partial\mathbb{D}} (f^n)^2 \left(\overline{a_r} a_l \overline{f^r} f^l + a_r \overline{a_l} f^r \overline{f^l} \right) dm,$$

$$F = 2 \sum_{n,s\in\mathcal{A}(k):\, s>n} a_n a_s \sum_{r,l\in\mathcal{A}(j),l>r} \int_{\partial\mathbb{D}} f^n f^s \left(\overline{a_r} a_l \overline{f^r} f^l + a_r \overline{a_l} f^r \overline{f^l} \right) dm.$$

By part (c) of Lemma 7 we have

$$\left| \int_{\partial\mathbb{D}} (f^n)^2 \overline{f^r} f^l \, dm \right| + \left| \int_{\partial\mathbb{D}} (f^n)^2 \overline{f^l} f^r \, dm \right| \leqslant C(f) |f'(0)|^{l-n}, \quad \text{if } n < r < l, r - n \geqslant 3.$$

We deduce that

$$|E| \leqslant C(f) \sum_{n\in\mathcal{A}(k)} |a_n|^2 \sum_{r,l\in\mathcal{A}(j),l>r} |a_r||a_l| |f'(0)|^{l-n}.$$

Denote $\psi(N) = \varphi(N)^{-1/2}/2$. According to (5.7), since $j > k$ we have $r - n \geqslant \psi(N)$ for any $r \in \mathcal{A}(j)$ and any $n \in \mathcal{A}(k)$. Hence we have

$$\sum_{r,l\in\mathcal{A}(j),l>r} |a_r||a_l| |f'(0)|^{l-n} \leqslant |f'(0)|^{\psi(N)} \sum_{t\geqslant 1} |f'(0)|^t \sum_{r\in\mathcal{A}(j):\, r+t\in\mathcal{A}(j)} |a_r||a_{r+t}|. \tag{5.26}$$

By Cauchy-Schwarz's inequality, last sum is bounded by $\sum_{r\in\mathcal{A}(j)} |a_r|^2 \lesssim \varphi(N)^{1/8} \sigma_N^2$. Hence

$$|E| \leqslant C(f) |f'(0)|^{\psi(N)} \varphi(N)^{1/4} \sigma_N^4. \tag{5.27}$$

Similarly, part (b) of Lemma 7 gives that

$$\left| \int_{\partial\mathbb{D}} f^n f^s \overline{f^r} f^l \, dm \right| + \left| \int_{\partial\mathbb{D}} f^n f^s f^r \overline{f^l} \, dm \right| \leqslant C(f) |f'(0)|^{l-n}, \quad n < s < r < l,$$

if $r - s \geqslant 3$. Then

$$|F| \leqslant 2C(f) \sum_{n,s\in\mathcal{A}(k):\, s>n} |a_n||a_s| \sum_{r,l\in\mathcal{A}(j),l>r} |f'(0)|^{l-n} |a_r||a_l|.$$

As before, since $j > k$ we have that $r - s \geqslant \psi(N)$ for any $r \in \mathcal{A}(j)$ and any $s \in \mathcal{A}(k)$. Hence $l - n = l - r + r - s + s - n \geqslant \psi(N) + l - r + s - n$, for any $n, s \in \mathcal{A}(k)$, $s > n$ and any $r, l \in \mathcal{A}(j)$, $l > r$. Hence

$$|F| \leqslant 2C(f) |f'(0)|^{\psi(N)} \sum_{n,s\in\mathcal{A}(k):\, s>n} |f'(0)|^{s-n} |a_n||a_s| \sum_{r,l\in\mathcal{A}(j),l>r} |f'(0)|^{l-r} |a_r||a_l|.$$

Repeating the argument in (5.26) we obtain

$$|F| \leqslant 2C(f) \sigma_N^4 |f'(0)|^{\psi(N)}. \tag{5.28}$$

Now, the exponential decay in (5.27) and (5.28) give that

$$\lim_{N\to\infty} \frac{C}{\sigma_N^4} = 0. \tag{5.29}$$

The corresponding estimate for D follows from the estimate

$$\left| \int_{\partial\mathbb{D}} \xi_k{}^2 \overline{\xi_j{}^2} \, dm \right| \leqslant C(f) S_N^4 |f'(0)|^{\psi(N)}, \quad k < j.$$

As before this last estimate follows from (5.7) and from

$$\left|\int_{\partial\mathbb{D}} f^n f^s \overline{f^l}\,\overline{f^t}\, dm\right| \leqslant C(f)|f'(0)|^{t-n}, n < s < l < t, l-s \geqslant 3,$$

which follows from part (b) of Lemma 7. This finishes the proof of (5.20).

4. The integral of f_N. In this last step we will prove (5.21). Observe that at almost every point in the unit circle we have

$$f_N = 1 + \sum_{k=1}^{Q_N} \frac{i^k 2^{k/2}}{\sigma_N^k} \sum \langle t, \xi_{i_1}\rangle \dots \langle t, \xi_{i_k}\rangle, \tag{5.30}$$

where the second sum is taken over all collections of indices $1 \leqslant i_1 < \ldots < i_k \leqslant Q_N$. Fix $1 \leqslant i_1 < \ldots < i_k \leqslant Q_N$. The integral

$$\int_{\partial\mathbb{D}} \langle t, \xi_{i_1}\rangle \dots \langle t, \xi_{i_k}\rangle\, dm = 2^{-k} \int_{\partial\mathbb{D}} \prod_{n=1}^{k} \left(\overline{t}\xi_{i_n} + t\overline{\xi_{i_n}}\right)\, dm$$

is a multiple of a sum of 2^k integrals of the form

$$\overline{t}^r t^l \int_{\partial\mathbb{D}} \xi_{i_1}^{\varepsilon_1} \dots \xi_{i_k}^{\varepsilon_k}\, dm,$$

where $r + l = k$, $\varepsilon_i = 1$ or $\varepsilon_i = -1$ for $i = 1, \dots, k$. We recall the notation ${\xi_i}^{-1}(z) = \overline{\xi_i(z)}$, $z \in \partial\mathbb{D}$. Now, each ξ_i is a linear combination of iterates of f, that is

$$\xi_j = \sum_{n \in \mathcal{A}(j)} a_n f^n.$$

Hence

$$\int_{\partial\mathbb{D}} \xi_{i_1}^{\varepsilon_1} \dots \xi_{i_k}^{\varepsilon_k}\, dm = \sum_{\boldsymbol{n}\in\mathcal{C}} \prod_{j=1}^{k} a_{n_j}^{\varepsilon_j} \int_{\partial\mathbb{D}} f^{n_1\varepsilon_1} \dots f^{n_k\varepsilon_k}\, dm,$$

where $\sum_{\boldsymbol{n}\in\mathcal{C}}$ means the sum over the collection $\mathcal{C}$ of all possible k-tuples $\boldsymbol{n} = \{n_j\}_{j=1}^k$ of indices such that $n_j \in \mathcal{A}(i_j)$ for $j = 1, \dots, k$. Since $|a_n| \leqslant S_N$, $n \leqslant N$, we have

$$\left|\int_{\partial\mathbb{D}} \xi_{i_1}^{\varepsilon_1} \dots \xi_{i_k}^{\varepsilon_k}\, dm\right| \leqslant S_N^k \sum_{\boldsymbol{n}\in\mathcal{C}} \left|\int_{\partial\mathbb{D}} f^{n_1\varepsilon_1} \dots f^{n_k\varepsilon_k}\, dm\right|. \tag{5.31}$$

Let $\boldsymbol{\varepsilon} = \{\varepsilon_j\}_{j=1}^{k-1}$ be fixed and consider $\Phi(\boldsymbol{n}) = \Phi(\boldsymbol{\varepsilon}, \boldsymbol{n}) = \sum_{j=1}^{k-1} \delta_j(n_{j+1} - n_j)$ where $\delta_j \in \{0, 1/2, 1\}$ for $j = 1, \dots, k-1$, with $\delta_1 = 1$ and $\delta_{k-1} \geqslant 1/2$, and with $\delta_j = 1$ if and only if $\delta_{j-1} = 0$ for $j = 2, \dots, k-1$, as introduced in Theorem 10. Let $a = |f'(0)|$. Theorem 10 gives

$$\left|\int_{\partial\mathbb{D}} \xi_{i_1}^{\varepsilon_1} \dots \xi_{i_k}^{\varepsilon_k}\, dm\right| \leqslant k! S_N^k C(f)^k \sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi(\boldsymbol{n})}. \tag{5.32}$$

We split the sum over $\boldsymbol{n} \in \mathcal{C}$ as follows. Let $\mathcal{D}$ denote the set of $(k-1)$-tuples $\boldsymbol{\delta} = \{\delta_j\}_{j=1}^{k-1}$ of coefficients that can appear in $\Phi(\boldsymbol{n})$ as one varies $\boldsymbol{n}$, that is, those tuples with $\delta_j \in \{0, 1/2, 1\}$ for $j = 1, \dots, k-1$, with $\delta_1 = 1$ and $\delta_{k-1} \geqslant 1/2$, and with $\delta_j = 1$ if and only if $\delta_{j-1} = 0$, for $j = 2, \dots, k-1$. Observe that $\mathcal{D}$ has at most 2^k elements. Given a k-tuple $\boldsymbol{n} \in \mathcal{C}$, let us denote by $\boldsymbol{\delta}(\boldsymbol{n})$ the $(k-1)$-tuple $\boldsymbol{\delta}$ of coefficients appearing in $\Phi(\boldsymbol{n})$ for that particular value of $\boldsymbol{n}$. Then we have that

$$\sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi(\boldsymbol{n})} = \sum_{\boldsymbol{\delta}\in\mathcal{D}} \sum_{\{\boldsymbol{n}\in\mathcal{C}\colon \boldsymbol{\delta}(\boldsymbol{n})=\boldsymbol{\delta}\}} a^{\Phi(\boldsymbol{n})}.$$

Given $\boldsymbol{\delta} = \{\delta_j\}_{j=1}^{k-1} \in \mathcal{D}$ fixed, we define $\Phi_{\boldsymbol{\delta}}(\boldsymbol{n}) = \sum_{j=1}^{k-1} \delta_j(n_{j+1} - n_j)$ for every $\boldsymbol{n} \in \mathcal{C}$. We clearly have that

$$\sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi(\boldsymbol{n})} \leqslant \sum_{\boldsymbol{\delta}\in\mathcal{D}} \sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi_{\boldsymbol{\delta}}(\boldsymbol{n})}. \tag{5.33}$$

Consider now a fixed $\boldsymbol{\delta} = (\delta_1, \dots, \delta_{k-1}) \in \mathcal{D}$ and let us compute the rightmost sum in (5.33). Recall that $\delta_1 = 1$. It may be useful for the following argument to have in mind that $\boldsymbol{\delta}$ might look like $\boldsymbol{\delta} = (1, 1/2, \dots, 1/2, 0, 1, 0, 1, 1/2, \dots, , 1/2)$. This is to say that $\boldsymbol{\delta}$ has consecutive chains of nonzero elements (at least one such chain), all of them starting by 1 and with the rest of the terms, if any, equal to 1/2. Moreover, there is exactly one null term between each consecutive pair of such chains. Let $l(1)$ be the minimum integer such that $\delta_{l(1)+1} = 0$ (we set $l(1) = k-1$ if $\delta_j \neq 0$ for all $1 \leqslant j \leqslant k-1$). This is to say that $l(1)$ is the last index of the first chain of nonzero elements. In particular, observe that if $l(1) > 1$, we have that $\delta_j = 1/2$ for $2 \leqslant j \leqslant l(1)$ because by Theorem 10 $\delta_j = 1$ if and only if $\delta_{j-1} = 0$ for $j > 1$. Assume now that we have determined $l(m-1)$. If $l(m-1) < k-1$, then let $l(m)$ be the minimum integer such that $l(m-1) < l(m) \leqslant k-1$ and such that $\delta_{l(m)+1} = 0$. As before, we set $l(m) = k-1$ if $\delta_j \neq 0$ for all $l(m-1) \leqslant j \leqslant k-1$. We iterate this process until we set $l(r) = k-1$ for some integer $1 \leqslant r \leqslant k$. Roughly speaking the indices $l(m)$ indicate the end of the strips of nonzero terms in the components of $\boldsymbol{\delta}$. Taking $l(0) = -1$, note that

$$\Phi_{\boldsymbol{\delta}}(\boldsymbol{n}) = \sum_{j=1}^{k-1} \delta_j (n_{j+1} - n_j) = \sum_{m=1}^{r} \left((n_{l(m-1)+3} - n_{l(m-1)+2}) + \frac{1}{2}(n_{l(m)+1} - n_{l(m-1)+3}) \right).$$

Hence the sum

$$\sum_{\boldsymbol{n} \in \mathcal{C}} a^{\Phi_{\boldsymbol{\delta}}(\boldsymbol{n})}$$

in the right-hand side of (5.33) becomes a product over $m = 1, \dots, r$, of sums of the form

$$\sum_{j=l(m-1)+2}^{l(m)+1} \sum_{n_j \in \mathcal{A}(i_j)} a^{(n_{l(m-1)+3} - n_{l(m-1)+2}) + \frac{1}{2}(n_{l(m)+1} - n_{l(m-1)+3})}. \tag{5.34}$$

Thus, to estimate

$$\sum_{\boldsymbol{n} \in \mathcal{C}} a^{\Phi_{\boldsymbol{\delta}}(\boldsymbol{n})},$$

we need to estimate sums of the form (5.34). Since the argument is identical for any m, we present it for $m = 1$. To simplify the notation write $l = l(1)$ and assume $l > 2$. Denote here $\overline{n_1} = \max \mathcal{A}(i_1)$, $\underline{n_2} = \min \mathcal{A}(i_2)$ and $\underline{n_{l+1}} = \min \mathcal{A}(i_{l+1})$, and observe that $\underline{n_2} - \overline{n_1} \geqslant \psi(N)$ because of (5.7). Here as before, $\psi(N) = \varphi(N)^{-1/2}/2$. Not that the exponent in (5.34) is

$$(n_2 - n_1) + \frac{1}{2}(n_{l+1} - n_2) = \frac{1}{2}(n_{l+1} - n_1) + \frac{1}{2}(n_2 - n_1) \geqslant \frac{1}{2}\psi(N) + \frac{1}{2}(n_{l+1} - n_1).$$

Summing over n_1 and n_2 we get that (5.34) is bounded by

$$C a^{\psi(N)/2} \sum_{j=3}^{l+1} \sum_{n_j \in \mathcal{A}(i_j)} a^{(n_{l+1} - \overline{n_1})/2}.$$

Next, summing over n_j for j up to l yields the factor $|\mathcal{A}(i_3)| + \ldots + |\mathcal{A}(i_l)|$, while summing over n_{l+1} we get the factor $a^{(\underline{n_{l+1}} - \overline{n_1})/2}$. In other words,

$$\sum_{j=3}^{l+1} \sum_{n_j \in \mathcal{A}(i_j)} a^{(n_{l+1} - \overline{n_1})/2} \leqslant a^{(\underline{n_{l+1}} - \overline{n_1})/2} (|\mathcal{A}(i_3)| + \ldots + |\mathcal{A}(i_l)|).$$

Here, $|\mathcal{A}(i_j)|$ denotes the number of indices in the set $\mathcal{A}(i_j)$. Recall that between each consecutive pair of sets $\mathcal{A}(i_j)$, there is a set $\mathcal{B}(i_j)$ with at least $\psi(N)$ terms. Thus, we have that $\underline{n_{l+1}} - \overline{n_1} \geqslant \psi(N) + |\mathcal{A}(i_2)| + \psi(N) + |\mathcal{A}(i_3)| + \ldots + \psi(N) + |\mathcal{A}(i_l)| > l\psi(N) + |\mathcal{A}(i_3)| + \ldots + |\mathcal{A}(i_l)|$. In the last step we used (5.7) again, so that we have $|\mathcal{A}(i_j)| \geqslant \varphi(N)^{-7/8} > \psi(N)$ for any $j = 1, \dots, k$. Hence, we find that

$$\sum_{j=1}^{l+1} \sum_{n_j \in \mathcal{A}(i_j)} a^{(n_2 - n_1) + (n_{l+1} - n_2)/2} \leqslant C a^{l\psi(N)/2} \tag{5.35}$$

because the factor $a^{|\mathcal{A}(i_3)|+\ldots+|\mathcal{A}(i_l)|}(|\mathcal{A}(i_3)|+\ldots+|\mathcal{A}(i_l)|)$ is bounded. Note that if $l=1$ or $l=2$, then (5.35) is obvious.

Now,the full sum

$$\sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi_{\boldsymbol{\delta}}(\boldsymbol{n})},$$

in the right-hand side of (5.33) becomes a product over $m=1,\ldots,r$, of sums of the form (5.34). Thus, applying the estimate (5.35) we get that

$$\sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi_{\boldsymbol{\delta}}(\boldsymbol{n})} \leqslant \prod_{m=1}^{r} Ca^{(l(m)-l(m-1)-1)\psi(N)/2} \leqslant C^k a^{k\psi(N)/5},$$

where in the last step we used that $l(m)\geqslant l(m-1)+2$ by Theorem 10 (so in particular $r\leqslant\lceil k/2\rceil$). Now, summing over $\boldsymbol{\delta}\in\mathcal{D}$ and using the fact that $\mathcal{D}$ has at most 2^k such tuples, we get that the sum in the left hand side of (5.33) can be estimated as

$$\sum_{\boldsymbol{n}\in\mathcal{C}} a^{\Phi(\boldsymbol{n})} \leqslant (2C)^k a^{k\psi(N)/5}.$$

Thus, using (5.31) and (5.32), there exists a constant $C(f)>0$ such that

$$\left|\int_{\partial\mathbb{D}} \xi_{i_1}^{\varepsilon_1}\ldots\xi_{i_k}^{\varepsilon_k}\,dm\right| \leqslant k!S_N^k C(f)^k a^{k\psi(N)/5}.$$

We deduce that

$$\left|\int_{\partial\mathbb{D}} \langle t,\xi_{i_1}\rangle\ldots\langle t,\xi_{i_k}\rangle\,dm\right| \leqslant k!S_N^k C(f)^k|t|^k a^{k\psi(N)/5}.$$

Since the total number of collections of indices $1\leqslant i_1<\ldots<i_k\leqslant Q_N$ is $\binom{Q_N}{k}$, using (5.30), it follows that

$$\left|\int_{\partial\mathbb{D}} f_N\,dm-1\right| \leqslant \sum_{k=1}^{Q_N}\binom{Q_N}{k}k!2^{k/2}\sigma_N^{-k}(C(f)S_N|t|)^k a^{k\psi(N)/5}.$$

Since $k!\leqslant Q_N^k$ for any integer $1\leqslant k\leqslant Q_N$, last sum is smaller than

$$\left(1+\frac{C(f)\sqrt{2}|t|S_NQ_Na^{\psi(N)/5}}{\sigma_N}\right)^{Q_N}-1,$$

which tends to 0 as $N\to\infty$ because by (5.4) we have

$$\lim_{N\to\infty}\frac{S_NQ_N^2a^{\psi(N)/5}}{\sigma_N}=0.$$

□

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