

Inner Functions with Derivatives in the Weak Hardy Space

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Abstract

It is proved that exponential Blaschke products are the inner functions whose derivative is in the weak Hardy space. As a consequence, it is shown that exponential Blaschke products are Frostman shift invariant. Exponential Blaschke products are described in terms of their logarithmic means and also in terms of the behavior of the derivatives of functions in the corresponding model space.

1 Introduction

For $0 < p < \infty$, let H^p be the Hardy space of analytic functions f in the unit disc \mathbb{D} of the complex plane for which

$$\|f\|_p^p = \sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Any function $f \in H^p$ has radial limits at almost every point of the unit circle $\partial\mathbb{D}$, that is, $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ exists a.e. $e^{i\theta} \in \partial\mathbb{D}$. An inner function I is a bounded analytic function in \mathbb{D} satisfying $|I(e^{i\theta})| = 1$ a.e. $e^{i\theta} \in \partial\mathbb{D}$. Any inner function I can be decomposed as $I = \gamma BS$ where γ is a unimodular constant,

$$B(z) = \prod_n \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}, \quad z \in \mathbb{D},$$

is a Blaschke product and

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right), \quad z \in \mathbb{D},$$

is a singular inner function. Here $\{z_n\}$ are the zeros of I ; $d\mu$ is a positive singular measure and we use the convention $\bar{z}/|z| = 1$ if $z = 0$. An inner function which extends continuously to the closed unit disc must be a finite Blaschke product, that is a Blaschke product with finitely many zeros. Any function whose derivative is in H^1 extends continuously to the closed unit disc. Actually the Hardy-Littlewood Theorem gives more precise information (see [Du, p. 78]). Hence the only inner functions I such that $I' \in H^1$ are the finite Blaschke products. Many authors have studied the problem of determining the Hardy space H^p , $0 < p < 1$, to which the derivative of an inner function

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belongs. See [AC], [Ah], [Cu], [CS], [Dy], [FM], [GGJ], [GPV], [P1], [P2], [P3], [Pe]. Ahern and Clark proved that if an inner function I satisfies $I' \in H^{1/2}$ then I must be a Blaschke product ([AC]). Let B be a Blaschke product with zeros $\{z_n\}$. Protas proved that the condition

$$\sum_n (1 - |z_n|)^{1-p} < \infty,$$

implies that $B' \in H^p$ if $1/2 < p < 1$ (see [P1]). The converse is not true but Ahern proved that for $1/2 < p < 1$, $B' \in H^p$ if and only if there exists $a \in \mathbb{D}$ such that

$$\sum (1 - |w_n|)^{1-p} < \infty,$$

where the sum is taken over all $w_n \in \mathbb{D}$ with $B(w_n) = a$. See Theorem 6.2 of [Ah]. The paper [Ah] has other very nice results in this direction but no geometrical description of the Blaschke products B such that $B' \in H^p$, $1/2 < p < 1$ in terms of the distribution of its zeros, is known. Results of this type have been recently gathered in the monograph [Ma].

For $0 < p < \infty$, let H_w^p be the weak Hardy space formed by those analytic functions f in the unit disc for which there exists a constant $C = C(f) > 0$ such that

$$|\{e^{i\theta} : |f(re^{i\theta})| > \lambda\}| \leq C\lambda^{-p}$$

for any $0 < r < 1$ and any $\lambda > 0$. Here $|E|$ denotes the length of the set $E \subset \partial\mathbb{D}$. Given an analytic function f in the unit disc, consider the non-tangential maximal function defined as

$$M_\alpha f(e^{i\theta}) = \sup\{|f(z)| : |z - e^{i\theta}| \leq \alpha(1 - |z|)\}$$

where $\alpha > 1$ is fixed. A fundamental result by Hardy and Littlewood (for $1 < p < \infty$) and by Burkholder, Gundy and Silverstein (for $0 < p \leq 1$) states that if f is analytic in \mathbb{D} then $f \in H^p$ if and only if $M_\alpha f \in L^p(\partial\mathbb{D})$. See [Ga, p. 111] or [Ko, p. 182]. For $0 < p < \infty$, let $L_w^p(\partial\mathbb{D})$ be the weak L^p space of measurable functions f defined on $\partial\mathbb{D}$ for which there exists a constant $C = C(f) > 0$ such that

$$|\{e^{i\theta} : |f(e^{i\theta})| > \lambda\}| \leq C\lambda^{-p}$$

for any $\lambda > 0$. An analytic function f in the unit disc belongs to H_w^p if and only if $M_\alpha f \in L_w^p(\partial\mathbb{D})$. See Remark 1 of ([A11]) or Theorem 2.1 of ([A12]). Functions in H_w^p have radial limits at almost every point and the boundary values are in $L_w^p(\partial\mathbb{D})$. One can show that an analytic function f in the Smirnov class whose boundary values are in $L_w^p(\partial\mathbb{D})$ belongs to H_w^p (see [A11] or [CMR, p. 36]).

In this paper we consider the extreme case $p = 1$ in the results by Protas and Ahern mentioned above and we will see that the Hardy space H^p should be replaced by the weak Hardy space H_w^1 . It is worth mentioning that the Blaschke products B for which $B' \in H_w^1$ can be described in terms of the distribution of their zeros, as it is stated in Theorem 1 below.

A Blaschke product B is called an exponential Blaschke product if there exists a constant $M = M(B) > 0$ such that for any $k = 1, 2, \dots$ one has $\#\{z : B(z) = 0, 2^{-k-1} \leq 1 - |z| \leq 2^{-k}\} \leq M$.

So zeros of exponential Blaschke products are finite unions of sequences which approach the unit circle exponentially. Verbitskii ([V2]) proved that exponential Blaschke products are precisely the inner functions whose boundary values are in certain Besov spaces. He also described exponential Blaschke products in terms of the decay of its Taylor coefficients, in terms of the smoothness of its boundary values and in terms of the integral means of its second derivative. See [V1].

Theorem 1. *Let I be an inner function. Then $I' \in H_w^1$ if and only if I is an exponential Blaschke product.*

Let I be an inner function. A classical result of Frostman (see [Fr]) tells us that there exists a set $E = E(I)$ of logarithmic capacity zero such that for any $a \in \mathbb{D} \setminus E$, the function $(I - a)/(1 - \bar{a}I)$ is a Blaschke product. A Blaschke product B is called indestructible if $E(B) = \emptyset$. This terminology was introduced in [Ml] and further results can be found in [Bi], [Mo], [Ro]. It is worth mentioning that no geometric description of indestructible Blaschke products in terms of the location of its zeros, is known. As a consequence of Theorem 1 we obtain that exponential Blaschke products are indestructible.

Corollary. *Let B be an exponential Blaschke product. Then for any $a \in \mathbb{D}$ the function $(I - a)/(1 - \bar{a}I)$ is also an exponential Blaschke product.*

Another classical result of Frostman (see [Fr]) tells that an inner function I is a Blaschke product if and only if

$$\lim_{r \rightarrow 1} \int_0^{2\pi} \log |I(re^{i\theta})| d\theta = 0.$$

Exponential Blaschke products can be described in similar terms. For $0 < r < 1$, consider

$$T(r) = T(I)(r) = \frac{1}{\log r} \frac{1}{2\pi} \int_0^{2\pi} \log |I(re^{i\theta})| d\theta, \quad 0 < r < 1.$$

It is easy to show that finite Blaschke products are precisely the inner functions for which $\sup\{T(r) : r \in [0, 1]\} < \infty$. The proof of this fact is given before the proof of Theorem 2. Exponential Blaschke products are the inner functions for which the corresponding $T(r)$ has a moderate growth, as it is stated in next result.

Theorem 2. *Let B be a Blaschke product. Then B is an exponential Blaschke product if and only if there exists a constant $M = M(B) > 0$ such that $|T(1 - 2^{-N-1}) - T(1 - 2^{-N})| \leq M$ for any $N \geq 1$.*

Given an inner function I , let $(IH^2)^\perp$ be the orthogonal complement of the subspace IH^2 in the Hardy space H^2 . For $2/3 < p < 1$, W. Cohn proved that $I' \in H^p$ if and only if $f' \in H^{2p/(p+2)}$ for any $f \in (IH^2)^\perp$. See [Co]. See also [Dy]. We have the following version in the extreme case $p = 1$.

Theorem 3. *Let B be a Blaschke product. Then $B' \in H_w^1$ if and only if there exists a constant $C = C(B) > 0$ such that for any $f \in (BH^2)^\perp$ and any $0 < r < 1$, one has*

$$|\{e^{i\theta} : |f'(re^{i\theta})| > \lambda \|f\|_2\}| \leq C\lambda^{-2/3} \tag{1.1}$$

for any $\lambda > 0$.

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2 Derivatives of exponential Blaschke products

This section is devoted to the proof of Theorem 1.

Proof of Theorem 1.

Necessity. Since $I' \in H_w^1 \subset H^{1/2}$, a result of Ahern and Clark already mentioned in the Introduction ([AC]) gives that I is a Blaschke product. We will use the notation $B = I$. Let $\{z_n\}$ be the zeros of B ordered so that $|z_n| \leq |z_{n+1}|$, $n = 1, 2, \dots$. Assume $B' \in H_w^1$. Then

$$|B'(\xi)| = \sum_n \frac{1 - |z_n|^2}{|\xi - z_n|^2}, \quad \text{a.e. } |\xi| = 1$$

(see [AC, Corollary 3]). We will prove that B is an exponential Blaschke product by contradiction. So, assume that there exists a sequence of integers n_k with $\lim_{k \rightarrow \infty} (n_{k+1} - n_k) = \infty$ such that $2^{-k-1} < 1 - |z_n| \leq 2^{-k}$ for any n with $n_k \leq n < n_{k+1}$. Let J_n be the arc on the unit circle centered at $z_n/|z_n|$ of length $2\pi(1 - |z_n|)$. For $\xi \in J_n$ we have $|\xi - z_n| \leq (\pi + 1)(1 - |z_n|)$. Hence

$$|B'(\xi)| \geq \frac{1}{(\pi + 1)^2} \frac{1}{1 - |z_{n_k}|}, \quad \xi \in F_k,$$

where

$$F_k = \bigcup_{n=n_k}^{n_{k+1}-1} J_n.$$

Since $B' \in H_w^1$, there exists a constant $c_1 > 0$, independent of k , such that $|F_k| \leq c_1 |J_{n_k}|$. Since $\lim_{k \rightarrow \infty} (n_{k+1} - n_k) = \infty$ and $|J_n| \geq |J_{n_k}|/2$ for any $n_k \leq n < n_{k+1}$, there exists $\xi_k \in F_k$ such that the set of indices $\mathcal{N}_k = \{n : n_k \leq n < n_{k+1}, \xi_k \in J_n\}$ satisfies $\#\mathcal{N}_k \rightarrow \infty$ as $k \rightarrow \infty$. Pick $m_k \in \mathcal{N}_k$. For any $n \in \mathcal{N}_k$ and any $\xi \in J_{m_k}$ we have $|\xi - \xi_k| \leq 2\pi(1 - |z_{m_k}|) \leq 4\pi(1 - |z_n|)$ and $|\xi_k - z_n| \leq (\pi + 1)(1 - |z_n|)$. Hence $|\xi - z_n| \leq (5\pi + 1)(1 - |z_n|)$. We deduce that for almost every $\xi \in J_{m_k}$ one has

$$|B'(\xi)| \geq \sum_{n \in \mathcal{N}_k} \frac{1 - |z_n|^2}{|\xi - z_n|^2} \geq \frac{1}{(5\pi + 1)^2} \sum_{n \in \mathcal{N}_k} \frac{1}{1 - |z_n|} \geq \frac{1}{2(5\pi + 1)^2} \frac{\#\mathcal{N}_k}{1 - |z_{m_k}|}.$$

Since $|J_{m_k}| = 2\pi(1 - |z_{m_k}|)$, the fact that $\#\mathcal{N}_k \rightarrow \infty$ as $k \rightarrow \infty$, contradicts that $B' \in H_w^1$.

The proof of the *sufficiency* uses the following auxiliary result.

Lemma 1. *Fix $\mu > 10$. Let $\{w_k\}$ be a sequence of points in the unit disk ordered so that $|w_k| \leq |w_{k+1}|$, $k = 1, 2, \dots$, satisfying*

$$(a) \quad 1 - |w_1| \leq 1/\mu.$$

(b) There exists $N > 0$ such that

$$C_0 = \sup_k \frac{1 - |w_{k+N}|}{1 - |w_k|} < 1.$$

Then there exist a sequence of numbers $n_k, n_k \nearrow \infty$ and a constant $K = K(C_0, N)$ such that

$$(c) \sum_k 2^{n_k} (1 - |w_k|) \leq K/\mu,$$

$$(d) \sum_k 2^{-2n_k} (1 - |w_k|)^{-1} \leq \mu.$$

Proof of Lemma 1. Choose n_k satisfying

$$\frac{1}{2^{2n_k} (1 - |w_k|)} = \frac{\mu}{100k^2}.$$

Thus (d) holds. Since $2^{2n_k} = 100k^2/\mu(1 - |w_k|)$, we have

$$\sum_k 2^{n_k} (1 - |w_k|) = \frac{10}{\mu^{1/2}} \sum_k k(1 - |w_k|)^{1/2}.$$

Condition (b) gives that $\{w_k\}$ may be split into at most N geometric progressions where, by (a), first term is smaller than $1/\mu$. Hence

$$\sum_k k(1 - |w_k|)^{1/2} \leq \frac{K(C_0, N)}{\mu^{1/2}}. \quad \square$$

We now prove the converse direction in Theorem 1. So let B be an exponential Blaschke product. The result of D. Protas mentioned in the Introduction ([P1]) gives that $B' \in H^p$ for any $p < 1$. Moreover Theorem 2 of [AC] gives

$$|B'(\xi)| = \sum_n \frac{1 - |z_n|^2}{|\xi - z_n|^2}, \quad \text{a.e. } \xi \in \partial\mathbb{D}.$$

Here $\{z_n\}$ is the sequence of zeros of B ordered so that $|z_n| \leq |z_{n+1}|, n = 1, 2, \dots$. So, by [CMR, p. 36] it will suffice to show that there exists a constant $C > 0$ such that for any $\lambda > 0$ one has

$$\left| \left\{ e^{i\theta} : \sum_n \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} > \lambda \right\} \right| \leq \frac{C}{\lambda}. \quad (2.1)$$

Fix $\lambda > 0$ and consider the set $E = E(\lambda) = \{k \in \mathbb{N} : |z_k| < 1 - M\lambda^{-1}\}$, where $M = M(\{z_n\})$ is a number depending on the sequence $\{z_n\}$ which will be fixed later. Since $|\xi - z_k| \geq 1 - |z_k|$ for any $\xi \in \partial\mathbb{D}$, we have

$$\sum_{k \in E} \frac{1 - |z_k|^2}{|\xi - z_k|^2} \leq 2 \sum_{k \in E} \frac{1}{1 - |z_k|}.$$

Let $A(l)$ be the set of indices k such that $2^{-l} \leq 1 - |z_n| < 2^{-l+1}$. Let $K = \ln_2(\lambda/M)$. Since B is an exponential Blaschke product, there exist a constant $N = N(B)$ such that for any $l \geq 1$, the number of points in $\{z_n\}$ with $2^{-l} \leq 1 - |z_n| \leq 2^{-l+1}$ is smaller than N . So

$$\sum_{k \in E} \frac{1}{1 - |z_k|} \leq \sum_{l=1}^K \sum_{n \in A(l)} \frac{1}{1 - |z_n|} < \frac{2N\lambda}{M}.$$

Choose $M = M(\{z_n\}) = 8N$ to deduce that for any $\xi \in \partial\mathbb{D}$ one has

$$\sum_{k \in E} \frac{1 - |z_k|^2}{|\xi - z_k|^2} \leq \frac{\lambda}{2}. \quad (2.2)$$

Apply Lemma 1 to the sequence $\{w_k\} = \{z_k : k \notin E\}$ and the parameter $\mu = \lambda/8N$ to get numbers $\{n_k\}$ satisfying (c) and (d). As before, let J_k be the arc on the unit circle centered at $z_k/|z_k|$ of length $2\pi(1 - |z_k|)$. Also if J is an arc in the unit circle, let NJ be the arc with the same center and length $N|J|$. Observe that

$$\left\{ e^{i\theta} : \sum_{k \notin E} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} > \frac{\lambda}{2} \right\} \subseteq \bigcup_{k \notin E} N^{-1/2} 2^{n_k} J_k. \quad (2.3)$$

Actually, if $e^{i\theta} \notin N^{-1/2} 2^{n_k} J_k$ one has $|e^{i\theta} - z_k| \geq N^{-1/2} 2^{n_k} (1 - |z_k|)$ and it follows that

$$\sum_{k \notin E} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} \leq 2N \sum_{k \notin E} \frac{1}{2^{2n_k} (1 - |z_k|)}$$

which by (d) is bounded by $\lambda/2$. So, (2.3) holds. Now observe that (c) gives that

$$\left| \bigcup_{k \notin E} 2^{n_k} J_k \right| \leq 2\pi \sum_{k \notin E} 2^{n_k} (1 - |z_k|) \leq 2\pi \frac{4KN}{\lambda}.$$

Hence, applying (2.2) and (2.3) we deduce (2.1) and the proof is completed. \square

It is worth mentioning that there exists no infinite Blaschke product B with $B' \in H_w^1$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda |\{e^{i\theta} : |B'(e^{i\theta})| > \lambda\}| = 0.$$

Actually if $\{z_n\}$ are the zeros of B , since

$$|B'(e^{i\theta})| = \sum_n \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} \quad \text{a.e. } e^{i\theta} \in \partial\mathbb{D},$$

one deduces $|B'(e^{i\theta})| \geq 1/4(1 - |z_n|)$ for any $e^{i\theta} \in \partial\mathbb{D}$ with $|e^{i\theta} - z_n| \leq 2(1 - |z_n|)$.

3 A Frostman type result

This section is devoted to the proof of Theorem 2.

Let B be a Blaschke with zeros $\{z_n\}$. Fix $0 < r < 1$. Recall the following classical calculation,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |B(re^{i\theta})| d\theta &= \sum_n \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{re^{i\theta} - z_n}{1 - \bar{z}_n r e^{i\theta}} \right| d\theta \\ &= (\log r) \#\{z_n : |z_n| \leq r\} + \sum_{n: |z_n| \geq r} \log |z_n|. \end{aligned}$$

Using the notation

$$T(r) = \frac{1}{\log r} \frac{1}{2\pi} \int_0^{2\pi} \log |B(re^{i\theta})| d\theta,$$

we have

$$T(r) = \#\{z_n : |z_n| \leq r\} + \frac{1}{\log r} \sum_{n:|z_n| \geq r} \log |z_n|. \quad (3.1)$$

Observe that $\sup\{T(r) : r \in [0, 1]\} < \infty$ if and only if B is a finite Blaschke product.

Proof of Theorem 2. Assume that B is an exponential Blaschke product. We will use the decomposition of $T(r)$ given in (3.1). Observe that there exists a constant $C > 0$ such that for $1/2 \leq r < 1$, one has

$$\frac{1}{\log r} \sum_{n:|z_n| \geq r} \log |z_n| \leq C \frac{1}{1-r} \sum_{n:|z_n| \geq r} 1 - |z_n|.$$

Since B is an exponential Blaschke product, its zeros $\{z_n\}$, ordered so that $|z_n| \leq |z_{n+1}|$, $n = 1, 2, \dots$, satisfy

$$\sup_n \frac{1 - |z_{n+K}|}{1 - |z_n|} < 1/2 \quad (3.2)$$

for a certain fixed integer $K > 0$. Thus

$$\frac{1}{1-r} \sum_{n:|z_n| \geq r} 1 - |z_n| \leq 2K, \quad 1/2 < r < 1.$$

Therefore

$$\frac{1}{\log r} \sum_{n:|z_n| \geq r} \log |z_n| \leq CK$$

for any $1/2 \leq r < 1$. So, we deduce that

$$|T(1 - 2^{-N-1}) - T(1 - 2^{-N})| \leq \#\{z_n : 1 - 2^{-N} \leq |z_n| \leq 1 - 2^{-N-1}\} + \frac{2CK}{1-\alpha}$$

for any $N = 1, 2, \dots$, which is uniformly bounded because B is an exponential Blaschke product.

Now let us show the converse. Let $\{z_n\}$ be the zeros of the Blaschke product B . Given $1/2 < r < 1$, choose r_1 such that $\log r_1^{-1} = (\log r^{-1})/2$. Using (3.1) one has

$$\begin{aligned} T(r_1) - T(r) &= \#\{z_n : r \leq |z_n| \leq r_1\} + \frac{2 \sum_{n:|z_n| \geq r_1} \log |z_n|^{-1} - \sum_{n:|z_n| \geq r} \log |z_n|^{-1}}{\log r^{-1}} \\ &= \#\{z_n : r \leq |z_n| \leq r_1\} - \frac{\sum_{n:r_1 \geq |z_n| \geq r} \log |z_n|^{-1}}{\log r^{-1}} + \frac{\sum_{n:|z_n| \geq r_1} \log |z_n|^{-1}}{\log r^{-1}}. \end{aligned}$$

Since

$$\sum_{r_1 \geq |z_n| \geq r} \log |z_n|^{-1} \leq (\log r^{-1}) \#\{z_n : r \leq |z_n| \leq r_1\}$$

we have

$$\#\{z_n : r \leq |z_n| \leq r_1\} - \frac{\sum_{r_1 \geq |z_n| \geq r} \log |z_n|^{-1}}{\log r^{-1}} \geq 0.$$

Hence

$$T(r_1) - T(r) \geq \frac{\sum_{n: |z_n| \geq r_1} \log |z_n|^{-1}}{\log r^{-1}}.$$

The estimate in the hypothesis and our choice of r_1 gives that there exists a constant $C > 0$ independent of r such that $|T(r_1) - T(r)| \leq C$. We deduce that

$$\frac{\sum_{n: |z_n| \geq r_1} \log |z_n|^{-1}}{\log r^{-1}} \leq C.$$

Pick r_2 such that $\log r_2^{-1} = (\log r^{-1})/8$ and observe

$$\#\{z_n : r_1 \leq |z_n| \leq r_2\} \leq 8 \frac{\sum_{n: r_2 \geq |z_n| \geq r_1} \log |z_n|^{-1}}{\log r^{-1}} \leq 8C.$$

Now, given $N > 0$ pick $r = (1 - 2^{-N})^2$. Then $r_1 = 1 - 2^{-N}$ and $r_2 = (1 - 2^{-N})^{1/4}$. Since $r_2 \geq 1 - 2^{-N-1}$ if N is large enough, we deduce that

$$\#\{z_n : 1 - 2^{-N} \leq |z_n| \leq 1 - 2^{-N-1}\} \leq \#\{z_n : r_1 \leq |z_n| \leq r_2\} \leq 8C.$$

Applying Theorem 1 we deduce that $B' \in H_w^1$. □

4 Derivatives of functions orthogonal to invariant subspaces

This section is devoted to the proof of Theorem 3.

Proof. Assume $B' \in H_w^1$. Let $\{z_k\}$ be the zeros of B ordered so that $|z_k| \leq |z_{k+1}|$, $k = 1, 2, \dots$. According to Theorem 1, there exists an integer $N > 0$ such that

$$\sup_k \frac{1 - |z_{k+N}|}{1 - |z_k|} < 1. \quad (4.1)$$

So, $\{z_k\}$ can be split into a finite union $\{z_k\} = \Lambda_1 \cup \dots \cup \Lambda_m$, $m = m(N)$, of sequences Λ_j satisfying

$$\sup_{k: z_k \in \Lambda_j} \frac{1 - |z_{k+1}|}{1 - |z_k|} < \frac{1}{2}, \quad \text{for any } j = 1, \dots, m.$$

Let B_j be the Blaschke product with zeros Λ_j . Since $B = B_1, \dots, B_m$, it is enough to prove the estimate for any $f \in (B_i H^2)^\perp$, $i = 1, \dots, m$. In other words, one can assume that $N = 1$ in equation (4.1). So, assume

$$\sup_k \frac{1 - |z_{k+1}|}{1 - |z_k|} < \frac{1}{2}. \quad (4.2)$$

Let $f(z) = \sum_{k=1}^M \beta_k (1 - |z_k|)^{1/2} / (1 - \bar{z}_k z) \in (BH^2)^\perp$. Since (4.2) holds, $\{z_k\}$ is an interpolating sequence. Then there exists a constant $C = C(\{z_n\})$ such that

$$C^{-2} \sum_{k=1}^M |\beta_k|^2 \leq \|f\|_2^2 \leq C^2 \sum_{k=1}^M |\beta_k|^2$$

(see Theorem B in [Co]). Hence

$$\begin{aligned} |f'(e^{i\theta})| &\leq \sum_{k=1}^M |\beta_k| \frac{(1 - |z_k|)^{1/2}}{|1 - \bar{z}_k e^{i\theta}|^2} \leq \left(\sum_{k=1}^M |\beta_k|^2 \right)^{1/2} \left(\sum_{k=1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} \right)^{1/2} \\ &\leq C \|f\|_2 \left(\sum_{k=1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} \right)^{1/2} \end{aligned}$$

Fix $\lambda > 0$. We have

$$\{e^{i\theta} : |f'(e^{i\theta})| > \lambda\} \subseteq \left\{ e^{i\theta} : \sum_{k=1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} > \frac{\lambda^2}{C^2 \|f\|_2^2} \right\}.$$

Let k_0 be the largest integer between 1 and M such that

$$\frac{4}{(1 - |z_{k_0}|)^3} \leq \frac{\lambda^2}{C^2 \|f\|_2^2}.$$

Since we can assume λ is large, the number k_0 exists. Observe that by (4.2),

$$\sum_{k=1}^{k_0} \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} \leq \sum_{k=1}^{k_0} \frac{1}{(1 - |z_k|)^3} \leq \frac{2}{(1 - |z_{k_0}|)^3}.$$

Hence

$$\{e^{i\theta} : |f'(e^{i\theta})| > \lambda\} \subseteq \left\{ e^{i\theta} : \sum_{k=k_0+1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} > \frac{\lambda^2}{2C^2 \|f\|_2^2} \right\}.$$

Pick $N_k > 0$ satisfying

$$\frac{1}{N_k^4 (1 - |z_k|)^3} = \frac{\lambda^2}{10C^2 \|f\|_2^2 (k - k_0)^2}, \quad k = k_0 + 1, \dots, M.$$

Let $N_k I_k$ denote the arc on the unit circle centered at $z_k/|z_k|$ of length $2N_k(1 - |z_k|)$. We claim that

$$\left\{ e^{i\theta} : \sum_{k=k_0+1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} > \frac{\lambda^2}{2C^2 \|f\|_2^2} \right\} \subseteq \bigcup_{k=k_0+1}^M N_k I_k. \quad (4.3)$$

Actually if $e^{i\theta} \notin \bigcup_{k=k_0+1}^M N_k I_k$, we have

$$\sum_{k=k_0+1}^M \frac{1 - |z_k|}{|1 - \bar{z}_k e^{i\theta}|^4} \leq \sum_{k=k_0+1}^M \frac{1}{N_k^4 (1 - |z_k|)^3} \leq \frac{\lambda^2}{2C^2 \|f\|_2^2}.$$

So, (4.3) holds. We deduce that

$$\begin{aligned} |\{e^{i\theta} : |f'(e^{i\theta})| > \lambda\}| &\leq \sum_{k_0+1}^M N_k(1 - |z_k|) \\ &= \frac{10^{1/4} C^{1/2} \|f\|_2^{1/2}}{\lambda^{1/2}} \sum_{k_0+1}^M (k - k_0)^{1/2} (1 - |z_k|)^{1/4}. \end{aligned}$$

Now (4.2) and the choice of k_0 gives

$$\begin{aligned} \sum_{k_0+1}^M (k - k_0)^{1/2} (1 - |z_k|)^{1/4} &\leq 10(1 - |z_{k_0+1}|)^{1/4} \\ &\leq 10 \left(\frac{4C^2 \|f\|_2^2}{\lambda^2} \right)^{1/12}. \end{aligned}$$

We deduce

$$|\{e^{i\theta} : |f'(e^{i\theta})| > \lambda\}| \leq \frac{20C^{2/3} \|f\|_2^{2/3}}{\lambda^{2/3}}$$

for any f which is a finite linear combination of $(1 - |z_k|)^{1/2}/(1 - \bar{z}_k z)$. Since these functions are dense in $(BH^2)^\perp$ we deduce that (1.1) holds.

Let us now prove the converse. Let $\{z_n\}$ be the sequence of zeros of B . For $m \geq 1$, let E_m be the annuli $E_m = \{z : 1 - 2^{-m} \leq |z| \leq 1 - 2^{-m-1}\}$. We will show that there exists a constant $K > 0$ such that $\#\{z_n : z_n \in E_m\} \leq K$ for any $m = 1, 2, \dots$. Then, according to Theorem 1, it would follow that $B' \in H_w^1$. Fix $m \geq 1$. Split E_m into 2^m truncated sectors Q_j , that is, $E_m = \bigcup_{j=0}^{2^m-1} Q_j$, where

$$Q_j = \{z = re^{i\theta} \in E_m : |\theta - 2\pi j 2^{-m}| < \pi 2^{-m}\}, \quad 0 \leq j \leq 2^m - 1.$$

The proof is organized in two steps. First we will show that there exists at most a fixed number (independent of m) of sectors Q_j which contain a point of the sequence $\{z_n\}$. Second, we will show that each Q_j can contain at most a fixed number (independent of j and m) of points of the sequence $\{z_n\}$.

Let us group the sectors $\{Q_j\}$ into ten families G_ℓ , $\ell = 1, \dots, 10$, defined as $G_\ell = \{Q_j\}_j$ where the index j runs over all indices j such that $j = \ell \pmod{10}$, that is, $j = \ell + k10$ for a certain integer k . See Figure 1.

Figure 1: G_1 consists of the two shadowed sectors.

For each sector Q_j with $Q_j \cap \{z_n\} \neq \emptyset$, pick a point in $Q_j \cap \{z_n\}$ and name it $z_j \in Q_j$. Fix $\ell = 1, \dots, 10$ and consider the function

$$f_\ell(z) = f_{\ell,m}(z) = \sum_{j: Q_j \in G_\ell} \frac{(1 - |z_j|^2)^{1/2}}{1 - \bar{z}_j z}.$$

Fix $z_k \in Q_k \in G_\ell$. If $z = e^{i\theta}$, $|\theta - 2\pi k 2^{-m}| < \pi 2^{-m}$, we have $|e^{i\theta} - z_k| \leq 52^{-m}$ and $|e^{i\theta} - z_j| > 8\pi 2^{-m}$ if $j \neq k$. Thus

$$\begin{aligned} |f'_\ell(z)| &\geq \frac{(1 - |z_k|)^{1/2} |\bar{z}_k|}{|1 - \bar{z}_k z|^2} - \sum_{j \neq k} \frac{2(1 - |z_j|)^{1/2}}{|1 - \bar{z}_j z|^2} \\ &\geq \frac{2^{-m/2} 1/2}{25 \cdot 2^{-2m}} - \sum_{i=1}^m \frac{2 \cdot 2^{-m/2}}{(8\pi 2^{-m_i})^2} \geq \frac{2^{3m/2}}{100}. \end{aligned} \quad (4.4)$$

Let H_ℓ be the subfamily of G_ℓ consisting of these sectors $Q_j \in G_\ell$ with $\{z_n\} \cap Q_j \neq \emptyset$. Estimate (4.4) gives that

$$\left\{ e^{i\theta} : |f'_\ell(e^{i\theta})| > \frac{2^{3m/2}}{10} \right\} \supseteq \bigcup \{e^{i\theta} : |\theta - 2\pi j 2^{-m}| < \pi 2^{-m}\} \quad (4.5)$$

where the union is taken over all $0 \leq j \leq 2^m - 1$ such that $Q_j \in H_\ell$. Since (1.1) holds, there exists a constant $C > 0$ such that

$$|\{e^{i\theta} : |f'_\ell(e^{i\theta})| > \lambda\}| \leq C \lambda^{-2/3} \|f_\ell\|_2^{2/3} \quad (4.6)$$

for any $\lambda > 0$. Since the points $\{z_j\}$ which appear in the definition of f_ℓ form an interpolating sequence with fixed constants (independent of ℓ and m), we have that $\|f_\ell\|_2^2$ is comparable to $\#H_\ell$ (see Theorem B in [Co]). Taking $\lambda = 2^{3m/2}/100$ in (4.6) and applying (4.5) we get

$$2^{-m} \#H_\ell \leq C_1 2^{-m} (\#H_\ell)^{1/3}.$$

Hence $\#H_\ell \leq C_1^{3/2}$. Adding over $\ell = 1, \dots, 10$ we deduce $\#\{Q_j : Q_j \cap \{z_n\} \neq \emptyset\} \leq 10C_1^{3/2}$.

Let N_j be the number of points of $\{z_n\}$ contained in Q_j . Next we will show that there exists a constant $K > 0$, independent of j and m , such that $N_j \leq K$. Fix $p < 1$ and observe that (1.1) gives that $h' \in H_w^{2/3} \subset H^{2p/(p+2)}$ for any $h \in (BH^2)^\perp$. So the result of Cohn ([Co]) gives that $B' \in H^p$. In particular B' has non-tangential limits at almost every point of the unit circle. We will use an idea of W. Cohn (see the proof of Theorem 2 in [Co]) which we collect in the following statement.

Claim. *Assume (1.1) holds. Then for any $h \in (BH^2)^\perp$ and any $\lambda > 0$ one has*

$$|\{e^{i\theta} : |B'(e^{i\theta})h(e^{i\theta})| > \lambda\}| \leq 3C \left(\frac{\|h\|_2}{\lambda} \right)^{2/3}.$$

Proof of the Claim. It is well known that any $f \in (BH^2)^\perp$ can be written as

$$f(z) = B(z) \frac{1}{z} \overline{h\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{C} \setminus \{z_k\} \cup \left\{ \frac{1}{\bar{z}_k} \right\} \quad (4.7)$$

where $h \in (BH^2)^\perp$. Hence

$$f'(z) = B'(z) \frac{1}{z} \overline{h\left(\frac{1}{\bar{z}}\right)} - B(z) \frac{1}{z^2} \overline{h\left(\frac{1}{\bar{z}}\right)} - B(z) \frac{1}{z^3} \overline{h'\left(\frac{1}{\bar{z}}\right)}.$$

Since $h \in (BH^2)^\perp \subset H_w^{2/3}$, (1.1) gives that

$$|\{e^{i\theta} : |B'(e^{i\theta})h(e^{i\theta})| > \lambda\}| \leq 3C \left(\frac{\|h\|_2}{\lambda} \right)^{2/3}.$$

Since the functions h which arise from functions f in (4.7) are dense in $(BH^2)^\perp$, the Claim is proved. \square

Fix a sector Q_j of the form $Q_j = \{re^{i\theta} : 1 - 2^{-m} \leq r < 1 - 2^{-m-1}, |\theta - 2\pi j 2^{-m}| < \pi 2^{-m}\}$ and recall that $N_j = \#\{n : z_n \in Q_j\}$. Pick a point $z_j \in Q_j \cap \{z_n\}$ and consider the function

$$h(z) = \frac{(1 - |z_j|^2)^{1/2}}{1 - \bar{z}_j z}.$$

Note that $h \in (BH^2)^\perp$, $\|h\|_2 = 1$ and $|h(e^{i\theta})| > 2^{m/2}/10$ if $|\theta - 2\pi j 2^{-m}| < \pi 2^{-m}$. Also, if $|\theta - 2\pi j 2^{-m}| < \pi 2^{-m}$, we have

$$e^{i\theta} \frac{B'(e^{i\theta})}{B(e^{i\theta})} = \sum_n \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} \geq \sum_{n: z_n \in Q_j} \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} \geq \frac{2^m N_j}{50}.$$

Choose $\lambda = 2^{\frac{3m}{2}} N_j / 500$ to obtain

$$\{e^{i\theta} : |\theta - 2\pi j 2^{-m}| < \pi 2^{-m}\} \subseteq \{e^{i\theta} : |B'(e^{i\theta})h(e^{i\theta})| > \lambda\}.$$

Applying the Claim we deduce

$$\pi 2^{-m} \leq C \left(\frac{500}{N_j 2^{\frac{3m}{2}}} \right)^{2/3}.$$

Hence

$$N_j \leq 500C^{3/2}.$$

This finishes the proof. \square

Observation. The only Blaschke products B for which $f' \in H^{2/3}$ for all $f \in (BH^2)^\perp$ are the finite ones.

Proof. We argue by contradiction. Let B be an infinite Blaschke product such that $f' \in H^{2/3}$ for any $f \in (BH^2)^\perp$. Let $\{z_n\}$ be the zeros of B . Taking a subsequence if necessary, we can assume that $\{z_n\}$ is an interpolating sequence. Arguing as in the previous Claim, one gets that

$$B'h \in H^{2/3} \text{ for any } h \in (BH^2)^\perp. \quad (4.8)$$

Pick a sequence $\{w_n\}$ of complex values such that

$$\begin{aligned} \sum |w_n|^2 (1 - |z_n|) &< \infty, \\ \sum |w_n|^{2/3} (1 - |z_n|)^{1/3} &= \infty. \end{aligned}$$

Since $\{z_n\}$ is an interpolating sequence, one can choose $h \in (BH^2)^\perp$ such that $h(z_n) = w_n$, $n \geq 1$. Since $\sum(1 - |z_n|)\delta_{z_n}$ is a Carleson measure, from (4.8) one deduces

$$\sum |B'(z_n)h(z_n)|^{2/3}(1 - |z_n|) < \infty.$$

Since $\{z_n\}$ is an interpolating sequence, one has $\inf_n |B'(z_n)|(1 - |z_n|) > 0$. Hence

$$\sum |w_n|^{2/3}(1 - |z_n|)^{1/3} < \infty$$

which gives the contradiction. □

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