CARLESON MEASURES, VANISHING MEAN OSCILLATION AND CRITICAL POINTS

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ABSTRACT. Given a finite positive Borel measure μ in the open unit disc of the complex plane, we construct a bounded outer function E whose boundary values have vanishing mean oscillation such that $|E|\mu$ is a vanishing Carleson measure. As an application it is shown that given any function in a Hardy space, there exists a bounded analytic function in the unit disc whose boundary values have vanishing mean oscillation, with the same critical points and multiplicities.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane and let $\partial\mathbb{D}$ be the unit circle. With dA, respectively dm, we denote the normalized Lebesgue measure in $\mathbb{D}$, respectively in $\partial\mathbb{D}$. For $0 < p < \infty$ let $\mathbb{H}^p$ be the Hardy space of analytic functions F in $\mathbb{D}$ such that

$$\|F\|_p^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |F(r\xi)|^p dm(\xi) < \infty,$$

and let $\mathbb{H}^\infty$ be the space of bounded analytic functions F in $\mathbb{D}$ with $\|F\|_\infty = \sup\{|F(z)| : z \in \mathbb{D}\}$. Any function $F \in \mathbb{H}^p$ with $0 < p \leq \infty$ has radial limit, denoted by $F(\xi)$, at m-almost every point $\xi \in \partial\mathbb{D}$ and factors as $F = BE$ where B is an inner function and E an outer function. We recall that B is called inner if $B \in \mathbb{H}^\infty$ and $|B(\xi)| = 1$ for m-almost every $\xi \in \partial\mathbb{D}$ and that the outer function E can be written as

$$E(z) = \exp\Big(\int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} h(\xi) dm(\xi)\Big), \quad z \in \mathbb{D},$$

where h is an integrable function in the unit circle. Actually $\log|E(\xi)| = h(\xi)$ for m-almost every $\xi \in \partial\mathbb{D}$. See Chapter II of [Gar06].

Given an arc $I \subset \partial\mathbb{D}$ of normalized length $m(I) = |I|$, let $Q = Q(I) = \{z \in \mathbb{D} : |z| \geq 1 - |I|, z/|z| \in I\}$ be the Carleson square based at I. It is customary to denote $\ell(Q) = |I|$. A finite positive Borel measure μ in $\mathbb{D}$ is

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called a Carleson measure if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |F(z)|^p d\mu(z) \leq C\|F\|_p^p,$$

for any $F \in \mathbb{H}^p$. A celebrated result of Carleson says that μ is a Carleson measure if and only if there exists a constant $C_1 > 0$ such that $\mu(Q) \leq C_1 \ell(Q)$ for any Carleson square Q. A Carleson measure μ is called a vanishing Carleson measure if

$$\frac{\mu(Q)}{\ell(Q)} \to 0 \text{ as } \ell(Q) \to 0.$$

The average of an integrable function h over an arc $I \subset \partial\mathbb{D}$, is denoted by

$$h_I = \fint_I h(\xi) dm(\xi) = \frac{1}{m(I)} \int_I h(\xi) dm(\xi).$$

An integrable function h in $\partial\mathbb{D}$ is in BMO if

$$\|h\|_{\mathrm{BMO}} = \sup \fint_I |h(\xi) - h_I| dm(\xi) < \infty,$$

where the supremum is taken over all arcs $I \subset \partial\mathbb{D}$. The subspace of functions $h \in \mathrm{BMO}$ such that

$$\fint_I |h(\xi) - h_I| dm(\xi) \to 0 \text{ as } m(I) \to 0$$

is denoted by VMO and coincides with the closure in the BMO semi-norm of the continuous functions on $\partial\mathbb{D}$. Given an integrable function h in $\partial\mathbb{D}$ we denote by $h(z)$ with $z \in \mathbb{D}$, its harmonic extension to $\mathbb{D}$. Functions in BMO and Carleson measures are intimately related. Indeed, an integrable function h is in BMO, respectively in VMO, if and only if $|\nabla h(z)|^2(1-|z|^2)dA(z)$ is a Carleson, respectively vanishing Carleson, measure. See Chapter VI [Gar06] for all these well known results.

Our work is inspired by the beautiful article [Wol82] of T. Wolff who considered the algebra QA of bounded analytic functions whose boundary values are in VMO. He proved the following deep result.

Theorem A. [Wol82, Theorem 1] *Given any bounded function f in $\partial\mathbb{D}$, there exists an outer function $E \in \mathrm{QA}$ such that $Ef \in \mathrm{VMO}$.*

We now state our main result.

Theorem 1.

(a) *Let μ be a Carleson measure in $\mathbb{D}$. Then, there exists an outer function $E \in \mathrm{QA}$ with $\log|E| \in \mathrm{VMO}$ such that $|E|\mu$ is a vanishing Carleson measure.*

(b) *Let μ be a finite positive Borel measure in $\mathbb{D}$. Then, there exists an outer function $E \in \mathrm{QA}$ with $\log|E| \in \mathrm{BMO}$ such that $|E|\mu$ is a vanishing Carleson measure.*

Let μ be a finite positive Borel measure in $\mathbb{D}$. Roughly speaking, for *most* Carleson squares Q we have $\mu(Q) < C(Q)\ell(Q)$ where $C(Q) \to 0$ as $\ell(Q) \to 0$. See Proposition 17. So for *most* Carleson squares, the contribution from $|E|$ is not needed. Consequently, $|E|$ only needs to be small on *few* Carleson squares, offering the flexibility to construct $E \in \mathrm{QA}$ such that $|E|\mu$ is a vanishing Carleson measure. The proof of Theorem 1 uses a decomposition of the measure μ, stopping time arguments yielding nested families of dyadic arcs on $\partial\mathbb{D}$ and a construction of certain BMO functions due to J. Garnett and P. Jones ([GJ78]).

Theorem 1 is sharp in several different senses. First, in the conclusion, the vanishing Carleson measure condition can not be replaced by a stronger condition of the same sort. Actually for any increasing function $\omega : [0,1] \to [0,\infty)$ with $\omega(0) = 0$, there exists a Carleson measure μ such that for any outer function $E \in H^\infty$ we have

$$\limsup_{\ell(Q)\to 0} \frac{\int_Q |E(z)| d\mu(z)}{\ell(Q)\omega(\ell(Q))} = \infty.$$

See Proposition 14. Second, one can not replace QA by the disk algebra $\mathrm{A}(\mathbb{D})$ of analytic functions in $\mathbb{D}$ which extend continuously to $\overline{\mathbb{D}}$. Actually a Carleson measure μ will be constructed for which $|E|\mu$ fails to be a vanishing Carleson measure for any non trivial $E \in \mathrm{A}(\mathbb{D})$. See Proposition 15. Finally, in part b) one can not have the sharper condition $\log|E| \in \mathrm{VMO}$, as it is in part a). See Proposition 16.

Theorem 1 has applications in three different contexts. First, using Theorem 1 one can prove Theorem A of T. Wolff. The second application of Theorem 1 concerns critical points of functions in Hardy spaces. Let BMOA, respectively VMOA, be the space of functions $F \in \mathbb{H}^2$ whose boundary values $F(\xi)$ with $\xi \in \partial\mathbb{D}$, are in BMO, respectively in VMO. W. Cohn proved in [Coh99, Theorem 1] that given $0 < p < \infty$ and $F \in \mathbb{H}^p$, there exists $G \in \mathrm{BMOA}$ such that the zeros (and the multiplicities) of F' and G' coincide. Later D. Kraus in [Kra13, Theorem 1.1] proved that given $0 < p < \infty$ and $F \in \mathbb{H}^p$, there exists a Blaschke product B such that the zeros (and the multiplicities) of F' and B' coincide. Using Theorem 1 and a beautiful technique developed by D. Kraus in [Kra13] we prove the following result.

Theorem 2. *Let $0 < p \leq \infty$ and $F \in \mathbb{H}^p$. Then there exists a function $G \in \mathrm{QA}$ such that the zeros (and the multiplicities) of F' and G' coincide.*

Our last application of Theorem 1 concerns generalized Volterra integral operators. Given an analytic function G in $\mathbb{D}$, the action of the generalized Volterra operator T_G on the analytic function F is defined as

$$T_G(F)(z) = \int_0^z F(w)G'(w)dw, \quad z \in \mathbb{D}.$$

Notice that if $G \in \mathrm{BMOA}$, then $T_G\colon \mathbb{H}^\infty \to \mathrm{BMOA}$ is continuous, see [AS95, Proposition 1]. As expected $T_G\colon \mathbb{H}^\infty \to \mathrm{BMOA}$ is compact if and only if $G \in \mathrm{VMOA}$. We will apply Theorem 1 to obtain the following result.

Theorem 3. *Let $G \in \mathrm{BMOA}$ be non constant. Then $T_G : \mathbb{H}^\infty \to \mathrm{BMOA}$ is not bounded from below and $T_G(\mathbb{H}^\infty)$ is not closed in* BMOA*.*

The rest of the paper is organized as follows. Section 2 contains auxiliary results that are used in the proof of Theorem 1 which is given in Section 3. In Section 4, Theorem 1 is applied to prove Theorems A, 2 and 3. The sharpness of Theorem 1 is discussed in the last section.

As usual, the notation $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$.

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2. Auxiliary Results

Our first result states that any finite positive Borel measure in $\mathbb{D}$ can be written as the sum of two measures which have small mass on a certain sequence of annuli.

Lemma 4. *Let μ be a finite positive Borel measure in $\mathbb{D}$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to zero. Then there exists an increasing sequence $\{r_n\}$ with $0 \leq r_n < 1$, $n = 0, 1, \dots$ and two positive measures μ_1, μ_2 with $\mu = \mu_1 + \mu_2$ such that*

$$\mu_1 \{z \in \mathbb{D} : |z| > r_{2n+1}\} \leq \varepsilon_{2n+1}(1 - r_{2n+1}) \tag{1}$$

and

$$\mu_2 \{z \in \mathbb{D} : |z| > r_{2n}\} \leq \varepsilon_{2n}(1 - r_{2n}), \tag{2}$$

for $n = 0, 1, \dots$. Moreover $\{r_n\}$ can be chosen of the form $r_n = 1 - 2^{-N(n)}$ for some integer $N(n) \geq 0$, $n = 0, 1, \dots$ and $\sum(1 - r_n) < \infty$.

Proof. By induction one can define an increasing sequence $\{r_n\}$ with $r_0 = 0$ and $0 \leq r_n < 1$, $n = 1, 2, \dots$, such that

$$\mu\{z \in \mathbb{D} \colon |z| \geq r_{n+1}\} \leq \varepsilon_n(1 - r_n), \quad n = 0, 1, 2, \dots .$$

It is clear that $\{r_n\}$ can be taken as described in the last part of the statement. Let $\mathbb{1}_n$ be the indicator function of the annulus $\{z \in \mathbb{D} : r_n \leq |z| < r_{n+1}\}$. Define the two measures μ_1, μ_2 as

$$\mu_1 = \mu \sum_{n=0}^{\infty} \mathbb{1}_{2n} \text{ and } \mu_2 = \mu \sum_{n=0}^{\infty} \mathbb{1}_{2n+1}.$$

We notice that $\mu = \mu_1 + \mu_2$. Moreover for $n = 0, 1, \dots$, we have that

$$\mu_1\{|z| \geq r_{2n+1}\} = \mu_1\{|z| \geq r_{2n+2}\} \leq \mu\{|z| \geq r_{2n+2}\} \leq \varepsilon_{2n+1}(1 - r_{2n+1})$$

and

$$\mu_2\{|z| \geq r_{2n}\} = \mu_2\{|z| \geq r_{2n+1}\} \leq \mu\{|z| \geq r_{2n+1}\} \leq \varepsilon_{2n}(1 - r_{2n}).$$

∎

The proof of Theorem 1 uses a beautiful construction due to P. Jones and J. Garnett ([GJ78]) of certain functions in BMO supported in a given arc which are large on certain subsets of the arc. A Lipschitz function $a\colon \partial\mathbb{D} \to \mathbb{R}$ is called B-adapted to the arc $I \subset \partial\mathbb{D}$ if the following three conditions hold: the support of a is contained in the dilated arc $3I$; $\sup |a| \leq 1$ and $|\nabla a_j(\xi)| \leq B/|I|$ for any $\xi \in \partial\mathbb{D}$.

Lemma 5. [GJ78] *Let $\{I_j\}$ be a sequence of arcs in $\partial\mathbb{D}$. Assume that there exists a constant $C_1 > 0$ such that for any arc $I \subseteq \partial\mathbb{D}$ we have*

$$\sum_{I_j \subset I} |I_j| \leq C_1 |I|. \tag{3}$$

Let a_j be a B-adapted function to the arc I_j for $j = 1, 2, \ldots$. Then $\sum_j a_j \in$ BMO and

$$\Big\| \sum_j a_j \Big\|_{\mathrm{BMO}} \lesssim C_1 B.$$

For the proof of Lemma 5, we refer to Lemma 2.1 in [GJ78] (see Lemma 3.2 of [NO00] for a VMO version). The following result may be known but, since it is not clearly stated in the literature, we provide a short proof based on an idea in Lemma 1.2 in [Wol82].

Corollary 6. *Let $\{I_j\}$ be a sequence of arcs in $\partial\mathbb{D}$ with $\sum_j |I_j| < \infty$. Then, there exists a positive function $f \in$ VMO such that*

$$\lim_{j\to\infty} \fint_{I_j} f\,dm = +\infty.$$

Proof. We use Lemma 2.2 of [GJ78], which says that given a measurable set $E \subset \partial\mathbb{D}$ there exists a positive function h with $\|h\|_{\mathrm{BMO}} \leq 1$ such that

$$-\log m(E) \lesssim h(\xi), \qquad \text{for all } \xi \in E.$$

Moreover, if E is a finite union of arcs, then h may be taken in $C^\infty(\partial\mathbb{D})$. Since $\sum_j |I_j| < \infty$, the collection $\{I_j\}$ can be split as $\{I_j\} = \cup_{n\geq 1} \mathcal{A}_n$ where $\mathcal{A}_n$ is a collection of finitely many arcs which satisfies

$$\sum_{I \in \mathcal{A}_n} |I| \lesssim e^{-n^3}, \quad n = 1, 2, \ldots$$

Let f_n be a positive smooth function in $\partial\mathbb{D}$ satisfying $\|f_n\|_{\mathrm{BMO}} \lesssim 1$ and

$$f_n(\xi) \geq n^3, \qquad \text{for all } \xi \in \bigcup_{I \in \mathcal{A}_n} I.$$

We set $f = \sum_n f_n / n^2$. Then $f \in$ VMO and

$$\lim_{j\to\infty} \frac{1}{|I_j|} \int_{I_j} f\,dm = \infty.$$

$\square$

Let $\mathcal{D}$ denote the family of dyadic arcs of the unit circle. The corresponding family of dyadic Carleson squares is defined as $\{Q(I) : I \in \mathcal{D}\}$. Note that two dyadic Carleson squares are either disjoint or one is contained into the other. Given a Carleson square $Q = Q(I)$ where $I \subset \partial\mathbb{D}$ is an arc centered at the point $\xi \in \partial\mathbb{D}$, consider $z_Q = (1 - \ell(Q))\xi$. The next preliminary result will be needed in the proof of Theorem A.

Lemma 7. *Let μ be a Carleson measure in $\mathbb{D}$. Given $\varepsilon > 0$ consider the collection $\mathcal{A} = \mathcal{A}(\varepsilon)$ of Carleson squares Q such that $\mu(Q) \geq \varepsilon\ell(Q)$. Let $E \in \mathrm{VMOA}$ such that $|E|\mu$ is a vanishing Carleson measure. Then*

$$\lim_{Q\in\mathcal{A},\, \ell(Q)\to 0} |E(z_Q)| = 0.$$

Proof. Given a Carleson square Q denote $I(Q) = \overline{Q} \cap \partial\mathbb{D}$. Fixed a constant $\eta > 0$ and a Carleson square $Q \in \mathcal{A}(\varepsilon)$, consider the family $\{Q_j\}$ of maximal dyadic Carleson squares contained in Q such that

$$\sup\{|E(w) - E(z_Q)| : w \in T(Q_j)\} \geq \eta.$$

Here $T(Q_j) = \{z \in Q_j : 1 - |z| \geq \ell(Q_j)/2\}$ is the top part of Q_j. Since $E \in \mathrm{VMOA}$, we have

$$\frac{1}{\ell(Q)} \sum \ell(Q_j) \to 0 \text{ as } \ell(Q) \to 0. \tag{4}$$

Consider the region $R = R(Q, \eta) = Q \setminus \cup Q_j$. Since μ is a Carleson measure and $Q \in \mathcal{A}(\varepsilon)$, from (4) we deduce that

$$\frac{\mu(R)}{\mu(Q)} \to 1 \text{ as } \ell(Q) \to 0. \tag{5}$$

Moreover, by construction we have that

$$\sup\{|E(w) - E(z_Q)| : w \in R\} \leq \eta.$$

Hence

$$\int_Q |E| d\mu \geq \int_R |E| d\mu \geq \frac{1}{2}(|E(z_Q)| - \eta)\varepsilon\ell(Q),$$

if $\ell(Q)$ is sufficiently small. Since $|E|\mu$ is a vanishing Carleson measure and $\eta > 0$ can be taken arbitrarily small, we deduce that $|E(z_Q)| \to 0$ as $\ell(Q)$ tends to 0. □

Next result will be used in the proof of Theorem 2. It can be understood as a hyperbolic analogue of the classical fact that a harmonic function u in $\mathbb{D}$ such that $|\nabla u(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure, has boundary values in BMO.

Lemma 8. *Let F be an analytic self-mapping of the unit disc.*

(a) Assume that

$$\frac{|F'(z)|^2(1 - |z|^2)}{(1 - |F(z)|^2)^2} dA(z) \tag{6}$$

is a Carleson measure. Then $\log(1-|F|^2) \in \mathrm{BMO}$.

(b) Assume that

$$\frac{|F'(z)|^2(1-|z|^2)}{(1-|F(z)|^2)^2}dA(z) \tag{7}$$

is a vanishing Carleson measure. Then $\log(1-|F|^2) \in \mathrm{VMO}$ *and consequently,*

$$\sup_{0<r<1}\int_{\partial\mathbb{D}}\frac{dm(\xi)}{(1-|F(r\xi)|)^s}<\infty,$$

for any $0<s<\infty$.

Proof. (a) Let $u(z)=-\log(1-|F(z)|^2)$, $z\in\mathbb{D}$. By Schwarz's Lemma there exists a constant $C_1>0$ such that

$$\sup\{|u(z)-u(w)| : z\in T(Q), w\in T(Q_1)\}\leq C_1, \tag{8}$$

for any pair of Carleson squares $Q_1\subset Q$ with $\ell(Q_1)=\ell(Q)/2$. Let K be the Carleson norm of the Carleson measure in (6). Let $C>2C_1+2K$ be a large constant to be determined later.

Let I be an arc of the unit circle and consider the dyadic decomposition of $Q(I)$. We now use a stopping time argument. Let $\mathcal{G}_1$ be the collection of maximal dyadic Carleson squares $Q_j^{(1)}\subset Q(I)$ such that

$$\sup\{|u(z)-u(z_{Q(I)})| : z\in T(Q_j^{(1)})\}\geq C. \tag{9}$$

The maximality and (8) give that

$$C-C_1\leq |u(z_{Q_j^{(1)}})-u(z_{Q(I)})|\leq C+C_1.$$

We continue by induction. More concretely, assume that the collection $\mathcal{G}_{n-1}=\{Q_l^{(n-1)} : l=1,2,\dots\}$ has been defined. For each $Q_l^{(n-1)}\in\mathcal{G}_{n-1}$ consider the collection $\mathcal{G}_n(Q_l^{(n-1)})$ of maximal dyadic Carleson squares $Q_j^{(n)}\subset Q_l^{(n-1)}$ such that

$$\sup\{|u(z)-u(z_{Q_l^{(n-1)}})| : z\in T(Q_j^{(n)})\}\geq C. \tag{10}$$

The collection $\mathcal{G}_n$ is defined as

$$\mathcal{G}_n=\cup_l\,\mathcal{G}_n(Q_l^{(n-1)}).$$

As before the maximality and (8) give that

$$C-C_1\leq |u(z_{Q_j^{(n)}})-u(z_{Q_j^{(n-1)}})|\leq C+C_1.$$

Observe that

$$|u(\xi)-u(z_I)|\leq (C+C_1)n, \quad \xi\in I\setminus\cup_j\overline{Q_j^{(n)}}, \quad n=1,2,\dots. \tag{11}$$

Let $Q \in \mathcal{G}_{n-1}$. Consider the region $\Omega = Q \setminus \cup Q_j^{(n)}$, where the union is taken over all Carleson squares $Q_j^{(n)} \in \mathcal{G}_n(Q)$. Note that

$$\Delta u(z) = \frac{|F'(z)|^2}{(1-|F(z)|^2)^2}, \quad z \in \mathbb{D} \tag{12}$$

and

$$|\nabla u(z)|^2 \leq \frac{|F'(z)|^2}{(1-|F(z)|^2)^2}, \quad z \in \mathbb{D}. \tag{13}$$

These facts follow from direct calculations and have been recently used in [IN24], [BN25] and [IN25]. Since $\Delta(u-u(z_Q))^2 = 2|\nabla u|^2 + 2(u-u(z_Q))\Delta u$ and $|u(z)-u(z_Q)| \leq C$ for any $z \in \Omega$, using (12) and (13) we deduce that

$$\Delta(u(z)-u(z_Q))^2 \leq \frac{2(C+1)|F'(z)|^2}{(1-|F(z)|^2)^2}, \quad z \in \Omega. \tag{14}$$

Apply Green's Formula to the functions $(u(z)-u(z_Q))^2$ and $\log|z|$, Then the estimate (14) gives

(15)

$$|\int_{\partial\Omega}(u(z)-u(z_Q))^2\,\partial_n \log|z| ds(z) - \int_{\partial\Omega} \log|z|\,\partial_n(u(z)-u(z_Q))^2 ds(z)| \lesssim$$
$$\lesssim \int_{\Omega} \frac{2(C+1)|F'(z)|^2(1-|z|^2)}{(1-|F(z)|^2)^2} dA(z) \leq 2(C+1)K\ell(Q).$$

Note that $|\nabla(u(z)-u(z_Q))^2| \leq 2C|\nabla u(z)|$ for any $z \in \Omega$ and that $(1-|z|^2)|\nabla u(z)| \leq 1$ for any $z \in \mathbb{D}$. We deduce that

$$\int_{\partial\Omega} |\log|z|\,\partial_n(u(z)-u(z_Q))^2| ds(z) \lesssim C\ell(Q). \tag{16}$$

Note that $\partial_n \log|z|$ is supported on the circular parts of the boundary of Ω, where it has values $\pm 1/|z|$. Since $|u-u(z_Q)| \leq C_1$ on $T(Q)$ and $|u(z)-u(z_Q)| > C - C_1$ for any $z \in T(Q_j^{(n)})$, from (15) and (16), we deduce

$$C^2 \sum_j \ell(Q_j^{(n)}) \lesssim CK\ell(Q) \tag{17}$$

Fix C sufficiently large such that

$$\sum_j \ell(Q_j^{(n)}) \lesssim \frac{K}{C}\ell(Q) \leq \frac{1}{2}\ell(Q). \tag{18}$$

Iterating this estimate we obtain

$$\sum_j \ell(Q_j^{(n)}) \lesssim \frac{1}{2^n}|I|, \quad n = 1, 2, \dots. \tag{19}$$

Next we show that $u \in \text{BMO}$. Fix $\lambda > 2C$ and let n be the integer part of $\lambda/(C+C_1)$. Note that (11) gives that

$$\{\xi \in I : |u(\xi) - u(z_I)| > \lambda\} \subset \cup_j \overline{Q_j^{(n)}} \cap I.$$

Then, estimate (19) gives that

$$m(\{\xi \in \partial\mathbb{D} : |u(\xi) - u(z_I)| > \lambda\}) \leq \frac{1}{2^n}|I|.$$

This implies that

$$\frac{1}{|I|}\int_I |u(\xi) - u(z_I)| dm(\xi) = \frac{1}{|I|}\int_0^\infty m(\{\xi \in I : |u(\xi) - u(z_I)| > \lambda\}) d\lambda$$

is bounded by a universal constant independent of I. This finishes the proof of (a).

(b) The proof of (b) only requires minor modifications. Actually one only needs to observe that if $|I|$ is sufficiently small, the constant K in (17) can be taken also small. This allows to fix also $C > 0$ such that K/C is also small. Hence given $\varepsilon > 0$, if $|I|$ is sufficiently small, one can replace the factor $1/2$ in (18) by ε. This gives that $\log(1 - |F|^2) \in \text{VMO}$. Last assertion in (b) follows from the well known fact that $u \in \text{VMO}$ implies that e^{su} is integrable for any $s > 0$.

□

3. Proof of Theorem 1

We now prove our main result.

Proof of Theorem 1. The proof is organized in three steps.

1. Splitting the measure. Let μ be a finite Borel measure on $\mathbb{D}$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to 0. We apply Lemma 4 to find two measures μ_1, μ_2 which satisfy (1) and (2) respectively. For $n \geq 0$, we pick the maximal dyadic squares $\{Q_k^n : k = 1, 2, \ldots\}$ with $1 - r_{2n+3} < \ell(Q_k^n) \leq 1 - r_{2n+1}$ such that

$$\frac{\mu_1(Q_k^n)}{\ell(Q_k^n)} \geq \varepsilon_{2n+1}. \tag{20}$$

We notice that if Q is a Carleson square with $\ell(Q) = 1 - r_{2n+1}$, then

$$\frac{\mu_1(Q)}{\ell(Q)} < \varepsilon_{2n+1}.$$

Indeed, applying (1), we have

$$\frac{\mu_1(Q)}{\ell(Q)} \leq \frac{\mu_1(\{z \in \mathbb{D} \colon \; |z| \geq r_{2n+1}\})}{1 - r_{2n+1}} \leq \varepsilon_{2n+1}. \tag{21}$$

Since $\{Q_k^n : k = 1, 2, \dots\}$ are pairwise disjoint, equation (20) gives that

$$\sum_k \ell(Q_k^n) \le \frac{1}{\varepsilon_{2n+1}} \mu_1(\{|z| \ge r_{2n+1}\}) \le 1 - r_{2n+1}$$

and we have

$$\sum_n \sum_k \ell(Q_k^n) \le \sum_n (1 - r_{2n+1}) < \infty. \tag{22}$$

We split each arc $I_k^n = \overline{Q_k^n} \cap \partial\mathbb{D}$ into finitely many smaller pairwise disjoint subarcs $\{J_{k,j}^n : j = 1, 2, \dots\}$ such that

$$|J_{k,j}^n| = 1 - r_{2n+5}, \quad j = 1, 2, \dots$$

Due to (22), we have that

$$\sum_{n,k,j} |J_{k,j}^n| = \sum_n \sum_k |I_k^n| < \infty.$$

A similar construction is applied to the measure μ_2.

2. Proof of (a). For $i = 1, 2$ we will construct an outer function $E_i \in \mathrm{QA}$ with $\log |E_i| \in \mathrm{VMO}$ such that $|E_i|\mu_i$ is a vanishing Carleson measure. Once this is done the result will follow easily. We will explicitly describe E_1. The function E_2 is constructed using the same procedure.

Apply Corollary 6 to find a positive function $f \in \mathrm{VMO}$ such that

$$\lim_{|J_{k,j}^n| \to 0} \fint_{J_{k,j}^n} f dm = +\infty. \tag{23}$$

Consider the outer function E_1 defined by $\log |E_1(\xi)| = -f(\xi)$, $\xi \in \partial\mathbb{D}$. Note that $\|E_1\|_\infty \le 1$. Since $f \in \mathrm{VMO}$ we have that $E_1 \in \mathrm{QA}$. Next we will show that $|E_1|\mu_1$ is a vanishing Carleson measure.

We first argue with dyadic Carleson squares. Given a dyadic Carleson square Q fix n such that $1 - r_{2n+3} < \ell(Q) \le 1 - r_{2n+1}$. If Q is not contained in any of the $\{Q_k^n : k = 1, 2, \dots\}$, we have that $\mu_1(Q) \le \varepsilon_{2n+1} \ell(Q)$ and

$$\int_Q |E_1(z)| d\mu_1(z) \le \mu_1(Q) \le \varepsilon_{2n+1} \ell(Q).$$

If $Q \subset Q_k^n$ for some k, then

$$\int_Q |E_1(z)| d\mu_1(z) = \int_{Q \cap \{|z| \ge r_{2n+3}\}} |E_1(z)| d\mu_1(z) + \int_{Q \cap \{|z| < r_{2n+3}\})} |E_1(z)| d\mu_1(z).$$

Applying (1), we have

$$\int_{Q \cap \{|z| \ge r_{2n+3}\}} |E_1(z)| d\mu_1(z) \le \mu_1(Q \cap \{|z| \ge r_{2n+3}\}) \le \varepsilon_{2n+3}(1 - r_{2n+3}) \le \varepsilon_{2n+3} \ell(Q).$$

Fix $z \in Q \cap \{z \in \mathbb{D} : |z| < r_{2n+3}\}$. Consider the arc $I(z) \subset \partial\mathbb{D}$ centered at $z/|z|$ of length $1 - |z|$. Note that $|I(z)| \ge 1 - r_{2n+3}$. Since $Q \subset Q_k^n$ we deduce that

$$\sum_{i:J_{k,i}^n \subset I(z)} |J_{k,i}^n| \geq \frac{|I(z)|}{4}.$$

Hence, by (23), given $\varepsilon > 0$ we have $|E_1(z)| < \varepsilon$ if n is sufficiently large. Thus

$$\int_{Q\cap\{|z|<r_{2n+3}\})} |E_1(z)| d\mu_1(z) \leq \varepsilon \mu_1(Q),$$

if n is sufficiently large. Hence, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_Q |E_1| d\mu_1 \leq \varepsilon \ell(Q)$$

when Q is a dyadic Carleson square with $\ell(Q) < \delta$. Since for any Carleson square Q one can find two dyadic Carleson squares Q_1, Q_2 such that $Q \subset Q_1 \cup Q_2$ and $\ell(Q_i) \leq 2\ell(Q)$, $i = 1, 2$, we deduce that $|E_1|\mu_1$ is a vanishing Carleson measure.

We repeat the above construction for μ_2 and we find another outer function E_2 with $\log|E_2| \in \mathrm{VMO}$ such that $|E_2|\mu_2$ is a vanishing Carleson measure. Now $E = E_1 E_2 \in \mathrm{QA}$ satisfies that $|E|\mu$ is a vanishing Carleson measure. Moreover $\log|E| = \log|E_1| + \log|E_2| \in \mathrm{VMO}$.

3. Proof of (b). We now prove part (b) of Theorem 1. Let $\mu = \mu_1 + \mu_2$ be the decomposition of Step 1. The Carleson squares Q_k^n of Step 1 will now be denoted as $Q_k^n = Q_k^{n,0}$, $k = 1, \ldots$; $n = 1, \ldots$. Note that (20) and the maximality gives that $\mu_1(Q_k^{n,0}) \leq 2\varepsilon_{2n+1}\ell(Q_k^{n,0})$. We will now use a stopping time argument in each $Q_k^{n,0}$. Fix n and k and pick the maximal dyadic Carleson squares $\{Q_j^{n,1} : j = 1, 2, \ldots\}$ contained in $Q_k^{n,0}$ such that

$$\frac{\mu_1(Q_j^{n,1})}{\ell(Q_j^{n,1})} \geq 10 \cdot \varepsilon_{2n+1}.$$

Note that the maximality gives that $\mu_1(Q_j^{n,1}) \leq 20 \cdot \varepsilon_{2n+1}\ell(Q_j^{n,1})$. We continue by induction, that is, if $i > 1$ is an integer and we have constructed a Carleson square $Q_j^{n,i-1}$ such that $10^{i-1} \cdot \varepsilon_{2n+1}\ell(Q_j^{n,i-1}) \leq \mu_1(Q_j^{n,i-1}) \leq 2 \cdot 10^{i-1} \cdot \varepsilon_{2n+1}\ell(Q_j^{n,1})$, we consider the maximal dyadic Carleson boxes $\{Q_l^{n,i} : l = 1, 2, \ldots\}$ contained in $Q_j^{n,i-1}$ such that

$$\frac{\mu_1(Q_l^{n,i})}{\ell(Q_l^{n,i})} \geq 10^i \cdot \varepsilon_{2n+1}.$$

As before, the maximality gives

$$\frac{\mu_1(Q_l^{n,i})}{\ell(Q_l^{n,i})} \leq 2 \cdot 10^i \cdot \varepsilon_{2n+1}, \quad l = 1, 2, \ldots.$$

We denote $I_l^{n,i} = \overline{Q_l^{n,i}} \cap \partial\mathbb{D}$. Since

$$\sum_{l:\ I_l^{n,i} \subset I_j^{n,i-1}} |I_l^{n,i}| \leq \frac{1}{10^i \cdot \varepsilon_{2n+1}} \sum_{l:\ I_l^{n,i} \subset I_j^{n,i-1}} \mu_1(Q_l^{n,i}) \leq \frac{\mu_1(Q_j^{n,i-1})}{10^i \cdot \varepsilon_{2n+1}} \leq \frac{1}{5}\ell(Q_j^{n,i-1}),$$

we obtain that

$$\sum_{l:\ I_l^{n,i} \subset I_j^{n,i-1}} |I_k^{n,i}| \leq \frac{1}{5}|I_j^{n,i-1}|, \quad i = 1, 2, \ldots; j = 1, 2, \ldots; n = 0, 1, \ldots. \tag{24}$$

Consequently, iterating (24), we have that

$$\sum_{l:\ I_l^{n,i} \subset I_k^{n,0}} |I_l^{n,i}| \leq \frac{1}{5^i}|I_k^{n,0}|, \quad i, k = 1, 2, \ldots; n = 0, 1, \ldots. \tag{25}$$

For each n, i, l we pick a smooth B-adapted function $a_l^{n,i}$ to the arc $I_l^{n,i}$ such that $a_l^{n,i}(\xi) = 1$ if $\xi \in I_l^{n,i}$. By Lemma 5 we have $\| \sum_l a_l^{n,i} \|_{\mathrm{BMO}} \lesssim B/5^i$. The function

$$h_n = \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} a_l^{n,i}$$

belongs to BMO and $\|h_n\|_{\mathrm{BMO}} \lesssim B$. Moreover, applying (25), (20) and (1) we obtain

$$\begin{aligned} \int_{\partial\mathbb{D}} h_n dm \lesssim & 6 \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} |I_l^{n,i}| \lesssim \sum_{k=1}^{\infty} |I_k^{n,0}| \lesssim \frac{1}{\varepsilon_{2n+1}} \sum_k \mu_1(Q_k^{n,0}) \\ \leq & \frac{1}{\varepsilon_{2n+1}} \mu_1(\{z \in \mathbb{D}\colon\ |z| \geq r_{2n+1}\}) \leq 1 - r_{2n+1}. \end{aligned}$$

Let E_1 be the outer function given by

$$\log |E_1(\xi)| = -4 \cdot \log(10) \sum_{n=1}^{\infty} h_n(\xi), \quad \xi \in \partial\mathbb{D}.$$

Note that $\|E_1\|_\infty \leq 1$. At this point, we verify that $|E_1|\mu_1$ is a vanishing Carleson measure. We first deal with dyadic Carleson squares. Let Q be a dyadic Carleson square. Pick the integer n such that $1 - r_{2n+3} < \ell(Q) \leq 1 - r_{2n+1}$. If Q lays outside $\bigcup_k Q_k^{n,0}$, then

$$\frac{\mu_1(Q)}{\ell(Q)} \leq \varepsilon_{2n+1}.$$

On the other hand, if $Q \subset Q_k^{n,i-1} \setminus \bigcup_l Q_l^{n,i}$ for some integer $i \geq 1$, $k = 1, 2, \ldots$, then

$$\frac{\mu_1(Q)}{\ell(Q)} \leq 10^i \varepsilon_{2n+1}.$$

Note that $h_n(\xi) \geq i$ for any $\xi \in I_k^{n,i-1}$. Then $\log|E_1(z)| \leq -i\log(10)$ for any $z \in Q_k^{n,i-1}$ and we obtain

$$\frac{\int_Q |E_1(z)| d\mu_1(z)}{\ell(Q)} \leq 10^{-i} \frac{\mu_1(Q)}{\ell(Q)} \leq \varepsilon_{2n+1}.$$

Hence given $\varepsilon > 0$ we have proved that

$$\int_Q |E_1| d\mu_1 \leq \varepsilon \ell(Q)$$

if Q is a dyadic Carleson square with $\ell(Q)$ sufficiently small. Since any Carleson square Q is contained in the union of two dyadic Carleson squares Q_1, Q_2 with $\ell(Q_i) \leq 2\ell(Q)$, $i = 1, 2$, we deduce that $|E_1|\mu_1$ is a vanishing Carleson measure. The same construction applied to μ_2 provides an outer function $E_2 \in H^\infty$ such that $|E_2|\mu_2$ is a vanishing Carleson measure.

We apply part (a) of Theorem 1 to the Carleson measure

$$|\,(E_1(z)E_2(z))'\,|^2(1-|z|^2)dA(z)$$

and find an outer function $F \in \mathrm{QA}$ with $\log|F| \in \mathrm{VMO}$ such that the measure

$$|F(z)||\,(E_1(z)E_2(z))'\,|^2(1-|z|^2)dA(z)$$

is a vanishing Carleson measure. Note that since $\log|F| \in \mathrm{VMO}$ we have that $F^{1/2} \in \mathrm{QA}$. Consequently, the function $E = F^{1/2}E_1E_2$ is an outer function in QA such that $|E|\mu$ is a vanishing Carleson measure.

Finally we show that $\log|E| \in \mathrm{BMO}$. It is sufficient to prove that $\log|E_1| \in \mathrm{BMO}$. Since

$$-\log|E_1(\xi)| = \sum_n h_n(\xi) = \sum_n \sum_i \sum_l a_l^{n,i}(\xi), \quad \xi \in \partial\mathbb{D},$$

it is sufficient to observe that $\{a_l^{n,i}\}_{n,i,l}$ are B-adapted functions to the arcs $\{I_l^{n,i}\}$ which satisfy the packing condition (3). □

4. Applications

4.1. **Theorem A.** Our first application is a proof of Theorem A of T. Wolff using Theorem 1.

Proof of Theorem A. Let $P_z(f) = f(z)$ denote the harmonic extension of f to $\mathbb{D}$. We apply case (a) of Theorem 1 to the Carleson measure $\mu(z) = |\nabla P_z(f)|^2(1-|z|^2)dA(z)$ to obtain an outer function $E \in \mathrm{QA}$ with $\log|E| \in \mathrm{VMO}$ such that $|E|\mu$ is a vanishing Carleson measure. Let $I \subset \partial\mathbb{D}$ be an arc. We have that

$$\begin{aligned}&\fint_I |E(\xi)f(\xi)-E(z_I)f(z_I)|dm(\xi)\\ &\le \fint_I |E(\xi)-E(z_I)||f(\xi)|dm(\xi)+\fint_I |f(\xi)-f(z_I)||E(z_I)|dm(\xi)\\ &\le \|f\|_\infty \fint_I |E(\xi)-E(z_I)|dm(\xi)+|E(z_I)|\fint_I |f(\xi)-f(z_I)|dm(\xi),\end{aligned}$$

where $z_I=(1-|I|)\xi_I$ and ξ_I is the center of I. Since $E\in$ VMOA, the first integral tends to 0 as $|I|$ tends to 0 and we only need to show that

$$\lim_{|I|\to 0}|E(z_I)|\fint_I |f(\xi)-f(z_I)|dm(\xi)=0. \tag{26}$$

By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}&\left(\fint_I |f(\xi)-f(z_I)|dm(\xi)\right)^2 \le \fint_I |f(\xi)-f(z_I)|^2dm(\xi)\lesssim \int_{\partial\mathbb{D}}|f(\xi)-f(z_I)|^2P_{z_I}(t)dm(\xi)\\ &\le \int_{\mathbb{D}}|\nabla f(w)|^2\frac{(1-|z_I|^2)(1-|w|^2)}{|1-\overline{z_I}w|^2}dA(w)=\int_{\mathbb{D}}\frac{(1-|z_I|^2)}{|1-\overline{z_I}w|^2}d\mu(w),\end{aligned}$$

where P_{z_I} is the Poisson kernel at the point z_I. We notice that

$$\frac{1-|z_I|^2}{|1-\overline{w}z_I|^2}\lesssim\frac{1}{2^{2n}|I|},\quad w\in 2^nQ(I)\setminus 2^{n-1}Q(I), n\ge 1.$$

Thus

$$\int_{\mathbb{D}}\frac{(1-|z_I|^2)}{|1-\overline{z_I}w|^2}d\mu(w)\lesssim\ \frac{\mu(Q(I))}{|I|}+\sum_{n\ge 1}\frac{\mu(2^nQ(I)\setminus 2^{n-1}Q(I))}{2^{2n}|I|}.$$

Since μ is a Carleson measure, for any $\varepsilon>0$ we have that

$$\sum_{n\ge\log(1/\varepsilon)}\frac{1}{2^{2n}|I|}\mu(2^nQ(I)\setminus 2^{n-1}Q(I))\lesssim\sum_{n\ge\log(1/\varepsilon)}\frac{1}{2^n}\le\varepsilon$$

and then

$$\fint_I |f(\xi)-f(z_I)|dm(\xi)\lesssim\Big(\sum_{n=1}^{\log(1/\varepsilon)}\frac{\mu(2^nQ(I))}{2^{2n}|I|}+\varepsilon\Big)^{1/2}\le\Big(\frac{\mu(\varepsilon^{-1}Q(I))}{|I|}+\varepsilon\Big)^{1/2}. \tag{27}$$

We are now going to show (26). Fix $\varepsilon>0$. Let us consider two cases. Assume first that $\mu\left(\varepsilon^{-1}Q(I)\right)\le\varepsilon|I|$. Then (27) gives that

$$\fint_I |f(\xi)-f(z_I))|dm(\xi)\lesssim\varepsilon^{1/2}$$

and (26) follows. Assume now that $\mu\left(\varepsilon^{-1}Q(I)\right)>\varepsilon|I|$. Since $|E|\mu$ is a vanishing Carleson measure, Lemma 7 applied to the family $\mathcal{A}(\varepsilon^2)$ gives that $|E(z(\varepsilon,I))|\le\varepsilon$, if $|I|$ is sufficiently small. Here $z(\varepsilon,I)$ denotes $z_{\varepsilon^{-1}I}$. Since $E\in$ VMOA we have $(1-|z|)|E'(z)|\to 0$ as $|z|\to 1$. We deduce that

$|E(z_I)| < 2\varepsilon$ if $|I|$ is sufficiently small. This proves (26) and finishes the proof

□

As usual $L^p(\partial\mathbb{D})$ denotes the classical Lebesgue spaces on the unit circle, $0 < p \leq \infty$.

Corollary 9. *Let $f \in L^p(\partial\mathbb{D})$, $0 < p \leq \infty$. Then, there exists an outer function $E \in \mathrm{QA}$ such that $Ef \in \mathrm{VMO} \cap L^\infty(\partial\mathbb{D})$.*

Proof. Consider the outer function E_0 defined as

$$E_0(z) = \exp\left(-\int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} \log^+ |f(\xi)| d\xi\right), \quad z \in \mathbb{D}.$$

It is clear that E_0 belongs to $\mathbb{H}^\infty$ and $E_0 f \in L^\infty(\partial\mathbb{D})$. We apply Theorem A twice. First, we find an outer function $E_1 \in \mathrm{QA}$ such that $E_1 E_0 \in \mathrm{QA}$. Since $E_1 E_0 f \in L^\infty(\partial\mathbb{D})$, another application of Theorem A provides a function $E_2 \in \mathrm{QA}$ such that $E_2 E_1 E_0 f \in \mathrm{QA}$ and one can take $E = E_2 E_1 E_0$. □

4.2. **Critical points of functions in Hardy spaces.** Theorem 2 follows from a convenient variant of a classical result by W. Cohn on factorization of derivatives of functions in Hardy spaces. Fix $0 < p < \infty$. W. Cohn proved in [Coh99, Theorem 1] that, given $F \in \mathbb{H}^p$, there exist a function $G \in \mathrm{BMOA}$ and an outer function $H \in \mathbb{H}^p$ such that $F' = G'H$. Conversely, for any $G \in \mathrm{BMOA}$ and $H \in \mathbb{H}^p$, the function $G'H$ is the derivative of a function in $\mathbb{H}^p$. See [Dya12] for a version in the Nevanlinna class. Next we apply Theorem 1, Lemma 8 and a nice technique of [Kra13] to show the following result which obviously implies Theorem 2.

Lemma 10. *Fix $0 < p < \infty$. For every $F \in \mathbb{H}^p$ there exist $G \in \mathrm{QA}$ and an outer function $H \in \mathbb{H}^q$ for any $q < p$ such that $F' = G'H$.*

Proof. Let $F \in \mathbb{H}^p$. According to Cohn's result, there exist $\Phi \in \mathrm{BMOA}$ and an outer function $R \in \mathbb{H}^p$ such that $F' = \Phi' R$. Applying Theorem 1 to the Carleson measure $|\Phi'(z)|^2(1-|z|^2)dA(z)$, one obtains an outer function $E \in \mathrm{QA}$ with $\log|E| \in \mathrm{VMO}$, such that $E^{1/2}\Phi'$ is the derivative of a function $G \in \mathrm{VMOA}$. Consequently

$$F' = G' \frac{R}{E^{1/2}}.$$

Since $\log|E| \in \mathrm{VMO}$, the John-Nirenberg Theorem gives that $E^{-1/2} \in \mathbb{H}^r$ for every $0 < r < \infty$. Holder's inequality gives that $RE^{-1/2} \in \mathbb{H}^q$ for any $0 < q < p$.

Note that the function $G \in \mathrm{VMOA}$ may be unbounded. Next we will apply the technique in [Kra13]. Consider the partial differential equation

$$\Delta u(z) = 4|G'(z)|^2 e^{2u(z)}, \quad z \in \mathbb{D}. \tag{28}$$

Since G is in BMOA the PDE (28) has a solution u_0 which is bounded on $\mathbb{D}$ (see Remark 3.4 of [Kra13]). By Liouville's Theorem (see Theorem 3.3 of [KR08]), there exists an analytic self-mapping I of $\mathbb{D}$ such that

$$u_0(z) = \log\Big(\frac{1}{|G'(z)|}\frac{|I'(z)|}{(1-|I(z)|^2)}\Big), \quad z \in \mathbb{D}.$$

Since u_0 is bounded in $\mathbb{D}$, we deduce that $|G'|$ is comparable to $|I'|/(1-|I|^2)$ on $\mathbb{D}$. Hence I and G have the same critical points with the same multiplicities. Since G is in VMOA, we deduce that $|I'(z)|^2(1-|z|)dA(z)/(1-|I(z)|)^2$ is a vanishing Carleson measure. In particular I is in VMOA and hence $I \in \mathrm{QA}$. Note that $|G'/I'|$ is comparable to $(1-|I|^2)^{-1} \geq 1$ on $\mathbb{D}$ and deduce that G'/I' is an outer function. Finally part (b) of Lemma 8 gives that G'/I' belongs to the Hardy space $\mathbb{H}^s$, for any $0 < s < \infty$.

□

We notice that the proof of the previous Lemma gives the following factorization for the derivatives of BMOA functions.

Corollary 11. *For every $F \in \mathrm{BMOA}$ there exist $G \in \mathrm{QA}$ and an function $H \in \mathbb{H}^q$ for any $q < \infty$ with $1/H \in \mathbb{H}^\infty$ such that $F' = G'H$.*

We close this section with two remarks. A. Aleksandrov and V. Peller proved in [AP96, Theorem 3.4] that for any $F \in \mathrm{BMOA}$ there exist $G_i, H_i \in \mathbb{H}^\infty$, $i = 1, 2$, such that $F' = G_1'H_1 + G_2'H_2$. The second remark concerns single generated ideals in the space A_1^2 of analytic functions F in $\mathbb{D}$ such that

$$\|F\|^2 = \int_{\mathbb{D}} |F(z)|^2(1-|z|^2)dA(z) < \infty.$$

O. Ivrii showed that any single generated invariant subspace of A_1^2 can be generated by the derivative of a bounded function (see Theorem 3.1 of [Ivr21]).

Corollary 12. *Let $F \in A_1^2$ and let $[F]$ denote the closure in A_1^2 of polynomial multiples of F. Then, there exists a function $G \in \mathrm{QA}$ such that $[G'] = [F]$.*

Proof. It is well known that there exists $I \in \mathrm{BMOA}$ such that $[F] = [I']$ (see [HKZ, Theorem 3.3]). Corollary 11 provides $G \in \mathrm{QA}$ and a function $H \in \mathbb{H}^2$ with $1/H \in \mathbb{H}^\infty$ such that $I' = G'H$. We now verify that $[I'] = [G']$. Let $W \in [I']$. Given $\varepsilon > 0$ there exists an analytic polynomial P such that $\|W - PI'\| \leq \varepsilon$. Consequently, if H_n is the Taylor polynomial of H of degree n, we have

$$\begin{aligned}\|W - PG'H_n\| &\leq \|W - PHG'\| + \|PHG' - PG'H_n\| \\ &\leq \|W - PI'\| + C\|G\|_{\mathrm{BMO}}^2\|P\|_\infty\|H - H_n\|_2,\end{aligned}$$

where $C > 0$ is an absolute constant. The last estimate follows from the fact that $|G'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure. Therefore $W \in [G']$. A similar argument using that $1/H \in \mathbb{H}^2$, proves the converse inclusion. □

4.3. **Generalized Volterra operators.** Given two analytic functions F, G in $\mathbb{D}$, $T_G(F)$ denotes the generalized Volterra operator with symbol G applied to F defined as

$$T_G(F)(z) = \int_0^z F(w)G'(w)dw, \quad z \in \mathbb{D}.$$

It is clear that if $G \in \mathrm{BMOA}$, the operator $T_G \colon \mathbb{H}^\infty \to \mathrm{BMOA}$ is continuous and $\|T_G\| \lesssim \|G\|_{\mathrm{BMO}}$. As it is expected, T_G is compact precisely when the symbol $G \in \mathrm{VMOA}$.

Lemma 13. *Let* $G \in \mathrm{BMOA}$*. Then* $T_G : \mathbb{H}^\infty \to \mathrm{BMOA}$ *is compact if and only if* $G \in \mathrm{VMOA}$*.*

Proof. Assume first that G is a polynomial. Note that $T_g = V \circ M_{G'}$, where $M_{G'} \colon \mathbb{H}^\infty \to \mathbb{H}^\infty$ and $V \colon \mathbb{H}^\infty \to \mathrm{BMOA}$ are respectively the operator of multiplication by G' and the classical Volterra operator. By [AJS14, Theorem 3.5], V acts compactly on $\mathbb{H}^\infty$. Hence T_G is compact.

Consider now an arbitrarily function $G \in \mathrm{VMOA}$. Note that there exist polynomials P_n, such that

$$\lim_{n\to\infty} \|G - P_n\|_{\mathrm{BMO}} = 0\,.$$

Thus

$$\lim_{n\to\infty} \|T_G - T_{P_n}\| \lesssim \lim_{n\to\infty} \|G - P_n\|_{\mathrm{BMO}} = 0\,.$$

Hence, $T_G \colon \mathbb{H}^\infty \to \mathrm{BMOA}$ is compact.

The converse is proved by contradiction. Assume that T_G is compact and that G is not in VMOA. Then there exist a constant $M > 0$ and a sequence of arcs $\{I_n\}$ in $\partial\mathbb{D}$ such that $|I_n| \to 0$ and

$$\frac{1}{|I_n|} \int_{Q(I_n)} |G'(z)|^2(1 - |z|^2)dA(z) > M, \quad n = 1, 2, \dots . \tag{29}$$

For $n = 1, 2, \dots$ pick the integer N_n with $|I_n|^{-1} \le N_n < |I_n|^{-1} + 1$. Then

$$\|T_G(z^{N_n})\|^2_{\mathrm{BMO}} \ge \frac{1}{|I_n|} \int_{Q(I_n)} |z|^{2N_n} |G'(z)|^2(1 - |z|^2)dA(z) \ge M/4.$$

Hence $T_G : \mathbb{H}^\infty \to \mathrm{BMOA}$ is not compact. □

The boundedness from below of the operators T_G acting on Hardy and Bergman spaces has already been studied in [And11] and [Pan22]. The core of our approach lies in establishing that for $G \in \mathrm{BMOA}$, there exists a non zero function $F \in \mathrm{VMOA} \cap T_G(\mathbb{H}^\infty)$.

Proof of Theorem 3. Let $H \in \mathbb{H}^\infty$. Because of Theorem 1, we can find an outer function $E \in \mathrm{QA}$ such that $|E(z)G'(z)H(z)|^2(1 - |z|^2)dA(z)$ is a vanishing Carleson measure. This implies that $F = T_G(EH) \in \mathrm{VMOA}$. Note that

$$EHG' = F'. \tag{30}$$

We argue by contradiction. Assume that $T_G : \mathbb{H}^\infty \to \text{BMOA}$ is bounded from below or that $T_G(\mathbb{H}^\infty)$ is closed in $BMOA$. Then, since T_G is also bounded, there exists a constant $C > 0$ such that

$$C^{-1}\|k\|_\infty \leq \|T_G(k)\|_{\text{BMO}} \leq C\|k\|_\infty \tag{31}$$

for every $k \in \mathbb{H}^\infty$. In particular, if $k_n(z) = E(z)H(z)z^n$, $z \in \mathbb{D}$, applying (30) we deduce that

$$\begin{aligned}\|T_G(k_n)\|^2_{\text{BMO}} &\sim \sup_I \frac{1}{|I|}\int_{Q(I)} |G'(z)E(z)H(z)|^2|z|^{2n}(1-|z|^2)\,dA(z)\\ &= \sup_I \frac{1}{|I|}\int_{Q(I)} |F'(z)|^2|z|^{2n}(1-|z|^2)\,dA(z).\end{aligned}$$

Since $F \in \text{VMOA}$ we deduce that $\|T_G(k_n)\|_{\text{BMO}} \to 0$ as $n \to \infty$ while $\|k_n\|_\infty$ is bounded below.

□

5. Sharpness of Theorem 1

We first show that in the conclusion of Theorem 1, one can not improve the vanishing Carleson measure condition.

Proposition 14. *For any increasing function $\omega\colon [0,1] \to [0,+\infty)$ with $\omega(0) = 0$, there exists a Carleson measure μ such that for any outer function $E \in \mathbb{H}^\infty$ we have*

$$\limsup_{\ell(Q)\to 0} \frac{\int_Q |E(z)|d\mu(z)}{\ell(Q)\omega(\ell(Q))} = +\infty.$$

Proof. Pick two sequences $\{\delta_k\}$ and $\{h_k\}$ of positive numbers such that

$$\sum_{k=1}^\infty \frac{h_k}{\delta_k} < \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{h_k}{\delta_k}\log(\omega(h_k)) = -\infty. \tag{32}$$

We can also assume that $N_k = 2\pi/\delta_k$ is an integer for any $k = 1, 2, \ldots$. For any $k = 1, 2, \ldots$, let $\Lambda_k = \{z_{k,j} : j = 1, \ldots, N_k\}$ be points uniformly distributed in the circle $\{z \in \mathbb{D} : 1 - |z| = h_k\}$. Then the sequence $\{z_n\}$ defined as

$$\{z_n\} = \bigcup_{k=1}^\infty \Lambda_k$$

is a Blaschke sequence. Actually

$$\mu = \sum_n (1 - |z_n|)\delta_{z_n},$$

is a Carleson measure. We argue by contradiction and assume that there exists an outer function $E \in \mathbb{H}^\infty$, $\|E\|_\infty \leq 1$, such that

$$\frac{\int_Q |E(z)|d\mu(z)}{\ell(Q)\omega(\ell(Q))} < 1 \tag{33}$$

for every Carleson square Q. We pick Carleson squares $Q_{k,j}$ with $\ell(Q_{k.j}) = h_k$ such that $z_{k,j}$ lies in the top part $T(Q_{k.j})$ of $Q_{k.j}$. The assumption (33) implies that for every k, j

$$|E(z_{k,j})|h_k \leq \ell(Q_{k,j})\omega(\ell(Q_{k,j})) = h_k\omega(h_k),$$

that is,

$$|E(z_{k,j})| \leq \omega(h_k), \quad j = 1, \dots, N_k;\ k = 1, 2, \dots. \tag{34}$$

Consider the discs $D_{k,j} = \{z \in \mathbb{D} : |z - z_{k,j}| \leq (1 - |z_{k,j}|)/2\}$. Harnack's inequality applied to the positive harmonic function $-\log|E|$ gives that

$$-\log|E(z)| \gtrsim -\log|E(z_{k,j})| \geq -\log\omega(h_k), \quad z \in D_{k,j}, \quad j = 1, \dots, N_k;\ k = 1, 2, \dots.$$

uniformly in k, j. Consequently, by subharmonicity, we have that

$$\log|E(0)| \leq \int_{\partial\mathbb{D}} \log|E((1-h_k)\xi)|dm(\xi) \lesssim \log(\omega(h_k))\frac{h_k}{\delta_k} \to -\infty, \text{ as } k \to \infty.$$

This is clearly a contradiction and finishes the proof. □

We recall that the disc algebra $\mathrm{A}(\mathbb{D})$ is the space of continuous functions in the closed unit disc which are analytic in $\mathbb{D}$. Our next result says that in Theorem 1 one can not replace the condition $E \in \mathrm{QA}$ by $E \in \mathrm{A}(\mathbb{D})$.

Proposition 15. *There exists a Carleson measure μ in $\mathbb{D}$ such that there are no non trivial functions $E \in \mathrm{A}(\mathbb{D})$ such that $|E|\mu$ is a vanishing Carleson measure.*

Proof. Consider a Carleson measure μ such that for any point $\xi \in \partial\mathbb{D}$ there exists a sequence of Carleson squares $\{Q_n(\xi)\}$ such that $\mu(Q_n(\xi)) > \ell(Q_n(\xi))$ and

$$\lim_{n\to\infty} \ell(Q_n(\xi)) = 0\,, \lim_{n\to\infty} \operatorname{dist}(\xi, Q_n(\xi)) = 0.$$

For instance one could consider a uniformly separated sequence Λ with $\partial\mathbb{D} \subset \overline{\Lambda}$ and

$$\mu = \sum_{z\in\Lambda}(1 - |z|)\delta_z.$$

Assume that there exists a function $E \in \mathrm{A}(\mathbb{D})$ such that $|E|\mu$ is a vanishing Carleson measure. Then

$$\lim_{n\to\infty} \frac{\int_{Q_n(\xi)} |E|d\mu}{\ell(Q_n(\xi))} = 0, \quad \xi \in \partial\mathbb{D}.$$

Since E is continuous in $\overline{\mathbb{D}}$, this implies that E vanishes identically. □

Our last remark concerns the sharpness of part (b) of Theorem 1: one can not replace the condition $\log|E| \in \mathrm{BMO}$ by the stronger one $\log|E| \in \mathrm{VMO}$.

Proposition 16. *There exists a finite positive Borel measure μ in $\mathbb{D}$ such that there are no function $E \in \mathrm{QA}$ with $\log|E| \in \mathrm{VMO}$ such that $|E|\mu$ is a vanishing Carleson measure.*

Proof. We argue by contradiction. Let $G \in \mathbb{H}^2$ and consider the measure $\mu(z) = |G'(z)|^2(1-|z|^2)dA(z)$. Assume there exists an outer function $E \in$ QA such that $|E|\mu$ is a vanishing Carleson measure and $\log|E| \in$ VMO. Then the function F defined by

$$F(z) = \int_0^z E(w)G'(w)dw, \quad z \in \mathbb{D}$$

belongs to VMOA and satisfies $F' = G'E$. Since $\log|E| \in VMO$, we have $1/E \in \mathbb{H}^p$ for $p > 2$. hence we obtain that for any function $G \in \mathbb{H}^2$ one can factor $G' = F'/E \in \{H' : H \in \mathbb{H}^p\}$ by the result of W.Cohn [Coh99], which is clearly a contradiction if $p > 2$. □

As explained in the introduction, the proof of Theorem 1 relies on the fact that for any finite positive Borel measure μ, the ratio $\mu(Q)/\ell(Q)$ is small for *most* Carleson squares $Q \subset \mathbb{D}$. Our last result points in this direction.

Proposition 17. *Let μ be a finite positive measure. Then*

$$\lim_{h\to 0} \frac{\mu\left(Q(\xi,h)\right)}{h} = 0\,,$$

for m-almost every $\xi \in \partial\mathbb{D}$. Here $Q(\xi,h) = Q(I(\xi,h))$ where $I(\xi,h)$ is the arc of $\partial\mathbb{D}$ centered at ξ of normalized length h.

Proof. We proceed by contradiction, that is, we assume that

$$m\left\{\xi \in \partial\mathbb{D} : \limsup_{h\to 0} \frac{\mu\left(Q(\xi,h)\right)}{h} > 0\right\} > 0\,. \tag{35}$$

By the regularity of the Lebesgue measure, there exists a constant $\eta > 0$ such that

$$m\left\{\xi \in \partial\mathbb{D} : \limsup_{h\to 0} \frac{\mu\left(Q(\xi,h)\right)}{h} > \eta\right\} > 0\,.$$

Let $\varepsilon > 0$ be a small number to be fixed later. Consider a compact set $K \subset \partial\mathbb{D}$ wih $m(K) > 0$ such that for every $\xi \in K$ there exists a sequence $\{h_n(\xi)\}_n$ tending to 0 as $n \to \infty$ with $0 < h_n(x) < \varepsilon$ and

$$\mu(Q(\xi, h_n(\xi))) > \frac{\eta}{2}\, h_n(\xi), \quad n = 1, 2, \dots .$$

We consider the collection of arcs $\{I(\xi, h_n(\xi)) : n = 1, 2, \dots; \xi \in K\}$. Using Vitali's covering lemma, we extract a family of pairwise disjoint arcs $\{I_k\}$ such that $K \subseteq \bigcup_k 5I_k$. Note that $\mu(Q(I_k)) \geq \eta|I_k|/2$. Consequently

$$\mu\left(\{z \in \mathbb{D} : 1-\varepsilon \leq |z| < 1\}\right) \geq \mu\left(\cup_k Q(I_k)\right) \geq \frac{\eta}{2}\sum_k |I_k| \geq \frac{\eta}{10}|K|\,.$$

However, the last inequality gives a contradiction if $\varepsilon > 0$ is taken sufficiently small since

$$\lim_{\varepsilon\to 0} \mu\left(\{z \in \mathbb{D} : 1-\varepsilon \leq |z| < 1\}\right) = 0\,.$$

□

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