FROSTMAN SHIFTS OF INNER FUNCTIONS

By

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Abstract. In this paper we study the class $\mathcal{M}$ of all inner functions whose non-zero Frostman shifts are Carleson–Newman Blaschke products. We present several geometric, measure theoretic and analytic characterizations of $\mathcal{M}$ in terms of level sets, distribution of zeros, and behaviour of pseudohyperbolic derivatives and observe that $\mathcal{M}$ is the set of all functions in H^∞ whose range on the set of trivial points in the maximal ideal space is $\partial\mathbb{D} \cup \{0\}$.

1 Introduction

Let H^∞ be the algebra of bounded analytic functions in the open unit disc $\mathbb{D}$ of the complex plane and let

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \quad z,w \in \mathbb{D}$$

be the pseudohyperbolic distance in $\mathbb{D}$. A pseudohyperbolic disc centered at $z \in \mathbb{D}$ of radius $0 < \delta < 1$ is given by $D_\rho(z,\delta) = \{w \in \mathbb{D} : \rho(w,z) \leqslant \delta\}$. A sequence of points (z_n) in $\mathbb{D}$ is called an interpolating sequence if for any bounded sequence (w_n) of complex numbers, there exists $f \in H^\infty$ such that $f(z_n) = w_n$ for all $n \in \mathbb{N}$. A celebrated result of L. Carleson [Ca] asserts that (z_n) is interpolating if and only if $\rho(z_n, z_m) \geq \delta > 0$ for all $n \neq m$ and $\mu = \sum_{n=1}^\infty (1-|z_n|^2)\delta_{z_n}$ is a Carleson measure (see [Ga]). Recall that μ is a Carleson measure if there exists a constant $C = C(\mu) > 0$ such that $\mu(Q) \leqslant C\ell$ for any box Q of the form

$$Q = Q(\theta_0, \ell) := \left\{re^{i\theta} : 0 \leqslant 1 - r \leqslant \ell,\, |\theta - \theta_0| \leqslant \ell\right\}.$$

A Blaschke product

$$B(z) = \prod_{n=1}^\infty \frac{|z_n|}{z_n}\frac{z_n - z}{1 - \overline{z}_n z}$$

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with distinct zeros z_n is called an interpolating Blaschke product if (z_n) is an interpolating sequence, or equivalently, if

$$\inf_n (1 - |z_n|^2)|B'(z_n)| > 0.$$

Interpolating Blaschke products play a crucial role in the theory of H^∞. For instance, they appear naturally when studying the structure of the maximal ideal space $M(H^\infty)$ of H^∞. The pseudohyperbolic distance ρ can be extended to $M(H^\infty)$ by defining

$$\rho(m, \tilde{m}) = \sup\{|f(m)| : f \in H^\infty, \|f\|_\infty \leqslant 1, f(\tilde{m}) = 0\}.$$

Here, as usual, we are identifying f with its Gelfand transform, that is, $f(m) = m(f)$. The Gleason part of a point $m \in M(H^\infty)$ is defined as

$$P(m) = \{\tilde{m} \in M(H^\infty) : \rho(m, \tilde{m}) < 1\}.$$

It is a fundamental result of K. Hoffman that either $P(m)$ is a singleton or $P(m)$ is an analytic disc. Moreover, for every $m \in M(H^\infty)$, there exists a continuous map L_m from $\mathbb{D}$ onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m \in H^\infty$ whenever $f \in H^\infty$ ([Ho$_2$]). The Hoffman map L_m has the form

$$L_m(z) = \lim_\alpha \frac{z + z_\alpha}{1 + \bar{z}_\alpha z},$$

where (z_α) is any net of points in $\mathbb{D}$ converging to m in the weak-$*$-topology of $M(H^\infty)$. Hoffman also proved that a point $m \in M(H^\infty)$ has a nontrivial Gleason part, that is, $P(m) \neq \{m\}$, if and only if $m \in \mathbb{D}$ or m lies in the weak-*-closure of an interpolating sequence in $\mathbb{D}$. In that case, L_m is a bijection. The set of points in $M(H^\infty)$ with nontrivial Gleason part is denoted by G.

A Blaschke product B with zero sequence (z_n) is called a Carleson–Newman Blaschke product if the measure

$$\nu = \sum_{n=1}^{\infty} (1 - |z_n|^2)\delta_{z_n}$$

is a Carleson measure or, equivalently, if B is the product of finitely many interpolating Blaschke products (see [MS], [McK]). It is well-known that ν is a Carleson measure if and only if

$$\sup_{\varphi \in \operatorname{Aut}\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi(z_n)|^2) < \infty,$$

where Aut $\mathbb{D}$ is the set of all conformal maps of $\mathbb{D}$ onto itself (see [Ga]).

A bounded analytic function I is called inner if for almost all $e^{i\theta} \in \partial\mathbb{D}$,

$$\left|\lim_{r\to 1} I(re^{i\theta})\right| = 1.$$

An inner function I can be decomposed as $I = e^{i\theta}BS_\mu$, where B is a Blaschke product and

$$S_\mu(z) = \exp\Big(-\int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} d\mu(\xi)\Big), \quad z \in \mathbb{D},$$

is a singular inner function associated with the positive finite singular Borel measure μ.

For $\alpha \in \mathbb{D}$, let τ_α denote the automorphism of $\mathbb{D}$ sending α to 0, defined as $\tau_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha}z)$, $z \in \mathbb{D}$. It is clear that $\tau_\alpha \circ I$ is inner if I is inner. Actually, a result of O. Frostman asserts that $\tau_\alpha \circ I$ is a Blaschke product for any $\alpha \in \mathbb{D}$ except, possibly, for a set of logarithmic capacity 0 ([Fr] and [Ga]).

Several authors ([To$_2$], [Ni], [GuIz], [MT]) have studied Carleson–Newman Blaschke products B for which $\tau_\alpha \circ B$ is also a Carleson–Newman Blaschke product for any $\alpha \in \mathbb{D}$. Let us denote this set of functions by $\mathcal{P}$ (see [To$_2$]). Obviously, every finite Blaschke product is in $\mathcal{P}$. But $\mathcal{P}$ is much bigger than that. In fact, let B be a thin Blaschke product; that is, a Blaschke product with zero sequence (z_n) satisfying

$$\lim_{n\to\infty} (1 - |z_n|^2)|B'(z_n)| = 1.$$

It is easy to check that thin Blaschke products are in $\mathcal{P}$. This class $\mathcal{P}$ plays an important role in studying closed subalgebras of $L^\infty(\mathbb{D})$ (see [GoIz]). Certain elements of $\mathcal{P}$ also appear in the study of almost isometries in the hyperbolic plane (see [GP]). It is well-known that the elements in $\mathcal{P}$ can be characterized in terms of the distribution of their zeros ([Ni]) or in terms of their behavior on the maximal ideal space ([To$_2$], [GuIz], [MT]). For example, $\mathcal{P}$ is the set of functions in H^∞ which are unimodular on the set of trivial points in $M(H^\infty)$.

The main purpose of this paper is to study the class $\mathcal{M}$ of inner functions I for which $\tau_\alpha \circ I$ is a Carleson–Newman Blaschke product for any $\alpha \in \mathbb{D} \setminus \{0\}$. Note that $\mathcal{P} \subseteq \mathcal{M}$. Whereas $\mathcal{P}$ only contains Blaschke products, it is easily seen that the singular inner function $S(z) = \exp\big((1+z)/(z-1)\big)$, which corresponds to the Dirac measure at the point 1, is in $\mathcal{M}$. Actually, a straightforward calculation shows that for any $\alpha \in \mathbb{D} \setminus \{0\}$, the zeros of $\tau_\alpha \circ S$ form an interpolating sequence (see [GoIz], [Mo]). It follows from Hoffman's theory that $\{m \in M(H^\infty) : 0 < |S(m)| < 1\} \subset G$. Since this property is preserved by finite products, we deduce that $S_\mu \in \mathcal{M}$ whenever μ is a finite linear combination of Dirac measures. The main goal of this paper is to present some other nontrivial examples of singular inner functions in $\mathcal{M}$. The construction of some of our examples is based on the following characterization of the functions in $\mathcal{M}$, a result which we prove in Section 3.

Theorem 1. *Let I be an inner function. Then the following assertions are equivalent.*

(a) *$I \in \mathcal{M}$, that is, $\tau_\alpha \circ I$ is a Carleson–Newman Blaschke product for any $\alpha \in \mathbb{D} \setminus \{0\}$.*

(b) *For every $0 < \varepsilon < 1$, there exists $c = c(\varepsilon, I) < 1$ such that the set $\{z \in \mathbb{D} : \varepsilon < |I(z)| < 1 - \varepsilon\}$ does not contain any pseudohyperbolic disc of radius bigger than c.*

(c) *For every $0 < \varepsilon < 1$, there exists $\eta \in]0, 1[$ and $\delta \in]0, 1[$ such that for any $z \in \mathbb{D}$ with $\varepsilon < |I(z)| < 1 - \varepsilon$, one can find $w \in \mathbb{D}$ with $\rho(z, w) < \delta$ such that $(1 - |w|^2)|I'(w)| > \eta$.*

(d) *For any trivial point $m \in M(H^\infty)$, that is $P(m) = \{m\}$, one has either $|I(m)| = 1$ or $I(m) = 0$.*

(e) *For any $m \in G$, either $I \circ L_m$ is an inner function or $I \circ L_m \equiv 0$. Here L_m is the Hoffman map corresponding to m.*

A positive measure is called discrete if it is a weighted, finite or countably infinite sum of point masses. A positive measure μ is called continuous if it does not give any mass to any singleton. A singular inner function S_μ is called discrete (respectively, continuous) if μ is a discrete (respectively, continuous) measure.

We use part (b) of Theorem 1 to find examples of nontrivial discrete singular inner functions in $\mathcal{M}$. We also give examples of both discrete and continuous singular inner functions not belonging to $\mathcal{M}$. Moreover, we characterize the compact sets $E \subseteq \partial\mathbb{D}$ for which $S_\mu \in \mathcal{M}$ for any positive measure μ supported on E. To this end, recall that a compact set $E \neq \emptyset$ of the unit circle is called porous if there exists $0 < \varepsilon < 1$ such that for any arc $J \subset \partial\mathbb{D}$ with $J \cap E \neq \emptyset$, there exists a subarc $\tilde{J} \subset J$, $|\tilde{J}| > \varepsilon|J|$, such that $\tilde{J} \cap E = \emptyset$. For example, the set $\{e^{i2^{-k}} : k \in \mathbb{N}\} \cup \{1\}$ is porous for $\varepsilon = (1/4)\frac{1}{2\pi}$, whereas $\{e^{ik^{-1}} : k \in \mathbb{N}\} \cup \{1\}$ is not porous. Moreover, the usual 1/3-Cantor set is porous. Finally, we mention that any porous subset of $\partial\mathbb{D}$ has one-dimensional Lebesgue measure zero.

Our result is the following:

Theorem 2. *Let E be a compact subset of the unit circle. Then, the following conditions are equivalent.*

(a) *For any positive singular measure μ supported on E, the inner function S_μ belongs to $\mathcal{M}$, that is, $\tau_\alpha \circ S_\mu$ is a Carleson–Newman Blaschke product for any $\alpha \in \mathbb{D} \setminus \{0\}$.*

(b) *E is porous.*

This allows us to give a first example of a continuous singular inner function in $\mathcal{M}$.

It is well-known that the behaviour of the modulus of an inner function (or, more generally, an arbitrary bounded analytic function) is related to the behaviour of the Poisson integral of a certain positive measure ([Bi]). More concretely, given an inner function I which decomposes as $I = B \cdot S_\sigma$, where B is a Blaschke product with zero sequence (b_n), we consider the measure

$$\mu = \mu_I = \frac{1}{2} \sum (1 - |b_n|^2)\delta_{b_n} + \sigma. \tag{1.1}$$

One of our major goals is to characterize the inner functions I in $\mathcal{M}$ in terms of the behaviour of their corresponding measures μ_I. To state our result, we need some more notation. Given a point $z \in \mathbb{D}$, denote by $J(z)$ the arc centered at $e^{i \arg z}$ of length $2(1-|z|)$, by $Q(z)$ the Carleson box $Q(e^{i \arg z}, 1-|z|)$ with base $J(z)$, that is,

$$J(z) = \left\{e^{i\theta} : |\theta - \arg z| \leqslant 1 - |z|\right\},$$
$$Q(z) = \left\{re^{i\theta} : 0 \leqslant 1 - r \leqslant 1 - |z|, |\theta - \arg z| \leqslant 1 - |z|\right\},$$

and by $\ell(Q(z)) := 1 - |z|$ (respectively $\ell(J(z))$) the length of the Carleson box $Q(z)$ (respectively, its base $J(z)$). Moreover, for $N > 0$, $NQ(z)$ is the Carleson box $Q(e^{i \arg z}, N(1-|z|)$. Note that $NQ(z) = \overline{\mathbb{D}}$ if N is sufficiently big. Arclength of an arc $J \subseteq \partial\mathbb{D}$ is denoted by $|J|$. Finally, $P_z(w) = (1-|z|^2)/|1-\bar{w}z|^2$ stands for the Poisson kernel. Here $z \in \mathbb{D}$ and $w \in \overline{\mathbb{D}}$.

Theorem 3. *Let I be an inner function and $\mu = \mu_I$ be the measure defined in (1.1). The following two conditions are equivalent.*

(a) *$I \notin \mathcal{M}$, that is, there exists $\alpha \in \mathbb{D} \setminus \{0\}$ such that $\tau_\alpha \circ I$ is not a Carleson–Newman Blaschke product.*

(b) *There exist $C > 0$, $z_n \in \mathbb{D}$, $|z_n| \to 1$, $0 < m_n < 1$, $m_n \to 1$ and integers $n(N) \to \infty$ as $N \to \infty$, such that*

(b.1)
$$\sup \left\{ \left| \frac{\mu(Q(z))}{\ell(Q(z))} - C \right| : \rho(z, z_n) \leqslant m_n \right\} \xrightarrow[n\to\infty]{} 0$$

and

(b.2)

$$\sup_{n \geq n(N)} \sup_{z \in D_\rho(z_n, m_n)} \int_{\overline{\mathbb{D}} \setminus NQ(z)} P_z(w) d\mu(w) \xrightarrow[N \to \infty]{} 0.$$

The proof of Theorem 3 uses several ideas from [Bi]. Condition (b.1) will be used to construct further examples of continuous singular inner functions in $\mathcal{M}$. Furthermore, condition (b.1) can also be applied to obtain examples of Blaschke products in $\mathcal{M}$, but not in $\mathcal{P}$.

Corollary 4. *Any Blaschke product whose zero set lies in a Stolz angle belongs to the class $\mathcal{M}$.*

The paper is organized as follows. In Section 2, some known results on Carleson–Newman Blaschke products are collected. Section 3 is devoted to the proof of Theorem 1, which is applied in Section 4 to present examples of discrete singular inner functions in $\mathcal{M}$. Theorem 2 is proved in Section 4, and a first example of a continuous singular inner function in $\mathcal{M}$ is given there. Section 5 contains the proof of Theorem 3, which is applied in Section 6 to present other examples of continuous singular inner functions in $\mathcal{M}$ and to prove Corollary 4. The paper concludes with some observations and questions.

As usual, the letter C denotes an absolute constant whose value may change from line to line. Also, constants which only depend on a given parameter m are denoted by $C_i(m)$, $i = 1, 2, \ldots$.

2 Carleson–Newman Blaschke products

We begin with a Lemma on the relation between pseudohyperbolic discs and Stolz domains.

Lemma 2.1. (a) *Let $\Gamma(M) = \{z \in \mathbb{D} : |z| > 1/2, |z-1| < M(1-|z|)\}$ be a Stolz domain. Then $\sup\{r > 0 : \exists\, D_\rho(z,r) \subseteq \Gamma(M)\} < 1$.*

(b) *Let $0 < r(M) < 1$ converge to 1 as M goes to infinity. Then there exist pseudohyperbolic discs D of radius $\rho(M)$, $\rho(M) \to 1$ as $M \to \infty$, such that $D \subseteq \Gamma(M) \cap \{z \in \mathbb{D} : |z| > r(M)\}$.*

Proof. Let us recall that $D_\rho(z,r)$ coincides with the euclidean disc K centered at $\frac{1-r^2}{1-r^2|z|^2} z$ and euclidean radius $R = r\frac{1-|z|^2}{1-r^2|z|^2}$. We may assume that $z \in [0,1[$ and that $r < |z|$. Hence, by ([Ga], p. 3), the smallest distance of a point in $D_\rho(z,r)$ to 0 is $A = \frac{|z|-r}{1-r|z|}$, the biggest is $B = \frac{|z|+r}{1+r|z|}$.

If $D_\rho(z,r) \subseteq \Gamma(M)$, then $R/(1-B) < \tan\alpha$ for some $\alpha = \alpha(M) \in]0, \pi/2[$. But $R/(1-B) = \frac{r(1+|z|)}{(1-r|z|)(1-r)}$. This yields both assertions (a) and (b). □

One can find in the literature many different descriptions of the Carleson–Newman Blaschke products. We collect some of them in the following result for further reference.

Theorem 2.2. [Ho_2], [KL], [McK], [GIS], [To_2] *Let I be an inner function. The following conditions are equivalent.*

(a) *I is a Carleson–Newman Blaschke product.*

(b) *I is a Blaschke product whose zero set is a finite union of interpolating sequences.*

(c) *There exist $\varepsilon = \varepsilon(I) > 0$ and $\delta = \delta(I) < 1$ such that the set $\{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ does not contain any pseudohyperbolic disc of radius bigger than δ.*

(d) *There exist $\varepsilon > 0$ and $\eta > 0$ such that for any $z \in \mathbb{D}$ with $|I(z)| < \varepsilon$, one can find $\tilde{z} \in \mathbb{D}$, $\rho(\tilde{z}, z) < 1 - \varepsilon$, such that*

$$(1 - |\tilde{z}|^2)|I'(\tilde{z})| > \eta.$$

(e) *The zeros of I on the maximal ideal space $M(H^\infty)$ lie in the set G of non-trivial points.*

Proof. The equivalence between (a) and (b) is due to [McK] and follows from the observation that if $\nu = \sum(1 - |z_n|^2)\delta_{z_n}$ is a Carleson measure, there exists a constant $N \in \mathbb{N}$ such that any pseudohyperbolic disc of radius $1/2$, say, contains at most N points of the sequence (z_n) (see also [MS]). The equivalence between (b) and (e) follows from Hoffman's theory (see [GIS] or [To_2]). The fact that (b) and (c) are equivalent can be found in [KL]. It remains to show that (d) is equivalent to (e) . Assume that (e) does not hold. Then there is a trivial point m with $I(m) = 0$. By [Ho_2], there exists a sequence (z_n) in $\mathbb{D}$ such that $I \circ L_{z_n}$ tends to zero locally uniformly in $\mathbb{D}$. Hence, for every $\delta \in]0, 1[$,

$$\sup_{z \in D_\rho(z_n, \delta)} (1 - |z|^2)|I'(z)| \to 0 \tag{2.1}$$

as $n \to \infty$. Thus (d) cannot hold.

On the other hand, if (d) does not hold, then we may choose for $\varepsilon = \eta = 1/n$ a point z_n with $|I(z_n)| < 1/n$ and such that $(1-|z|^2)|I'(z)| < 1/n$ on $D_\rho(z_n, 1-1/n)$. In particular, (2.1) holds. Thus $(I \circ L_m)' \equiv 0$ for any cluster point m of the (z_n). Since $I(m) = 0$, we have $I \circ L_m \equiv 0$, and so, for any $0 < \delta < 1$, $\sup_{z \in D_\rho(z_n, \delta)} |I(z)| \to 0$ as $n \to \infty$. This contradicts (c), which was shown to be equivalent to (e) . □

From part (c) of Theorem 2.2, it is obvious that the set $\alpha \in \mathbb{D}$ for which $\tau_\alpha \circ I$ is a Carleson–Newman Blaschke product is open. Observe also that as a consequence of (e), given an inner function I, the set of $\alpha \in \mathbb{D}$ for which $\tau_\alpha \circ I$ is not a Carleson–Newman Blaschke product coincides with the intersection of $\mathbb{D}$ with the range of I on the set of trivial points.

It is well-known that Carleson–Newman Blaschke products cannot have radial limit 0. Indeed, if

$$\lim_{r \to 1} |I(re^{i\theta})| = 0,$$

then $I(z)$ tends to 0 whenever z approaches $e^{i\theta}$ within any Stolz angle

$$\Gamma(M) = \left\{ z : |z - e^{i\theta}| \leqslant M(1 - |z|) \right\}.$$

Fix $\varepsilon > 0$. Let $0 < r(M) < 1$ be such that $|I| < \varepsilon$ on $\Gamma(M) \cap \{|z| > r(M)\}$. By Lemma 2.1, $\Gamma(M) \cap \{|z| > r(M)\}$ contains pseudohyperbolic discs of radii close to 1 as $M \to \infty$. Applying (c) , we would get that I is not a Carleson–Newman Blaschke product.

3 Proof of Theorem 1

We first need some auxiliary results, interesting in their own right. Let dA denote area measure and let

$$\text{BMOA} = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{BMOA} := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\tau_a(z)|^2)\, dA(z) \ < \infty \right\}$$

be the usual space of analytic functions of bounded mean oscillation. It is well-known that $\|f \circ \Phi\|_{BMOA} = \|f\|_{BMOA}$ whenever Φ is a conformal automorphism of $\mathbb{D}$ and $f \in$ BMOA. Note also that

$$\frac{1}{2}\|f - f(0)\|_2^2 \le \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)\, dA(z) \le \|f - f(0)\|_2^2,$$

where $\|f\|_2$ is the usual H^2 norm of f (see [Sh], p. 39).

Lemma 3.1. *Let I be a function in BMOA. Given $\varepsilon > 0$ and $0 < m < 1$, there exists $c = c(\varepsilon, m, \|I\|_{BMOA}) < 1$ such that*

$$\sup_{w \in D_\rho(z,m)} (1 - |w|)|I'(w)| > \varepsilon \quad \textit{for all } |z| \le r$$

implies that $r \le c$.

Moreover, c can be chosen so that for every $r_n \to 1$, there exists $\varepsilon_n \to 0$ and $m_n \to 1$ satisfying $c(\varepsilon_n, m_n, \|I\|_{BMOA}) < r_n$.

Proof. Assume that $\sup_{w\in D_\rho(z,m)}(1-|w|)|I'(w)| > \varepsilon$ for all $|z| \le r$. Let m_* be defined by $m_* = (m+1/2)/(1+m/2)$. We claim that

$$(3.1) \qquad \int_{w\in D_\rho(z,m_*)} (1-|w|^2)|I'(w)|^2 dA(w) > \widetilde{C}(m)\varepsilon^2(1-|z|),$$

where $\widetilde{C}(m)$ can be chosen to be $(1-m)C$, for some constant C. To show this, observe that for any $w \in D_\rho(z,m_*)$, the quantity $(1-|w|^2)$ is comparable to $(1-|z|)$; more precisely, we have

$$1-|w|^2 \ge \frac{1-m_*^2}{4}(1-|z|^2) \ge \frac{1}{12}(1-m)(1-|z|).$$

So we only need to show that

$$(3.2) \qquad \int_{w\in D_\rho(z,m_*)} |I'(w)|^2 dA(w) > C_1\varepsilon^2.$$

To prove (3.2), let $w(z) \in D_\rho(z,m)$ be such that

$$(1-|w(z)|^2)|I'(w(z))| = \sup_{w\in D_\rho(z,m)} (1-|w|^2)|I'(w)|.$$

Observe that the definition of m_* gives $D_\rho(w(z),1/2) \subset D_\rho(z,m_*)$. Hence,

$$(3.3) \qquad \begin{aligned}\int_{w\in D_\rho(z,m_*)} |I'(w)|^2 dA(w) &> \int_{w\in D_\rho(w(z),1/2)} |I'(w)|^2 dA(w)\\ &> C_1|I'(w(z))|^2(1-|w(z)|)^2 > C_1\varepsilon^2.\end{aligned}$$

Here the second inequality holds because $|I'|^2$ is subharmonic and $D_\rho(w(z),1/2)$ contains an euclidean disc centered at $w(z)$ of euclidean radius comparable to $(1-|w(z)|)$. This proves (3.2) and so (3.1).

Next, we multiply both sides of (3.1) by $1/(1-|z|)^2$ and integrate with respect to area measure $dA(z)$ in the disk centered at the origin and radius r. Hence

$$(3.4) \qquad \begin{aligned}&\int_{|z|\le r}\int_{w\in D_\rho(z,m_*)} (1-|z|)^{-2}(1-|w|^2)|I'(w)|^2 dA(w)dA(z)\\ &> \widetilde{C}(m)\int_{|z|\le r} \varepsilon^2(1-|z|)^{-1}dA(z) = 2\pi\tilde{C}(m)\varepsilon^2\left[\log(1/(1-r)) - r\right].\end{aligned}$$

Let Ξ_E denote the characteristic function of the set E. By Fubini, the left hand term in (3.4) is bounded above by

$$(3.5) \qquad \int_{w:|w|<1} (1-|w|^2)|I'(w)|^2 \left[\int_{|z|<1} \Xi_{D_\rho(z,m_*)}(w)/(1-|z|)^2 dA(z)\right] dA(w).$$

Using the facts that

$$C_2(1-m_*) < \frac{1-|z|}{1-|w|} < [C_2(1-m_*)]^{-1}$$

whenever $z \in D_\rho(w, m_*)$, and that the euclidean radius of $D_\rho(w, m_*)$ is bounded above by $\frac{2}{1-m_*}(1-|w|)$, we can further estimate the interior integral by

$$\int_{z:z\in D_\rho(w,m_*)} (1-|z|)^{-2}dA(z) \le \frac{C_3}{(1-m_*)^2}(1-|w|)^{-2}\mathcal{A}\big(D_\rho(w,m_*)\big) \le C(m),$$

where $C(m) = C_4(1-m)^{-4}$ and where $\mathcal{A}$ is the euclidean area. Thus the left hand term in (3.4) is bounded above by

$$C(m)\int_{|w|<1}(1-|w|^2)|I'(w)|^2dA(w) \le C(m)\,||I||^2_{BMOA}.$$

Accordingly, we have by (3.4)

$$C(m)\,||I||^2_{BMOA} > 2\pi\widetilde{C}(m)\varepsilon^2\left[\log(1/(1-r)) - r\right]. \tag{3.6}$$

From this we conclude that r has to be bounded away from 1; thus $r \le c(\varepsilon, m, ||I||^2_{BMOA}) < 1$. More precisely,

$$\log\frac{1}{1-r} - r < \frac{C_6||I||^2_{BMOA}}{\varepsilon^2(1-m)^5}.$$

Note that the function $\log\frac{1}{1-r} - r$ is strictly increasing to infinity for $r \to 1$. It is now easy to check that if $r_n \to 1$, then we may chose $\varepsilon_n \to 0$ and $m_n \to 1$ such that

$$\frac{C_6||I||^2_{BMOA}}{\varepsilon_n^2(1-m_n)^5} < \log\frac{1}{1-r_n} - r_n.$$

□

Lemma 3.2. *Let $f \in BMOA$ and let $D_n = D_\rho(a_n, \rho_n)$ be a sequence of pseudohyperbolic disks of pseudohyperbolic center a_n and pseudohyperbolic radius $\rho_n, \rho_n \to 1$ as $n \to \infty$. Then there exist pseudohyperbolic disks $D_n^* \subset D_n$ of pseudohyperbolic radius also tending to 1 such that*

$$\sup_{w\in D_n^*}(1-|w|^2)|f'(w)| \to 0 \quad \textit{as } n \to \infty. \tag{3.7}$$

Proof. Let $\widetilde{D}_n = D_\rho(a_n, r_n)$, where $0 < r_n < 1$ is defined by $2r_n/(1+r_n^2) = \rho_n$. We now apply Lemma 3.1 to the functions $f \circ \tau_n$, where τ_n is the automorphism of the unit disk interchanging a_n and the origin, $(\tau_n)^{-1} = \tau_n$. According to the second assertion in Lemma 3.1, we may choose $\varepsilon_n \to 0$ and $m_n \to 1$ such that $c(\varepsilon_n, m_n, ||f||_{\text{BMOA}}) < r_n$. Note that $||f \circ \tau_n||_{\text{BMOA}} = ||f||_{\text{BMOA}}$. By Lemma 3.1, one can find $z_n, |z_n| < r_n$, such that

$$\sup_{w \in D_\rho(z_n, m_n)} (1 - |w|^2)|(f \circ \tau_n)'(w)| < \varepsilon_n.$$

Since $(1 - |w|^2)|(f \circ \tau_n)'(w)| = (1 - |\tau_n(w)|^2)|f'(\tau_n(w))|$ and the pseudohyperbolic distance is invariant under automorphisms of the disk, we deduce

$$\sup_{\xi \in D_\rho(\tau_n(z_n), m_n)} (1 - |\xi|^2)|f'(\xi)| < \varepsilon_n.$$

Choose $D_n^* = D_\rho(\tau_n(z_n), \min\{r_n, m_n\})$. We claim that D_n^* is contained in D_n. This follows from the fact that if $w \in D_n^*$, then

$$\rho(w, a_n) < \frac{\rho(w, \tau_n(z_n)) + \rho(\tau_n(z_n), a_n)}{1 + \rho(w, \tau_n(z_n))\rho(\tau_n(z_n), a_n)} < 2r_n/(1 + r_n^2) = \rho_n,$$

because $\rho(\tau_n(z_n), a_n) = |z_n| < r_n$. □

Proof of Theorem 1. (a) $\Longrightarrow$ (b) : Let $I \in \mathcal{M}$. Assume that (b) does not hold. Then there exists $\varepsilon > 0$ such that the set $E = \{\varepsilon < |I(z)| < 1 - \varepsilon\}$ contains arbitrarily large pseudohyperbolic discs, say $D_n = D_\rho(z_n, r_n) \subseteq E$, where $r_n \to 1$. By Lemma 3.2, there exist $\alpha_n \in D_n$ and $D_n^* := D(\alpha_n, \rho_n) \subseteq D_n$, $\rho_n \to 1$, such that $\lim_n \sup_{z \in D_n^*} (1 - |z|^2)|I'(z)| = 0$. Taking a subsequence, if necessary, we may assume that $I(\alpha_n) \to w$ for some $w \in \mathbb{D} \setminus \{0\}$. Since

$$(1 - |z|^2)\,|(\tau_w \circ I)'(z)| \leq \frac{2}{1 - |w|}(1 - |z|^2)|I'(z)| \to 0,$$

uniformly on D_n^* as $n \to \infty$, we obtain by Theorem 2.2 (d) that $\tau_w \circ I$ cannot be a Carleson–Newman Blaschke product. This contradicts the assumption that $I \in \mathcal{M}$.

(b) $\Longrightarrow$ (c) : Let $z \in \mathbb{D}$ with $\varepsilon < |I(z)| < 1 - \varepsilon$. By (b) applied to $\varepsilon/2$, there exist $\delta = \delta(\varepsilon/2)$ and $\tilde{z} \in \mathbb{D}$ with $\rho(z, \tilde{z}) < \delta < 1$, such that $|I(\tilde{z})| < \varepsilon/2$ or $|I(\tilde{z})| > 1 - \varepsilon/2$. In both cases, we deduce $|I(z) - I(\tilde{z})| > \varepsilon/2$. Let γ be the hyperbolic geodesic curve joining z and $\tilde{z}$. Since

$$\begin{aligned} |I(z) - I(\tilde{z})| &\leqslant \int_\gamma |I'(w)||dw| \leqslant \big(\sup_\gamma (1 - |w|^2)|I'(w)|\big) \int_\gamma \frac{|dw|}{1 - |w|^2} \\ &< \sup_\gamma (1 - |w|^2)|I'(w)|\, C(\delta), \end{aligned}$$

there exists $w \in \gamma$ so that

$$(1 - |w|^2)|I'(w)| > \frac{\varepsilon}{2C(\delta)}$$

and (c) holds.

(c) $\Longrightarrow$ (a) : Let $\alpha \in \mathbb{D} \setminus \{0\}$. For $u = \tau_\alpha \circ I$, we show that condition (d) in Theorem 2.2 holds. Choose $\varepsilon \in]0, |\alpha|/2[$, to be determined later, sufficiently small. Then $|u(z)| < \varepsilon$ implies that

$$\frac{|\alpha| - \varepsilon}{1 - \varepsilon|\alpha|} < |I(z)| < \frac{|\alpha| + \varepsilon}{1 + \varepsilon|\alpha|}.$$

Choose $\varepsilon' > 0$ so small that

$$\varepsilon' < \frac{|\alpha| - |\alpha|/2}{1 - |\alpha||\alpha|/2} \quad \text{and} \quad \frac{|\alpha| + |\alpha|/2}{1 + |\alpha||\alpha|/2} < 1 - \varepsilon'.$$

By (c), applied to ε', there exists $\delta'(\varepsilon') > 0$ and $\eta'(\varepsilon') > 0$ such that for any $z \in \mathbb{D}$ with $\varepsilon' < |I(z)| < 1 - \varepsilon'$, there exists $w \in \mathbb{D}$ with $\rho(w, z) \leq \delta'(\varepsilon')$ and

$$(1 - |w|^2)|I'(w)| > \eta'(\varepsilon').$$

This implies that

$$(1 - |w|^2)|u'(w)| \geq \frac{1 - |\alpha|^2}{4}(1 - |w|^2)|I'(w)| > \eta > 0.$$

If $\varepsilon > 0$ is chosen so small that $\delta'(\varepsilon') < 1 - \varepsilon$, we see that condition (d) in Theorem 2.2 is fulfilled. Hence u is a Carleson–Newman Blaschke product.

That (a) is equivalent to (d) is easily seen by using Theorem 2.2, (a) $\Longleftrightarrow$ (e) and the fact that $I(x) = \alpha$ if and only if $(\tau_\alpha \circ I)(x) = 0$ whenever $\alpha \in \mathbb{D}$.

It remains to show that (d) is equivalent to (e) . Suppose that (d) does not hold. Then there exist a trivial point x and $\alpha \in \mathbb{D} \setminus \{0\}$ such that $I(x) = \alpha$. The inner function $u = \tau_\alpha \circ I$ vanishes at x. By [GM], there exists a nontrivial part $P(m)$ such that u vanishes identically on $P(m)$. Hence $I \equiv \alpha$ on $P(m)$; and so $I \circ L_m$ is the constant function α, which is surely not inner.

To show that (d) implies (e) , let $m \in G$ and consider the inner-outer factorization of $f = I \circ L_m$. Then $f = uF$, where u is inner and F is outer. We may assume that f is not identically zero. Let $x \in M(L^\infty)$. Note that $L_m(x)$ is trivial whenever x is trivial (see [Bu]). Then $f(x) = I(L_m(x)) \in \{0\} \cup \partial\mathbb{D}$. Since $|u(x)| = 1$, we have $F(x) \in \{0\} \cup \partial\mathbb{D}$. Thus, the range R of F on the Shilov boundary ∂H^∞ of H^∞, is contained in $\{0\} \cup \partial\mathbb{D}$. Note that R coincides with the essential range of the L^∞ function F on $\partial\mathbb{D}$ (see [Ho_1], p. 171). Hence, $0 \in R$ isolated would imply that F has radial value 0 on a set of positive Lebesgue measure, a contradiction. Thus F has modulus one everywhere on ∂H^∞ and so is inner, hence a constant. □

As an immediate consequence of Theorem 1 and Theorem 2.2, we get that an inner function I (or equivalently a Blaschke product) belongs to $\mathcal{P}$ if and only if for every $\varepsilon \in]0, 1[$ the level set $\{|I| < \varepsilon\}$ does not contain arbitrary large pseudo-hyperbolic discs.

4 Proof of Theorem 2

We first need two well-known auxiliary results. The second of these is a comparison between the modulus of an inner function and the Poisson integral of its corresponding measure. See [Bi]. For the reader's convenience, we include the short, elementary proofs. Recall that

$$P_z(w) = \frac{1-|z|^2}{|1-\overline{w}z|^2}$$

is the Poisson kernel and that

$$P_z(\mu) = \int_{\overline{\mathbb{D}}} P_z(w)d\mu(w),$$

where μ is a Borel measure on $\overline{\mathbb{D}}$. Finally, for a set $E \subseteq \mathbb{D}$, $E \neq \emptyset$, let $\rho(z,E) = \inf\{\rho(z,x) : x \in E\}$ be the pseudohyperbolic distance of E to $z, z \in \mathbb{D}$. If $E = \emptyset$, then we let $\rho(z,E) = 1$.

Lemma 4.1. *If z and w are in $\mathbb{D}$, then*

$$|1-\overline{w}z| \leq 3(1-|z|) + 1 - |w| + |\arg z - \arg w|; \tag{4.1}$$

and if $|\arg w - \arg z| \leq \pi$, then

$$\frac{1}{\pi}|\arg w - \arg z| \leq |e^{i\arg z} - e^{i\arg w}| \leq |\arg w - \arg z| \tag{4.2}$$

and

$$2|z-w| \geq |z||e^{i\arg w} - e^{i\arg z}|. \tag{4.3}$$

Moreover,

$$|e^{it} - z| \geq \frac{1}{\pi}|\arg z - t| \tag{4.4}$$

if $|\arg z - t| \leq \pi$.

Proof. Inequalities (4.2) and (4.4) are well-known. To prove (4.1), we proceed as follows:

$$|1-\overline{w}z| = |(1-\overline{z}z) + \overline{z}z - \overline{w}z| \leq 1-|z|^2 + |z-w| \leq 2(1-|z|) + |z-w|$$

and

$$|z-w| \leq \left||z|e^{i\arg z} - |w|e^{i\arg z}\right| + \left||w|e^{i\arg z} - |w|e^{i\arg w}\right|$$

$$\le \big||z| - |w|\big| + |e^{i\arg z} - e^{i\arg w}| \le (1 - |w|) + (1 - |z|) + |e^{i\arg z} - e^{i\arg w}|$$
$$\le (1 - |w|) + (1 - |z|) + |\arg z - \arg w|.$$

To prove (4.3), we note that

$$|z - w| \ge |z||e^{i\arg z} - e^{i\arg w}| - \big||w| - |z|\big| \ge |z||e^{i\arg w} - e^{i\arg z}| - |z - w|.$$

Hence

$$2|z - w| \ge |z||e^{i\arg w} - e^{i\arg z}|. \qquad \square$$

Lemma 4.2. *Let I be an inner function with zero sequence (z_n). Let $\mu = \mu_I$ be defined as*

$$\mu = \frac{1}{2}\sum_n (1 - |z_n|^2)\delta_{z_n} + \sigma,$$

where σ is the measure associated with the singular part of I. Then, for $E = \{z_n : n \in \mathbb{N}\}$ and any $z \in \mathbb{D}$, one has

$$P_z(\mu) \leqslant \log|I(z)|^{-1} \leqslant \frac{1}{\rho(z, E)^2} P_z(\mu), \tag{4.5}$$

and

$$\begin{aligned} &C^{-1}\left(\frac{\mu(Q(z))}{\ell(Q(z))} + \sum_{n=1}^{\infty} \frac{\mu(2^n Q(z) \setminus 2^{n-1}Q(z))}{2^n \ell(2^n Q(z))}\right) \leqslant \log|I(z)|^{-1} \\ &\leqslant \frac{C}{\rho(z, E)^2}\left(\frac{\mu(Q(z))}{\ell(Q(z))} + \sum_{n=1}^{\infty} \frac{\mu(2^n Q(z) \setminus 2^{n-1}Q(z))}{2^n \ell(2^n Q(z))}\right). \end{aligned} \tag{4.6}$$

Here $C > 1$ is a universal constant.

Note that when I has no zeros, $P_z(\mu) = \log|I(z)|^{-1}$.

Proof. Observe that

$$\log|I(z)|^{-1} = \sum_n \log\left|\frac{z - z_n}{1 - \bar{z_n} z}\right|^{-1} + P_z(\sigma).$$

Now, using that $1 - x^2 \leqslant \log x^{-2} \leqslant (1 - x^2)/x^2$, which holds for $0 < x < 1$, and the identity

$$1 - \left|\frac{z - z_n}{1 - \bar{z_n} z}\right|^2 = \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - \bar{z_n} z|^2} =: k(z_n, z),$$

we deduce

$$k(z_n, z) \leqslant \log\left|\frac{z - z_n}{1 - \bar{z_n} z}\right|^{-2} \leqslant \frac{1}{\rho(z, E)^2} k(z_n, z).$$

Now observe that $\sum_n k(z_n, z)$ is the Poisson integral of the measure $\sum(1-|z_n|^2)\delta_{z_n}$ at the point z. So

$$P_z(\mu) \leqslant \log|I(z)|^{-1} \leqslant \frac{P_z(\mu)}{\rho(z,E)^2}.$$

To prove (4.6), we decompose the integral as

$$P_z(\mu) = \int_{Q(z)} P_z(w)d\mu(w) + \sum_{n\geq 1}\int_{2^nQ(z)\setminus 2^{n-1}Q(z)} P_z(w)d\mu(w)$$

and use the fact that there exists a universal constant $C > 1$ such that

$$\text{(4.7)} \qquad \frac{C^{-1}}{1-|z|} \leqslant P_z(w) \leqslant \frac{C}{1-|z|} \quad \text{for } w \in Q(z)$$

and

$$\text{(4.8)} \qquad \frac{C^{-1}}{2^{2n}(1-|z|)} \leqslant P_z(w) \leqslant \frac{C}{2^{2n}(1-|z|)} \quad \text{for } w \in 2^nQ(z)\setminus 2^{n-1}Q(z).$$

To prove (4.7) and (4.8), first note that $P_z(w) \leq \frac{2}{1-|z|}$ always holds. To show the left inequalities, let $w \in 2^nQ(z)$, $n = 0, 1, 2, \ldots$. Then the result follows from the observations that by Lemma 4.1,

$$\begin{aligned}|1-\overline{w}z| &\leq (1-|w|) + 3(1-|z|) + |\arg z - \arg w| \\ &\leq 2^n(1-|z|) + 3(1-|z|) + 2^n(1-|z|).\end{aligned}$$

To show the right inequality in (4.8) (which is trivial for $n = 1$), note that for $n \geq 2$ and $2^n(1-|z|) \geq 1-|w| \geq 2^{n-1}(1-|z|)$, we have

$$|1-\overline{w}z| \geq |z-w| \geq (1-|w|)-(1-|z|) \geq 2^{n-1}(1-|z|)-(1-|z|) \geq 2^{n-2}(1-|z|);$$

also, for $\pi \geq |\arg w - \arg z| \geq 2^{n-1}(1-|z|)$ and $|z| \geq 1/2$, we have by (4.3),

$$2|1-\overline{w}z| \geq 2|z-w| \geq |z|\frac{1}{\pi}|\arg z - \arg w| \geq \frac{1}{2}\frac{1}{\pi}2^{n-1}(1-|z|) \geq 2^{n-4}(1-|z|).$$

The case $|z| \leq 1/2$ offers no problem, since $2^3Q(z) = \mathbb{D}$, and so we may restrict to $n \leq 3$. Now (4.6) follows from (4.7) and (4.8) by noticing that $\ell(2^nQ(z)) = 2^n\ell Q(z)$ whenever $2^n(1-|z|) \leq \pi$, that $\mu(2^nQ(z)\setminus 2^{n-1}Q(z)) = 0$ whenever $2^{n-1}(1-|z|) \geq \pi$ and that for n_0 with $2^{n_0-1}(1-|z|) < \pi \leq 2^{n_0}(1-|z|)$, we have that

$$\frac{1}{2} < \frac{\ell(2^{n_0}Q(z))}{2^{n_0}\ell(Q(z))} \leq 1. \qquad \square$$

As mentioned in the introduction, any singular inner function whose singular set is finite belongs to the class $\mathcal{M}$. The situation is different when the singular set is infinite. We first present two ways of constructing discrete singular inner functions which do not belong to the class $\mathcal{M}$.

For short, let us write $[\theta_1, \theta_2]$ for the arcs $\{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$.

Example 1. Given $\xi_n \in \partial\mathbb{D}$, $\xi_n \to 1$, one can use E. Decker's result [De] to construct a discrete measure μ such that

$$\lim_{r\to 1} S_\mu(r\xi_n) = 1/2, \quad n = 1, 2, \ldots .$$

Then $S_\mu \notin \mathcal{M}$ because, by the remarks at the end of Section 2, $\tau_{1/2} \circ S_\mu$ is not a Carleson–Newman Blaschke product.

Example 2. Let μ be a positive singular measure on $\partial\mathbb{D}$. Suppose that the derivative D of μ at $e^{i\theta}$ is strictly positive (and finite). Then S_μ does not belong to $\mathcal{M}$.

In fact, if $D = \lim_{t\to\theta^-} \mu([t,\theta])/|t-\theta| = \lim_{t\to\theta^+} \mu([\theta,t])/|t-\theta| > 0$, then, by a theorem of Fatou (see [Ko], p. 11),

$$\lim_{\substack{z\in\Gamma(M)\\ z\to\exp(i\theta)}} \int_{|w|=1} P_z(w)d\mu(w) = 2\pi D$$

for every Stolz domain $\Gamma(M)$ at θ. Hence

$$\lim_{\substack{z\in\Gamma(M)\\ z\to\exp(i\theta)}} |S_\mu(z)| = \exp(-2\pi D).$$

Since by Lemma 2.1, $\Gamma(M)$ contains with $M \to \infty$ arbitrary large pseudohyperbolic discs, we see that by Theorem 1(b) , $S_\mu \notin \mathcal{M}$.

As a concrete example, we mention $\mu = \sum_{n=1}^{\infty} \ell_n(\delta_{a_n} + \delta_{\overline{a}_n})$, where $a_n = e^{i/n}$ and $\ell_n = \frac{1}{n} - \frac{1}{n+1}$. We could replace a_n by any point of the form $e^{i\theta_n}$, $\theta_n > 0$, where θ_n is strictly decreasing to 0 and satisfies $\theta_{n+1}/\theta_n \to 1$ and ℓ_n by $\theta_n - \theta_{n+1}$. In fact, this implies that the derivative of μ at $z = 1$ is 1. Just note that for $t > 0$, there is a unique $k \in \mathbb{N}$ with $\theta_{k+1} < t \leq \theta_k$. Then

$$\mu([0,t])/t \leq \mu([0,\theta_k])/\theta_{k+1} = \theta_k/\theta_{k+1}$$

and

$$\mu([0,t])/t \geq \mu([0,\theta_{k+1}])/\theta_k = \theta_{k+1}/\theta_k.$$

Another concrete example could be built upon the following construction. Let $M_k \in \mathbb{N}$ tend to infinity and assume that within the arc $\left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]$ we have $M_k + 1$

equidistant points θ_n (including the endpoints). Let $\ell_n = |\theta_{n+1} - \theta_n|$. Note that $\ell_n = \frac{1}{2^k+1} M_k$ if $\frac{1}{2^{k+1}} \le \theta_{n+1} < \theta_n \le \frac{1}{2^k}$. Then the measure

$$\mu = \sum_{n=1}^{\infty} \ell_n \left(\delta_{e^{i\theta_n}} + \delta_{e^{-i\theta_n}}\right)$$

has derivative 1 at the point $z = 1$. Thus the singular inner function associated with μ is not in $\mathcal{M}$.

In contrast with these examples, it is worth mentioning that there exist sequences $a_n \in \partial\mathbb{D}$ which converge (fast) to 1 such that any singular inner function whose singularities lie in $\{a_n\} \cup \{1\}$ belong to $\mathcal{M}$. This is a consequence of Theorem 2, which is proved below.

Proof of Theorem 2. (a) $\Longrightarrow$ (b) : Assume that (b) does not hold, that is, that E is not porous. Then for any $\varepsilon_n > 0$, $\varepsilon_n \to 0$, there exists an arc J_n, $J_n \cap E \ne \emptyset$, such that for any arc $\tilde{J}_n \subseteq J_n$ with $|\tilde{J}_n| \ge \varepsilon_n |J_n|$, we have $\tilde{J}_n \cap E \ne \emptyset$. Let us call this the np-property. We assume without loss of generality that $\varepsilon_n n^2 \to 0$, $\varepsilon_n < 1/4$ and that $1/\varepsilon_n$ is an integer. We claim that by taking convenient subarcs, respectively subsequences, we may assume that the J_n are pairwise disjoint, that the centers converge monotonically to some point on the circle (hence the lengths $|J_n|$ converge to zero), that $|J_{n+1}| < |J_n| < 1$ and that

$$\operatorname{dist}(J_{n+1}, J_n) > |J_n|. \tag{4.9}$$

To begin the construction, let $J_0 = \partial\mathbb{D}$. Assume that $J_1, \dots, J_n$ have been constructed. We now construct J_{n+1}. Let $\varepsilon'_{n+1} = |J_n|/8\pi$, $\varepsilon''_{n+1} = \frac{1}{8}\varepsilon'_{n+1}\varepsilon_{n+1}$. By the np-property applied for ε''_{n+1}, we obtain an arc I_{n+1}, $I_{n+1} \cap E \ne \emptyset$ such that for any $\tilde{I}_{n+1} \subseteq I_{n+1}$ with $|\tilde{I}_{n+1}| \ge \varepsilon''_{n+1}|I_{n+1}|$, we have $\tilde{I}_{n+1} \cap E \ne \emptyset$. Let J_{n+1} be a subarc of I_{n+1} with $|J_{n+1}| = \varepsilon'_{n+1}|I_{n+1}|$. Since $\varepsilon'_{n+1} \ge \varepsilon''_{n+1}$, we have $J_{n+1} \cap E \ne \emptyset$. Moreover, $|J_{n+1}| \le 2\pi\varepsilon'_{n+1} < \frac{1}{4}|J_n|$. It is clear that whenever $\tilde{J}_{n+1} \subseteq J_{n+1}$ with $|\tilde{J}_{n+1}| \ge \frac{1}{8}\varepsilon_{n+1}|J_{n+1}|$, we have $|\tilde{J}_{n+1}| \ge \varepsilon''_{n+1}|I_{n+1}|$, so $\tilde{J}_{n+1} \cap E \ne \emptyset$. If $J_{n+1} \cap J_n = \emptyset$, we skip the next step.

So suppose that $J_n \cap J_{n+1} \ne \emptyset$. Choose $\hat{J}_{n+1} \subseteq J_{n+1}$, same centers, with $|\hat{J}_{n+1}| = \frac{1}{4}|J_{n+1}|$. Since $|J_{n+1}| \le |J_n|$, there exists $\hat{J}_n \subseteq J_n$ with $\hat{J}_n \cap \hat{J}_{n+1} = \emptyset$ and $|\hat{J}_n| = \frac{1}{4}|J_n|$. By our construction, $\hat{J}_k \cap E \ne \emptyset$ for $k = n, n+1$. Now replace J_n by $\hat{J}_n$ and J_{n+1} by $\hat{J}_{n+1}$. This ends the inductive construction of a preliminary set of arcs (J_n) with the np-property for $(\varepsilon_n/2)$.

Now choose a monotone subsequence of the centers of the J_n. The associated arcs are again denoted by J_n. Note that $|J_n| \to 0$. We may suppose that for all n, the arc J_n lies to the left of J_{n+1}. Now suppose that $\operatorname{dist}(J_{n+1}, J_n) > |J_n|$ is not satisfied. Then choose as new J_n the left half of the old J_n. Clearly, (4.9) is then satisfied. Of course, we also have that for any arc $\tilde{J}_n \subseteq J_n$ with $|\tilde{J}_n| \geq \varepsilon_n |J_n|$, $\tilde{J}_n \cap E \neq \emptyset$. This ends the construction of the sequence of arcs (J_n).

Now we split each J_n into ε_n^{-1} adjacent subarcs $\{J(n,k) : k = 1, \dots, \varepsilon_n^{-1}\}$ of length $\varepsilon_n |J_n|$. By assumption, each $J(n,k)$ meets E. Let $a(n,k) \in E \cap J(n,k)$. Let $\lambda(n,k)$ be the distance of $a(n,k)$ to $a(n,k+1)$ and let

$$\mu_n = \sum_{k=1}^{\varepsilon_n^{-1}-1} \lambda(n,k)\delta_{a(n,k)}, \quad \mu = \sum_{n=1}^{\infty} \mu_n.$$

Note that μ is supported on E. We show that $S_\mu \notin \mathcal{M}$. To this end, by Theorem 1(b), it is sufficient to construct points $z_n \in \mathbb{D}$ such that for constants $c > 0$ and $C > 0$, we have $c < P_z(\mu) < C$ on the pseudohyperbolic discs $D_\rho(z_n, 1 - 1/n)$.

Choose $z_n \in \mathbb{D}$ such that $J(z_n) = \frac{1}{4n} J_n$. Then $1 - |z_n| = |J_n|/8n$. We claim that for any $z \in D_\rho(z_n, 1 - 1/n)$ and n large enough, we have

$$J(z) \subseteq J_n. \tag{4.10}$$

To see this, let $\rho(z, z_n) < m_n$, where $m_n = 1 - 1/n$. Note that $|z_n| > m_n$. Let

$$Q = \left\{ w \in \mathbb{D} : |w| \geq \frac{|z_n| - m_n}{1 - |z_n| m_n}, |\arg w - \arg z_n| < \frac{1 - |z_n|}{|z_n| - m_n} \right\}.$$

Observe that

$$\frac{|J_n|}{7} > \frac{1 - |z_n|}{|z_n| - m_n} = \frac{|J_n|}{8 - |J_n|} > \frac{|J_n|}{8}.$$

We claim that $D_\rho(z_n, m_n) \subseteq Q$. In fact, by ([Ga], p. 3), $\rho(z, z_n) < m_n$ implies that $|z| \geq \frac{|z_n| - m_n}{1 - |z_n| m_n}$. Moreover, if (without loss of generality) z_n is positive, and

$$A = \frac{|z_n| - m_n}{1 - |z_n| m_n} \exp\left(i \frac{1 - |z_n|}{|z_n| - m_n} \right),$$

then for n large enough, the imaginary part of A is bigger than the euclidean radius R of $D_\rho(z_n, m_n)$, where

$$R = m_n \frac{1 - |z_n|^2}{1 - m_n^2 |z_n|^2}.$$

Hence $D_\rho(z_n, m_n) \subseteq Q$. Thus we have found an estimate of the arguments for the elements in $D_\rho(z_n, m_n)$. Since for every $z \in D_\rho(z_n, m_n)$,

$$1 - \frac{|z_n| - m_n}{1 - |z_n| m_n} + \frac{1 - |z_n|}{|z_n| - m_n} < \frac{(1 + m_n)(1 - |z_n|)}{1 - m_n} + \frac{|J_n|}{7} \leq 2n(1 - |z_n|) + \frac{|J_n|}{7}$$
$$= \left(\frac{1}{4} + \frac{1}{7} \right) |J_n| < \frac{1}{2} |J_n|,$$

we conclude that $J(z) \subseteq J_n$ (see Figure1).

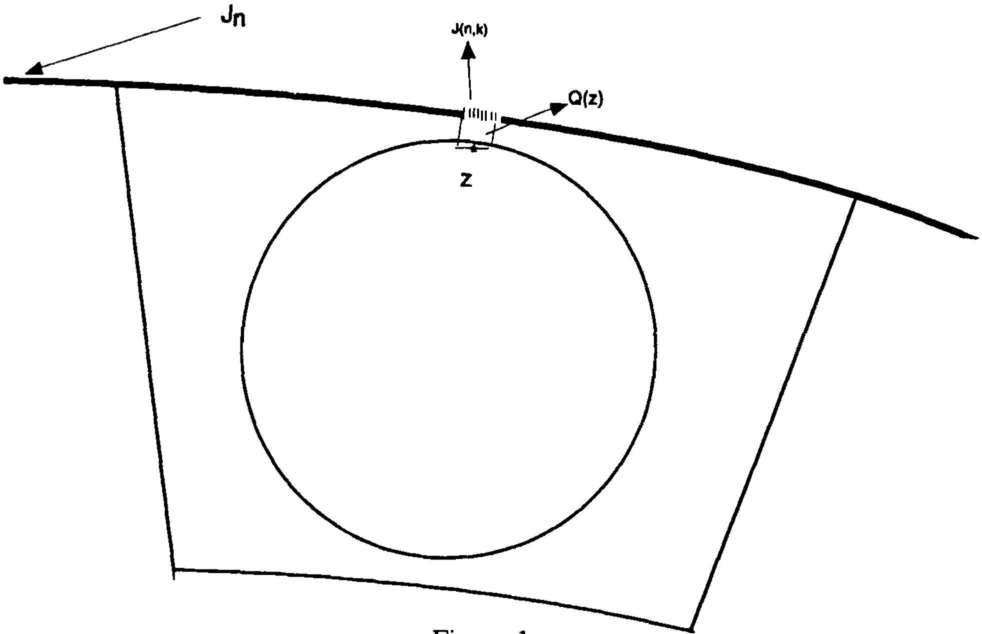

Figure 1.

Next we show that for large n and every $z \in D_\rho(z_n, m_n)$, the number of points $a(n,k)$ within the arc $J(z)$ tends to infinity with $n \to \infty$. In fact, for these z, we have by ([Ga], p. 3),

$$\begin{aligned} |J(z)| = 1 - |z| \geq 1 - \frac{|z_n| + m_n}{1 + m_n|z_n|} &= \frac{(1 - m_n)(1 - |z_n|)}{1 + m_n|z_n|} \\ &\geq \frac{1}{2n}(1 - |z_n|) = \frac{|J_n|}{(4n)^2}. \end{aligned} \tag{4.11}$$

Since $\varepsilon_n n^2 \to 0$, we see that the number of intervals $J(n,k)$ (which have length $\varepsilon_n|J_n|$) belonging to $J(z)$ tends to infinity as $n \to \infty$. This yields the assertion above.

Next we claim that for $z \in D_\rho(z_n, m_n)$,

$$\lim_{n\to\infty} \frac{\mu_n(J(z))}{|J(z)|} = 1; \tag{4.12}$$

in other words, μ_n looks like linear measure at $J(z)$. To see this, we note that at most two of the arcs $J(n,k)$ which meet $J(z)$ are not entirely contained in $J(z)$. Hence

$$\mu_n(J(z)) = \sum_{k:a(n,k)\in J(z)} \lambda(n,k) \geq \sum_{k:J(n,k)\subseteq J(z)} \lambda(n,k) \geq |J(z)| - 2\varepsilon_n|J_n|.$$

Thus, by (4.11),

$$\frac{\mu_n(J(z))}{|J(z)|} \geq 1 - \frac{2\varepsilon_n|J_n|}{|J(z)|} \geq 1 - \frac{32\varepsilon_n|J_n|n^2}{|J_n|} = 1 - 32\varepsilon_n n^2 \to 1.$$

By the same argument, we also see that

$$\frac{\mu_n(J(z))}{|J(z)|} \leq 1 + 32\varepsilon_n n^2 \to 1.$$

This proves (4.12). Applying Lemma 4.2, we obtain that

$$P_z(\mu) \geqslant P_z(\mu_n) \geqslant C^{-1}/2$$

whenever n is large and $z \in D_\rho(z_n, m_n)$.

On the other hand, if $p \in \mathbb{N}$, as above, we see that

$$\frac{\mu_n(2^p J(z))}{|2^p J(z)|} \leq 2$$

whenever n is large and $z \in D_\rho(z_n, m_n)$. Observe that $2^p J(z) = \partial\mathbb{D}$ and $|2^p J(z)| > 2^p|J(z)|$ whenever p is big. Hence, by Lemma 4.2,

$$P_z(\mu_n) \leqslant C \sum_{p=0}^{\infty} \frac{\mu_n(2^p J(z))}{2^{2p}|J(z)|} \leqslant 4C.$$

We next estimate the contribution coming from $\sum_{k\neq n} \mu_k$. Let $z \in D_\rho(z_n, m_n)$. Since $\mu_k(J_k) \leq |J_k|$, we have

$$\sum_{k\neq n} P_z(\mu_k) = \sum_{k\neq n} \int_{J_k} \frac{1-|z|^2}{|\xi - z|^2} d\mu_k(\xi) \leqslant \sum_{k\neq n} \frac{1-|z|^2}{|\xi_k - z|^2}|J_k| =: I, \tag{4.13}$$

where ξ_k is the point in J_k closest to z.

Now

$$\int_{J_k} \frac{1-|z|^2}{|\xi - z|^2} d\xi = \frac{1-|z|^2}{|\eta_k - z|^2}|J_k| \quad \text{for some } \eta_k = \eta_k(z) \in J_k.$$

We show that for $z \in D_\rho(z_n, m_n)$ and $k \neq n$, we have

$$\frac{|\eta_k - z|}{|\xi_k - z|} \leq 3. \tag{4.14}$$

In fact, if we assume that $\Delta := |\arg z - \arg \xi_k| \leq \pi/2$ and notice that $J(z) \subseteq J_n$, then by (4.9)

$$\frac{|\eta_k - z|}{|\xi_k - z|} \leq \frac{|\eta_k - \xi_k| + |\xi_k - z|}{|\xi_k - z|} \leq 1 + \frac{|J_k|}{|\xi_k - z|}$$

$$\leq 1+\frac{|J_k|}{||z|-e^{i\Delta}|}\leq 1+\frac{|J_k|}{|\sin\Delta|}\leq 1+\frac{\pi}{2}\frac{|J_k|}{\Delta}\leq 1+\frac{\pi|J_k|}{2\text{dist}\,(J_n,J_k)}.$$

But by (4.9), $\text{dist}(J_n, J_k) \geq \text{dist}(J_{k+1}, J_k) \geq |J_k|$ if $k < n$ and $\text{dist}(J_n, J_k) \geq \text{dist}(J_{n+1}, J_n) \geq |J_n|$ if $k > n$. Thus, since $|J_k| < |J_n|$ if $k > n$, we obtain

$$\frac{|\eta_k - z|}{|\xi_k - z|}\leq 1+\frac{\pi}{2}\leq 3.$$

If $\pi/2 \leq \Delta \leq \pi$, then $|\xi_k - z| \geq 1$, and so

$$\frac{|\eta_k - z|}{|\xi_k - z|}\leq 2.$$

Hence, in both cases,

$$I\leq 3\sum_{k\neq n}\int_{J_k}\frac{1-|z|^2}{|\xi-z|^2}d\xi\leq 6\pi\leq 20. \tag{4.15}$$

We conclude that for any $z \in D_\rho(z_n, m_n)$,

$$4C+20\geqslant P_z(\mu)\geqslant C^{-1}/2.$$

Since $m_n = 1 - 1/n \to 1$, we deduce that property (b) of Theorem 1 is not satisfied and therefore $S_\mu \notin \mathcal{M}$.

(b) $\Longrightarrow$ (a) : Suppose that E is ε-porous. Let μ be a positive singular measure supported in E. Fix $0 < \eta < 1/2$. Consider the set

$$L=\{z\in\mathbb{D}:\eta<|S_\mu(z)|<1-\eta\}.$$

We show that this level set does not contain arbitrary large pseudohyperbolic discs. Thus, by Theorem 1(b) , $S_\mu \in \mathcal{M}$.

First we claim that there exists a constant $C = C(\varepsilon)$ such that for any $z \in L$, there exists $\tilde{z} \in \mathbb{D}$ with $\rho(z, \tilde{z}) \leqslant C(\varepsilon) < 1$ such that $J(\tilde{z}) \cap E = \emptyset$. If $J(z) \cap E = \emptyset$, then we let $\tilde{z} = z$. If $J(z) \cap E \neq \emptyset$, there exists by (b) an arc $\tilde{J} \subset J(z)$, $|\tilde{J}| > \varepsilon|J(z)|$, with $\tilde{J} \cap E = \emptyset$. Choose $\tilde{z} \in \mathbb{D}$ such that $J(\tilde{z}) = \tilde{J}$. Note that $1-|z| \geq 1-|\tilde{z}| \geq \varepsilon(1-|z|)$. Hence we get by Lemma 4.1,

$$1-\rho^2(z,\tilde{z})=\frac{(1-|z|^2)(1-|\tilde{z}|^2)}{|1-\bar{z}\tilde{z}|^2}$$

$$\geq\frac{\varepsilon(1-|z|)^2}{\left[3(1-|z|)+(1-|\tilde{z}|)+|\arg\tilde{z}-\arg z|\right]^2}\geq\frac{\varepsilon}{25}.$$

Thus $\rho(z, \tilde{z}) \leq \sqrt{1-\varepsilon/25} =: C(\varepsilon)$.

If $|S_\mu(\tilde{z})| < \eta$ or $|S_\mu(\tilde{z})| > 1-\eta$, then we are done, since the disc $D_\rho(z, C(\varepsilon))$ is then not contained in $\{w : \eta < |S_\mu(w)| < 1-\eta\}$. So we may suppose that $|S_\mu(\tilde{z})| \geq \eta$. We now construct $z^* := re^{i\arg\tilde{z}}$ with $\rho(z^*, z) < C(\varepsilon,\eta)$ for which $|S_\mu(z^*)| > 1-\eta$. Let

$$\delta = \frac{\log\frac{1}{1-\eta}}{50\log\frac{1}{\eta}}.$$

Note that $\delta < 1/2$. Choose $r \in]0,1[$ such that $1-r = \delta(1-|\tilde{z}|)$ and let $z^* = re^{i\arg\tilde{z}}$. Thus $1-|\tilde{z}| \geq 1-|z^*| = \delta(1-|\tilde{z}|)$.

By Lemma 4.1, we have

$$1-\rho^2(\tilde{z},z^*) = \frac{(1-|\tilde{z}|^2)(1-|z^*|^2)}{|1-\overline{z^*}\tilde{z}|^2}$$

$$\geq \frac{\delta(1-|\tilde{z}|)^2}{\left[3(1-|\tilde{z}|)+(1-|z^*|)+|\arg\tilde{z}-\arg z^*|\right]^2} \geq \frac{\delta}{16}.$$

Thus $\rho(z^*,\tilde{z}) \leq \sqrt{1-\delta/16} =: C(\eta)$. Combining this with $\rho(z,\tilde{z}) \leq C(\varepsilon)$, we see that $\rho(z,z^*) \leq C(\varepsilon,\eta)$.

Now we evaluate S_μ at z^*. Note that for $e^{it} \in \partial\mathbb{D}\setminus J(\tilde{z})$, $|e^{it}-\tilde{z}|$ is comparable to $|e^{it}-z^*|$; more precisely, since $\arg\tilde{z} = \arg z^*$ and $1-|z^*| = \delta(1-|\tilde{z}|)$, we have by (4.4),

$$\frac{|e^{it}-\tilde{z}|}{|e^{it}-z^*|} \leq \frac{|e^{it}-z^*|+|\tilde{z}-z^*|}{|e^{it}-z^*|} \leq 1+\frac{(1-|\tilde{z}|)-(1-|z^*|)}{\frac{1}{\pi}|\arg z^*-t|}$$

$$\leq 1+\frac{(1-\delta)(1-|\tilde{z}|)}{\frac{1}{\pi}(1-|\tilde{z}|)} \leq 1+\pi \leq 5.$$

Hence

$$\begin{aligned}
\log|S_\mu(z^*)|^{-1} &= \int_{t:e^{it}\in\partial\mathbb{D}\setminus J(\tilde{z})} \frac{1-|z^*|^2}{|e^{it}-z^*|^2}\,d\mu(t)\\
&\leq 2\delta\int_{t:e^{it}\in\partial\mathbb{D}\setminus J(\tilde{z})} \frac{1-|\tilde{z}|}{|e^{it}-\tilde{z}|^2}\left|\frac{|e^{it}-\tilde{z}|^2}{|e^{it}-z^*|^2}\right|d\mu(t)\\
&\leq 50\delta\int_{t:e^{it}\in\partial\mathbb{D}} \frac{1-|\tilde{z}|^2}{|e^{it}-\tilde{z}|^2}\,d\mu(t)\\
&= 50\delta\log|S_\mu(\tilde{z})|^{-1} \leq 50\delta\log\frac{1}{\eta} = \log\frac{1}{1-\eta}.
\end{aligned}$$

Thus we have found a point z^* with $\rho(z,z^*) \leq C(\varepsilon,\eta)$ such that $|S_\mu(z^*)| \geq 1-\eta$. Therefore, the level set L does not contain any hyperbolic disc of radius larger than or equal to $C(\varepsilon,\eta)$. Hence condition (b) in Theorem 1 is satisfied. Therefore, $S_\mu \in \mathcal{M}$. □

We remark that if the set $E \subseteq \partial\mathbb{D}$ contains a sequence (α_n) converging to α such that

$$(4.16) \qquad \frac{\alpha_n - \alpha}{\alpha_{n+1} - \alpha} \to 1 \quad \text{as } n \to \infty,$$

then it is quite easy to find a subset F of E and a singular inner function supported on F which does not belong to $\mathcal{M}$ (see the following paragraph). In particular, such a set is not porous. Trivial examples are sets containing a continuum.

To construct F, we may assume that $\alpha = 1$ and that $\arg \alpha_n > 0$ for all n. Put $F = \{\alpha_n : n \in \mathbb{N}\} \cup \{1\}$. Let ℓ_n be the length of the arc $[\alpha_{n+1}, \alpha_n]$. Let μ be the measure

$$\mu = \sum_{n=1}^{\infty} \ell_n \delta_{\alpha_n} + \sum_{n=1}^{\infty} \ell_n \delta_{\overline{\alpha}_n}.$$

Since the derivative of μ at 1 is 1, by Example 2, we see that the associated singular inner function $S_\mu \notin \mathcal{M}$. Now $S_\mu = S_1 S_2$, where S_1 is associated to $\mu_1 = \sum_{n=1}^{\infty} \ell_n \delta_{\alpha_n}$. Obviously, μ_1 is supported on E and $S_2(z) = \overline{S_1(\overline{z})}$. Since $I(z) \in \mathcal{M} \iff \overline{I(\overline{z})} \in \mathcal{M}$, we obtain that $S_1 \notin \mathcal{M}$.

Let us also mention that there exist compact sets which are, roughly speaking, "δ-porous at every point" (the δ depending on the point), but which are not porous. In fact, let $\theta_k = 1/2^{2^k}$ and consider the points $\beta_{k,n} = \theta_k + 1/(1+1/k)^n$ for $n > n(k)$. Then for all k, the sets $E_k = \{e^{i\beta_{k,n}} : n > n(k)\} \cup \{e^{i\theta_k}\}$ are δ_k-porous, but since $\delta_k \to 0$, $E = \bigcup_{k=1}^{\infty} E_k$ is not porous.

This example also shows that there exist nonporous sets which do not have property (4.16).

On the other hand, we have the following result.

Proposition 4.3. *Let $E \subseteq \mathbb{R}$ be a compact set. Suppose there exist $\delta > 0$ and for every $\theta \in E$ some $\varepsilon = \varepsilon(\theta) > 0$ such that for every $(x, y) \in E^2$ with $x \neq y$, $0 < |y - \theta| < \varepsilon$ and $0 < |x - \theta| < \varepsilon$, we have*

$$(4.17) \qquad \left| \frac{x - \theta}{y - \theta} - 1 \right| > \delta.$$

Then E is $C\delta$-porous for some universal constant $C > 0$.

Proof. By a compactness argument, it is easily seen that we have only to test open arcs J containing a given $\theta \in E$ and of length less than $\varepsilon(\theta)$.

So let J be such an arc with $J \cap E \neq \emptyset$. If J contains only one point of E, then there trivially exists $J' \subseteq J$ with $|J'| = \frac{1}{3}|J|$ such that $J' \cap E = \emptyset$. Without loss of generality, we may assume that $\theta < \theta' = \sup\{x : x \in E \cap J\}$ and that θ lies in the left half of J. If $|\theta - \theta'| < \frac{1}{4}|J|$, then there exists $J' \subseteq J$ with $J' \cap E = \emptyset$ and $|J'| = \frac{1}{4}|J|$.

So let $|\theta - \theta'| \geq \frac{1}{4}|J|$. We shall construct $J' \subseteq J_1 := [\theta, \theta']$ with $J' \cap E = \emptyset$ and $|J'| \geq \frac{1}{2}\delta|J_1| \geq \frac{1}{8}\delta|J|$.

Note that $J_1 \subseteq J$ implies that $|J_1| < \varepsilon(\theta)$. By (4.17), we have for every $J_2 = [\theta, \varphi] \subseteq J_1$ with $\varphi \in E$ and $\varphi < \theta'$ that

$$\left|\frac{|J_1|}{|J_2|} - 1\right| > \delta.$$

Thus $|J_1| - |J_2| \geq \delta|J_2|$, and hence

$$|J_2| \leq \frac{|J_1|}{1+\delta}.$$

Hence, for the supremum of those φ, $]\varphi, \theta'[\, \cap E = \emptyset$. Let $J' =]\varphi, \theta'[$. Then $|J'| = |J_1| - |J_2| \geq \delta|J_2|$. If $|J_2| \geq \frac{1}{2}|J_1|$, then $|J'| \geq \frac{1}{2}\delta|J_1|$; if $|J_2| < \frac{1}{2}|J_1|$, then $|J'| \geq \frac{1}{2}|J_1|$ (since $J_1 = J_2 \cup J'$). □

Example 3. Using Theorem 2 and the fact that the usual $1/3$-Cantor set E is porous, we obtain a continuous singular inner function $S_\mu \in \mathcal{M}$. Just take for μ the singular measure given by the Cantor-function associated to E (this is a continuous, weakly increasing function on $[0,1]$ which has zero-derivative at each point of $[0,1] \setminus E$).

At the end of Section 6, we give an example of a continuous singular inner function in $\mathcal{M}$ whose support is *not* porous.

It would be interesting to know a characterization of those *non-porous* sets E for which there exists a singular inner function in $\mathcal{M}$ whose set of singularities is E. We guess that for every compact set $E \subseteq \partial\mathbb{D}$, there is a singular inner function S whose set of singularities equals E and which belongs to $\mathcal{M}$.

5 Proof of Theorem 3

Proof. (a) $\Longrightarrow$ (b) : Let $\alpha \in \mathbb{D} \setminus \{0\}$ be such that $\tau_\alpha \circ I$ is not a Carleson–Newman Blaschke product. Then, by Theorem 2.2, there exist $z_n \in \mathbb{D}$, $|z_n| \to 1$, and $0 < m_n^* < 1$, $m_n^* \to 1$ such that

$$\sup\{|I(z) - \alpha| : \rho(z, z_n) \leqslant m_n^*\} \xrightarrow[n\to\infty]{} 0. \tag{5.1}$$

In particular, I has no zeros on $\{z : \rho(z, z_n) \leqslant m_n^*\}$ if n is sufficiently large. Then, considering $\tilde{m}_n = m_n^* - (1 - m_n^*)^{1/2}$, one can check that

$$\rho(D_\rho(z_n, \tilde{m}_n), \mathbb{D} \setminus D_\rho(z_n, m_n^*)) \xrightarrow[n\to\infty]{} 1.$$

Hence

$$\rho(D_\rho(z_n, \tilde{m}_n), \{z \in \mathbb{D} : I(z) = 0\}) \xrightarrow[n\to\infty]{} 1. \tag{5.2}$$

Therefore, applying (4.5) and (5.1), one obtains

$$\sup\left\{\left|\int_{\overline{\mathbb{D}}} P_z(w)d\mu(w) - \log|\alpha|^{-1}\right| : \rho(z, z_n) \leqslant \tilde{m}_n\right\} \xrightarrow[n\to\infty]{} 0. \tag{5.3}$$

The constant C in part (b) of the statement will be $C = (\log|\alpha^{-1}|)/\pi$ and m_n will be chosen such that $m_n < \tilde{m}_n$, but still with $m_n \to 1$.

We first show that (b.2) holds. For $N > 0$ and $z \in \mathbb{D}$, let us consider $z(N) = (1 - N(1 - |z|))\exp(i \operatorname{Arg} z)$. Then if $N > 0$ is fixed and z is sufficiently close to the unit circle, $z(N) \in \mathbb{D}$. Moreover, $z(N)$ and z lie on the same radius and $1 - |z(N)| = N(1 - |z|)$. Fix $N > 0$. We claim that $\rho(z, z(N)) \leqslant 1 - 1/N$. In fact, since $z(N)$ and z have the same argument, $1 - 1/N < |z|$ and $|z(N)| < |z|$, we have

$$\rho(z, z(N)) = \frac{|z| - |z(N)|}{1 - |z(N)||z|} = \frac{(1 - |z(N)|) - (1 - |z|)}{1 - |z|(1 - N(1 - |z|))}$$

$$= \frac{(N-1)(1 - |z|)}{1 - |z| + |z|N(1 - |z|)} = \frac{N-1}{1 + N|z|} < \frac{N-1}{1 + N(1 - \frac{1}{N})} = \frac{N-1}{N} = 1 - \frac{1}{N}.$$

Choose $\hat{m}_n$ such that

$$\frac{2\hat{m}_n}{1 + \hat{m}_n^2} = \tilde{m}_n.$$

There exists $n(N)$ such that $\hat{m}_n > 1 - 1/N$ for every $n \geq n(N)$. Hence, for every $z \in D_\rho(z_n, \hat{m}_n)$,

$$\rho(z(N), z_n) \leq \frac{\rho(z(N), z) + \rho(z, z_n)}{1 + \rho(z(N), z)\rho(z, z_n)} \leq \frac{1 - 1/N + \hat{m}_n}{1 + (1 - 1/N)\hat{m}_n} \leq \frac{2\hat{m}_n}{1 + \hat{m}_n^2} \leq \tilde{m}_n.$$

Thus

$$z(N) \in D_\rho(z_n, \tilde{m}_n) \quad \text{whenever } z \in D_\rho(z_n, \hat{m}_n) \text{ and } n \geq n(N). \tag{5.4}$$

Next we claim that if $w \notin 2NQ(z)$, then

$$|1 - \bar{z}w| \geqslant c|1 - \overline{z(N)}w|, \tag{5.5}$$

where c is an absolute constant.

In fact, observe that by (4.1) we have for $N \geq 3$,

$$\begin{aligned} |1 - \overline{z(N)}z| &\leq (1 - |z(N)|) + 3(1 - |z|) + |\arg z - \arg z(N)| \\ &\leq (N + 3)(1 - |z|) \leq 2N(1 - |z|). \end{aligned} \tag{5.6}$$

Hence, if $1-|w| > 2N(1-|z|)$, we obtain

$$|1-\overline{z(N)}z| \leq 2N(1-|z|) < 1-|w| \leq |1-\bar{z}w|.$$

On the other hand, if $\pi/2 \geq |\theta| := |\arg z - \arg w| > 2N(1-|z|)$, then

$$\frac{2}{\pi}|\theta| \leq |\sin\theta| = \left|\operatorname{Im}(e^{-i\theta}-|zw|)\right| \leq \left|e^{-i\theta}-|zw|\right| \leq \left|1-e^{i\theta}|z||w|\right| = |1-\bar{z}w|;$$

and if $\pi/2 < |\theta| \leq \pi$, then

$$|1-\bar{z}w| = \left|e^{-i\theta}-|zw|\right| \geq \sqrt{|zw|^2+1} \geq 1 \geq \frac{1}{\pi}\pi \geq \frac{1}{\pi}|\arg z - \arg w|.$$

Hence, in both cases (by (5.6)),

$$|1-\overline{z(N)}z| \leq 2N(1-|z|) \leq |\arg z - \arg w| \leq \pi|1-\bar{z}w|.$$

Therefore

$$\begin{aligned} |1-\overline{z(N)}w| &\leq |1-\overline{z(N)}z| + |z\overline{z(N)} - \overline{z(N)}w| \leq \pi|1-\bar{z}w| + |z-w| \\ &\leq \pi|1-\bar{z}w| + \frac{|z-w|}{|1-\bar{z}w|}|1-\bar{z}w| \leq \tilde{c}|1-\bar{z}w|. \end{aligned} \tag{5.7}$$

This finishes the proof of (5.5).

It follows immediately from (5.5) that

$$P_z(w) \leqslant \frac{C}{N}P_{z(N)}(w) \quad \text{if } w \notin 2NQ(z).$$

Thus

$$\begin{aligned} \int_{\overline{\mathbb{D}}\setminus 2NQ(z)} P_z(w)d\mu(w) &\leq \frac{C}{N}\int_{\overline{\mathbb{D}}\setminus 2NQ(z)} P_{z(N)}(w)d\mu(w) \\ &\leq \frac{C}{N}\int_{\overline{\mathbb{D}}} P_{z(N)}(w)d\mu(w). \end{aligned}$$

Now, if $\rho(z, z_n) \leqslant \hat{m}_n$ and $n \geq n(N)$, then by (5.4) we have $\rho(z(N), z_n) \leqslant \tilde{m}_n$; this means that we may apply (5.3) to obtain

$$\int_{\overline{\mathbb{D}}} P_{z(N)}(w)d\mu(w) \leqslant \log|\alpha|^{-1} + 1,$$

if n is sufficiently large. Therefore, for $n \geq n(N)$ sufficiently large, and any $z \in \mathbb{D}$ with $\rho(z, z_n) \leqslant \hat{m}_n$, we have

$$\int_{\overline{\mathbb{D}}\setminus 2NQ(z)} P_z(w)d\mu(w) \leqslant \frac{C}{N}(\log|\alpha|^{-1} + 1),$$

which gives (b.2).

We now prove (b.1). Let $\varepsilon > 0$. Choose, according to (b.2), an integer $N = N(\varepsilon)$ such that

$$\sup_{w \in D_\rho(z_n, \hat{m}_n)} \int_{\overline{\mathbb{D}} \setminus NQ(w)} P_w(\xi) d\mu(\xi) < \varepsilon \tag{5.8}$$

for all $n \geq n(N)$. Define m_n by

$$\frac{2m_n}{1 + m_n^2} = \hat{m}_n.$$

Let $\eta_n = 1 - m_n^2$. We may assume that $\eta_n \leq 1/25$. For every $z \in D_\rho(z_n, m_n)$, let $r = r(z)$ satisfy

$$\frac{1}{25} \frac{1 - r(z)}{1 - |z|} = \eta_n. \tag{5.9}$$

In particular, $1 - |r(z)| \leq 1 - |z|$. For each $z \in \mathbb{D}$, with $\rho(z, z_n) \leqslant m_n$, consider the arc

$$L(z) = \left\{ r(z) e^{i\theta} : |\theta - \arg z| \leqslant (1 - \delta)(1 - |z|) \right\},$$

where $\delta = \delta(z) > 0$ is chosen so that

$$NQ(w) \subseteq Q(z) \quad \text{if } w \in L(z) \tag{5.10}$$

and $z \in D_\rho(z_n, m_n)$.

To show that this is possible, observe that if $e^{i\theta} \in NQ(w)$, $w \in L(z)$, we have

$$\begin{aligned} |\theta - \arg z| &\leqslant |\theta - \arg w| + |\arg w - \arg z| \\ &\leqslant N(1 - |w|) + (1 - \delta)(1 - |z|) = N(1 - r(z)) + (1 - \delta)(1 - |z|), \end{aligned}$$

which is smaller than $1 - |z|$ because

$$\frac{N(1 - r(z))}{1 - |z|} + (1 - \delta) < 1$$

if $\delta = \delta(z)$ is chosen so that

$$\frac{N(1 - r(z))}{1 - |z|} < \delta.$$

So (5.10) holds.

We claim that the choice of $r(z)$ implies that for every $z \in D_\rho(z_n, m_n)$ we have

$$L(z) \subset \{ w : \rho(w, z_n) \leqslant \hat{m}_n \} \tag{5.11}$$

(see Figure 2).

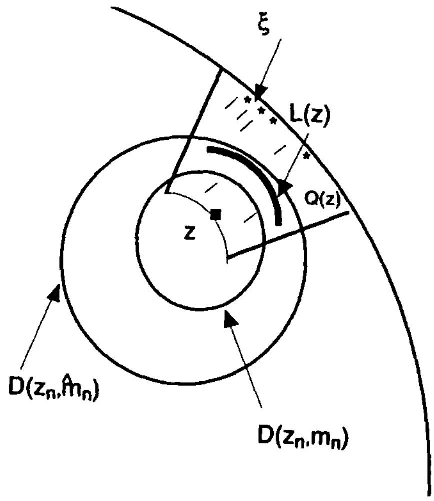

Figure 2. The set $L(z)$.

To see this, we first compute for $w \in L(z)$ the distance $\rho(w,z)$ using (4.1):

$$1-\rho^2(w,z) \geq \frac{(1-|z|)(1-|w|)}{((1-|z|)+3(1-|w|)+|\arg z-\arg w|)^2}$$

$$\geq \frac{(1-|z|)(1-|r(z)|)}{((1-|z|)+3(1-|z|)+1-|z|)^2} = \frac{1}{25}\frac{1-|r(z)|}{1-|z|} = \eta_n.$$

Hence $\rho(w,z) \leq \sqrt{1-\eta_n} = m_n$. We conclude that

$$\rho(w,z_n) \leq \frac{\rho(w,z)+\rho(z_n,z)}{1+\rho(w,z)\rho(z_n,z)} \leq \frac{m_n+m_n}{1+m_n m_n} = \hat{m}_n.$$

By (5.8), (5.10) and (5.11), we obtain

$$\sup_{w\in L(z)} \int_{\overline{\mathbb{D}}\setminus Q(z)} P_w(\xi)d\mu(\xi) < \varepsilon. \tag{5.12}$$

By (5.3), (5.11) and the fact that $m_n < \hat{m}_n < \tilde{m}_n$, we see that for $z \in D_\rho(z_n, m_n)$,

$$\sup_{w\in L(z)} \left| \int_{\overline{\mathbb{D}}} P_w(\xi)d\mu(\xi) - \log|\alpha|^{-1} \right| < \varepsilon$$

for n sufficiently large. Hence, for every $z \in D_\rho(z_n, m_n)$, (5.12) yields

$$\sup_{w\in L(z)} \left| \int_{Q(z)} P_w(\xi)d\mu(\xi) - \log|\alpha|^{-1} \right| < 2\varepsilon \tag{5.13}$$

for n sufficiently large.

Let $|dw|$ be the usual linear measure (arc-length) on $L(z)$. We integrate (5.13) along $L(z)$ to obtain

$$\left|\frac{1}{|L(z)|}\int_{Q(z)}\int_{L(z)} P_w(\xi)|dw|d\mu(\xi) - \log|\alpha|^{-1}\right| \leq 2\varepsilon \tag{5.14}$$

for all $z \in D_\rho(z_n, m_n)$.

Let $J = J(z) = \{\theta : |\theta - \arg z| \leq (1-\delta(z))(1-|z|)\}$. Observe that for $w \in L(z)$,

$$\begin{aligned}\int_{L(z)} P_w(\xi)|dw| &= \int_{\theta\in J} \frac{1-r(z)^2}{|e^{i\theta}-\xi r(z)|^2} r(z)d\theta \\ &= \frac{1-r(z)^2}{1-r(z)^2|\xi|^2}\int_{\theta\in J}\frac{1-r(z)^2|\xi|^2}{|e^{i\theta}-\xi r(z)|^2} r(z)d\theta \leq 2\pi.\end{aligned} \tag{5.15}$$

By (5.2) and (5.11), we get that

$$\rho(L(z), \{z^* \in \mathbb{D} : I(z^*) = 0\}) \to 1 \quad \text{as } n \to \infty \tag{5.16}$$

uniformly in $z \in D_\rho(z_n, m_n)$.

To obtain a lower estimate in (5.15), we first need to show that (5.16) implies that for fixed $t > 0$ and $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$,

$$\lim \frac{1-r(z)}{1-r(z)|\xi|} = 1 \quad \text{uniformly for } z \in D_\rho(z_n, m_n) \text{ as } n \to \infty. \tag{5.17}$$

To show this, we make two observations. First, we note that if $p \in Q(z)$ satisfies $1 - r(z) \leq 1 - |p| \leq 1 - |z|$, where $z \in D_\rho(z_n, m_n)$, then $p \in D_\rho(z_n, \tilde{m}_n)$. In fact, $|\arg p - \arg z| \leq 1 - |z|$; hence by (4.1),

$$\begin{aligned}1 - \rho^2(z,p) &\geq \frac{(1-|p|)(1-|z|)}{(1-|p| + 3(1-|z|) + |\arg p - \arg z|)^2} \\ &\geq \frac{1-r(z)}{25(1-|z|)} = \eta_n = 1 - m_n^2,\end{aligned}$$

and therefore

$$\rho(p, z_n) \leq \frac{\rho(p,z) + \rho(z,z_n)}{1 + \rho(p,z)\rho(z,z_n)} \leq \frac{2m_n}{1+m_n^2} = \hat{m}_n < \tilde{m}_n.$$

Since by (5.2), $\{z : I(z) = 0\} \cap D_\rho(z_n, \tilde{m}_n) = \emptyset$, every point $\xi \in Q(z) \cap \operatorname{supp}\mu$ then has modulus bigger than $r(z)$, provided n is sufficiently large.

Next observe that with $r = r(z)$,

$$\frac{1-r}{1-r|\xi|} = \frac{1}{1 + r\frac{1-|\xi|}{1-r}}.$$

To show (5.17), assume that $\frac{1-|\xi|}{1-r}$ does not tend to 0, say $1 \geq \frac{1-|\xi|}{1-r} \geq \eta > 0$. Then we choose $w \in L(z)$ with $\arg w = \arg \xi$, where $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$. Here we have used that $\delta(z) \to 0$ as $|z| \to 1$. Since $|w| = r(z) \leq |\xi|$, we obtain, by (4.1),

$$1 - \rho^2(w, \xi) \geq \frac{(1-|w|)(1-|\xi|)}{\left((1-|w|) + 3(1-|\xi|) + |\arg w - \arg \xi|\right)^2}$$

$$\geq \frac{\eta(1-|w|)^2}{\left((1-|w|) + 3(1-|\xi|)\right)^2} = \frac{\eta}{\left[1 + 3\frac{1-|\xi|}{1-r(z)}\right]^2} \geq \frac{\eta}{64} > 0.$$

But the left hand side of this inequality tends to zero by (5.16). Thus (5.17) is proved.

Our next step to prove a lower estimate of the Poisson integral will be to show that for $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$ we have $r(z)\xi \in Q(z)$. Indeed, by (5.17) and (5.9),

$$\frac{1 - r|\xi|}{1 - |z|} = \frac{1 - r|\xi|}{1 - r} \frac{1 - r}{1 - |z|} \to 1 \times 0 = 0$$

as $n \to \infty$. Hence $1 - r|\xi| < 1 - |z|$. Since r is real we get that $r\xi \in Q(z)$.

Finally, in our last step, we show that $|e^{i\theta} - z|/|e^{i\theta} - r\xi|$ is bounded by a constant depending only on t whenever $z \in D_\rho(z_n, m_n)$, $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$ and $\theta \in [0, 2\pi[\, \setminus J$. We proceed as follows. Choose n so large, and hence $|z|$ so close to 1, that $1 - t/2 < 1 - \delta(z)$. Hence, for $\theta \in [0, 2\pi[\, \setminus J$, we have

$$|e^{i\theta} - \xi r| \geq \frac{2}{\pi}|\theta - \arg \xi| \geq \frac{2}{\pi}(t - \delta(z))(1 - |z|) \geq \frac{t}{\pi}(1 - |z|).$$

Since $r\xi \in Q(z)$, we get

$$\begin{aligned} \frac{|e^{i\theta} - z|}{|e^{i\theta} - r\xi|} &\leq \frac{|e^{i\theta} - r\xi| + |r\xi - z|}{|e^{i\theta} - r\xi|} = 1 + \frac{|r\xi - z|}{|e^{i\theta} - r\xi|} \\ &\leq 1 + \frac{\operatorname{diam} Q(z)}{\frac{t}{\pi}(1 - |z|)} \leq 1 + \frac{4(1 - |z|)}{\frac{t}{\pi}(1 - |z|)} \leq 1 + 4\pi/t. \end{aligned} \tag{5.18}$$

We are now able to prove that we can take $2\pi - \varepsilon$ as a lower bound of $\int_{L(z)} P_w(\xi)|dw|$ whenever n is large. More precisely, we claim that for fixed $t > 0$ and $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$, the integral $\int_{L(z)} P_w(\xi)|dw|$ converges uniformly for $z \in D_\rho(z_n, m_n)$ to 2π whenever n goes to infinity. This can been seen as follows.

First, we see that by (5.17),

$$\int_{|w|=r} P_w(\xi)|dw| = r\frac{1 - r^2}{1 - |r\xi|^2} \int_{[0,2\pi[} P_{r\xi}(e^{i\theta})d\theta = 2\pi r \frac{1 - r^2}{1 - |r\xi|^2} \to 2\pi$$

as $n \to \infty$. On the other hand, whenever $z \in D_\rho(z_n, m_n)$, $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$ and $w = re^{i\theta}$, we obtain by (5.9) and (5.18),

$$\int_{[0,2\pi[\setminus J} P_w(\xi) r \, d\theta \leq 2 \int_{[0,2\pi[\setminus J} \frac{1 - r}{|e^{i\theta} - r\xi|^2} \, d\theta$$

$$\leq 50 \int_{[0,2\pi[\setminus J} \frac{(1-|z|)\eta_n}{|e^{i\theta}-z|^2} \frac{|e^{i\theta}-z|^2}{|e^{i\theta}-r\xi|^2}\, d\theta \leq C(t)\eta_n \int_{\partial\mathbb{D}} P_z(\zeta)d\zeta = 2\pi C(t)\eta_n.$$

Since $\eta_n \to 0$, we finally get that $\int_{L(z)} P_w(\xi)|dw|$ converges uniformly to 2π for $z \in D_\rho(z_n, m_n)$ and $\xi \in (1-t)Q(z) \cap \operatorname{supp}\mu$ as $n \to \infty$.

Hence, by (5.14), (5.15) and the preceding paragraph,

$$2\pi \frac{\mu(Q(z))}{|L(z)|} \geqslant \frac{1}{|L(z)|} \int_{Q(z)} \int_{L(z)} P_w(\xi)|dw|d\mu(\xi) \geqslant \log|\alpha|^{-1} - 2\varepsilon;$$

and for fixed $t > 0$,

$$(2\pi - \varepsilon) \frac{\mu((1-t)Q(z))}{|L(z)|} \leqslant \frac{1}{|L(z)|} \int_{Q(z)} \int_{L(z)} P_w(\xi)|dw|d\mu(\xi) \leqslant \log|\alpha|^{-1} + 2\varepsilon$$

uniformly for $z \in D_\rho(z_n, m_n)$ if n is sufficiently large.

The second estimate applied to a point $z^* = z(t)$ such that $(1-t)Q(z^*) = Q(z)$ reads

$$(2\pi - \varepsilon) \frac{\mu(Q(z))}{|L(z^*)|} \leqslant \log|\alpha|^{-1} + 2\varepsilon.$$

By the continuity of the functions $r(z(t))$ and $\delta(z(t))$ within a fixed disc $D_\rho(z_n, m_n)$, we have $|L(z(t))| \to |L(z)|$ if $t \to 0$. We deduce

$$(2\pi - \varepsilon) \frac{\mu(Q(z))}{|L(z)|} \leqslant \log|\alpha|^{-1} + 2\varepsilon$$

if $z \in D_\rho(z_n, m_n)$ and n is sufficiently large.

Since $|L(z)| = 2r(z)(1-\delta(z))(1-|z|) = 2r(z)(1-\delta(z))\ell(Q(z))$ and $\delta(z) \to 0$, respectively $r(z) \to 1$, as $|z| \to 1$, we obtain

$$\frac{2}{2\pi}\left[\log|\alpha|^{-1} - 2\varepsilon\right] \leqslant \frac{\mu(Q(z))}{\ell(Q(z))} \leqslant \frac{2}{2\pi - \varepsilon}\left[\log|\alpha|^{-1} + 2\varepsilon\right]$$

for $z \in D_\rho(z_n, m_n)$ and n sufficiently large. Since $\varepsilon > 0$ is arbitrary, we finally obtain

$$\sup_{z \in D_\rho(z_n, m_n)} \left| \frac{\mu(Q(z))}{\ell(Q(z))} - \frac{\log|\alpha|^{-1}}{\pi} \right| \to 0 \quad \text{as } n \to \infty.$$

This finishes the proof of (b.1).

(b) $\Longrightarrow$ (a) : We show that for any $0 < m < 1$, one has

$$\sup\left\{ \left|\log|I(z)|^{-1} - \pi C\right| : \rho(z, z_n) \leqslant m \right\} \xrightarrow[n\to\infty]{} 0. \tag{5.19}$$

This implies that for any $\varepsilon > 0$, the set

$$\{z \in \mathbb{D} : \exp(-\pi C - \varepsilon) \leq |I(z)| \leq \exp(-\pi C + \varepsilon)\}$$

contains arbitrarily large pseudohyperbolic discs. Applying part (b) of Theorem 1 we deduce that $I \notin \mathcal{M}$.

Observe that condition (b.1) implies that μ cannot have a point mass on $\{z : \rho(z, z_n) < m_n\}$ for large n. In fact, suppose that

$$\sup_{z \in D_\rho(z_n, m_n)} \left| \mu(Q(z))/(1-|z|) - C \right| < \tilde{\delta} < 1,$$

where $0 < \tilde{\delta} \ll C$ is a small number to be fixed later, and that μ has point mass at b with $\rho(b, z_n) < m_n$. Note that $\mu(\text{b}) = (1 - |b|^2)/2$. Choose $a \in D_\rho(z_n, m_n)$, $a \neq b$, such that $\arg a = \arg b$ and $1 - |b| > 1 - |a| \geq (1 - \tilde{\delta})(1 - |b|)$. Let $R = Q(b) \setminus \{b\}$. Then $Q(\text{a}) \subseteq Q(b)$ implies

$$C + \tilde{\delta} \geq \frac{\mu(Q(b))}{1-|b|} \geq \frac{\mu(R)}{1-|b|} \geq \frac{\mu(Q(\text{a}))}{1-|a|} \frac{1-|a|}{1-|b|} \geq (C - \tilde{\delta})(1 - \tilde{\delta}). \tag{5.20}$$

Since

$$\frac{\mu(Q(b))}{1-|b|} = \frac{\mu(R)}{1-|b|} + \frac{1+|b|}{2},$$

we get a contradiction to $(C + \tilde{\delta}) - (C - \tilde{\delta})(1 - \tilde{\delta}) < 1/2$ if $\tilde{\delta}$ is sufficiently small.

We deduce that for fixed $0 < m < 1$,

$$\rho\big(\{z : \rho(z, z_n) \leqslant m\}, \{z : I(z) = 0\}\big) \to 1 \quad \text{as } n \to \infty.$$

In order to prove (5.19), it is therefore sufficient, by (4.5), to show

$$\sup \left\{ \left| \int_{\overline{\mathbb{D}}} P_z(w) d\mu(w) - \pi C \right| : \rho(z, z_n) \leqslant m \right\} \xrightarrow[n\to\infty]{} 0. \tag{5.21}$$

The proof proceeds by discretizing the Poisson integral of μ. Given $\varepsilon > 0$, apply (b.2) to obtain $N = N(\varepsilon)$ and $n(\varepsilon) := n(N)$ such that for all $n \geq n(\varepsilon)$,

$$\left| \int_{\overline{\mathbb{D}}} P_z(w) d\mu(w) - \int_{NQ(z)} P_z(w) d\mu(w) \right| < \varepsilon \tag{5.22}$$

uniformly in $z \in D_\rho(z_n, m_n)$. For such a z, given a small number $\delta > 0$ (to be fixed later and depending only on $\varepsilon > 0$) such that N/δ is an integer, we split the arc $NJ(z)$ into N/δ disjoint subarcs $\{J_k : k = 1, \ldots, N/\delta\}$ of length $|J_k| = \delta |J(z)|$. We denote by $Q_k = \{re^{i\theta} : e^{i\theta} \in J_k,\ 1 - r \leq \ell(J_k)\}$ the Carleson box which meets the unit circle at the arc J_k. We claim that μ has no mass on $S_n := NQ(z) \setminus \bigcup Q_k$ for $\rho(z, z_n) \leq m$ and n big. To see this, note that

$$S_n = \big\{ w \in \mathbb{D} : \delta(1 - |z|) \leq 1 - |w| < N(1 - |z|), |\arg w - \arg z| \leq N(1 - |z|) \big\}.$$

Hence, for $w \in S_n$, we have by (4.1) that $1 - \rho^2(z,w) \geq \delta/(2N+3)^2$; and so $\rho(w,z) \leq c = C(N,\delta) < 1$. Thus

$$\rho(w,z_n) \leq \frac{\rho(w,z) + \rho(z,z_n)}{1 + \rho(w,z)\rho(z,z_n)} \leq \frac{c+m}{1+cm} \leq m_n$$

whenever n is large, since $m_n \to 1$. Since μ has no mass on $D_\rho(z_n, m_n)$, we get

$$\mu(NQ(z) \setminus \bigcup Q_k) = 0. \tag{5.23}$$

Next we claim that

$$|P_z(w) - P_z(w_k)| \leqslant \tilde{C}\delta P_z(w) \quad \text{if } w, w_k \in Q_k. \tag{5.24}$$

To see this, note that $|w - w_k| \leq \text{diam } Q_k$, hence $|w - w_k| \leq 4\delta(1 - |z|)$. Therefore,

$$\frac{|1 - \overline{w}_k z|}{|1 - \overline{w} z|} \leq \frac{|1 - \overline{w} z| + |\overline{w} z - \overline{w}_k z|}{|1 - \overline{w} z|} \leq 1 + \frac{4\delta(1-|z|)}{1-|z|} = 1 + 4\delta. \tag{5.25}$$

Hence

$$P_z(w) = \frac{1-|z|^2}{|1-\overline{w}z|^2} \leq \frac{1-|z|^2}{|1-\overline{w}_k z|^2}\frac{|1-\overline{w}_k z|^2}{|1-\overline{w}z|^2} \leq (1+4\delta)^2 P_z(w_k).$$

From this it is easy to conclude that $|P_z(w) - P_z(w_k)| \leq \tilde{C}\delta P_z(w)$, which is (5.24).

From (5.24) and (5.23), we deduce that for $z \in D_\rho(z_n, m)$, n large,

$$\left| \int_{NQ(z)} P_z(w)d\mu(w) - \sum_{k=1}^{N/\delta} P_k(w_k)\mu(Q_k) \right| \leqslant \tilde{C}\delta \int_{NQ(z)} P_z(w)d\mu(w). \tag{5.26}$$

Here w_k is any point in Q_k. Later we take $w_k \in Q_k$ so that $Q(w_k) = Q_k$.

We want to show that the right side term of (5.26) is small if $\delta > 0$ is small. We use the standard estimates of the Poisson kernel given in Lemma 4.2 to show

$$\int_{NQ(z)} P_z(w)d\mu(w) \leqslant C_1 \sum_{k=0}^{M} \frac{\mu(2^k Q(z))}{2^k \ell(2^k Q(z))}. \tag{5.27}$$

Here $M = [\log_2 N] + 1$ and, without loss of generality, $N(1 - |z|) \leq 1/4$, so that $\ell(2^k Q(z)) = 2^k \ell(Q(z))$.

Let $z(k) \in \mathbb{D}$ be defined so that $Q(z(k)) = 2^k Q(z)$. Note that $1-|z_k| = 2^k(1-|z|)$. Hence $\rho(z(k), z) \leq \sqrt{1 - (1/16)(1/2^k)}$. Since $k \leqslant M = [\log_2 N]+1$, which is fixed, and $\rho(z, z_n) \leqslant m < 1$, we have

$$\rho(z(k), z_n) \leqslant \frac{\rho(z(k), z) + \rho(z, z_n)}{1 + \rho(z, z_n)\rho(z(k), z)} \leqslant C(N, m) < 1.$$

Hence $\rho(z(k), z_n) < m_n$ if n is large. Thus (b.1) gives

$$\frac{\mu(2^k Q(z))}{\ell(2^k Q(z))} < C+1, \quad k=0,\dots,M,$$

and (5.27) yields

$$\int_{NQ(z)} P_z(w)d\mu(w) \leqslant 2C_1(C+1).$$

Hence (5.26) gives

$$\left|\int_{NQ(z)} P_z(w)d\mu(w) - \sum_{k=1}^{N/\delta} P_z(w_k)\mu(Q_k)\right| \leqslant 2\delta C_1(C+1)\tilde{C} \tag{5.28}$$

uniformly on $D_\rho(z_n, m)$ for n sufficiently large. Now is the time to specify δ; fix $\delta \in]0, 1/4[$ so that $2\delta C_1(C+1)\tilde{C} < \varepsilon$. Choose w_k so that $Q(w_k) = Q_k$. Observe that

$$1-|z| \geq 1-|w_k| = \ell(Q_k) = \ell(J_k) = \delta\ell(J(z)) = \delta(1-|z|)$$

and that $|\arg w_k - \arg z| \leq N(1-|z|)$. Hence $\rho(w_k, z) \leq \sqrt{1-\delta/(4+N)^2}$.

We conclude that for $z \in D_\rho(z_n, m)$,

$$\rho(w_k, z_n) \leqslant \frac{\rho(w_k, z) + \rho(z, z_n)}{1+\rho(z, z_n)\rho(w_k, z)} \leqslant C(\delta, N, m) < 1,$$

which is smaller than m_n if n is sufficiently large. Thus (b.1) gives

$$\left|\frac{\mu(Q_k)}{\ell(Q_k)} - C\right| < \varepsilon, \quad k = 1, \dots, N/\delta$$

if n is sufficiently large. Therefore, for all z such that $\rho(z, z_n) \leqslant m$,

$$\left|\sum_{k=1}^{N/\delta} P_z(w_k)\mu(Q_k) - C\sum_{k=1}^{N/\delta} P_z(w_k)\ell(Q_k)\right| \leq \varepsilon \sum_{k=1}^{N/\delta} P_z(w_k)\ell(Q_k). \tag{5.29}$$

To further estimate the Poisson sums, we have to show that for every $\xi, \eta \in Q_k$,

$$1-4\delta \leq \frac{|1-\bar{\xi}z|}{|1-\bar{\eta}z|} \leq 1+4\delta. \tag{5.30}$$

Indeed,

$$\frac{|1-\bar{\xi}z|}{|1-\bar{\eta}z|} \geq \frac{|1-\bar{\eta}z| - |\bar{\eta}z - \bar{\xi}z|}{|1-\bar{\eta}z|} \geq 1 - \frac{|\xi-\eta|}{|1-\bar{\eta}z|} \geq 1 - \frac{\operatorname{diam} Q_k}{1-|z|} \geq 1-4\delta.$$

The upper estimate was proven in (5.25).

Now applying this to the upper and lower Riemann–Darboux sums of the Poisson integral given by

$$\mathcal{U} = \sum_{k=1}^{N/\delta} \frac{1-|z|^2}{|\xi_k - z|^2}|J_k| \quad \text{and} \quad \mathcal{L} = \sum_{k=1}^{N/\delta} \frac{1-|z|^2}{|\eta_k - z|^2}|J_k|,$$

where $(\xi_k, \eta_k) \in J_k \times J_k$, we obtain by (5.30),

$$\begin{aligned}\mathcal{R} := \sum_{k=1}^{N/\delta} P_z(w_k)|J_k| = \sum_{k=1}^{N/\delta} \frac{1-|z|^2}{|1-\overline{w}_k z|^2}|J_k| &\le \sum_{k=1}^{N/\delta} \frac{1-|z|^2}{|\eta_k - z|^2}|J_k| \frac{|\eta_k - z|^2}{|1-\overline{w}_k z|^2} \\ &\le (1+4\delta)^2 \mathcal{L} \le 2\pi(1+4\delta)^2 \le 32\end{aligned}$$

as well as

$$\mathcal{R} \ge (1-4\delta)^2 \mathcal{U} \ge 2\pi(1-4\delta)^2.$$

Hence

$$|\mathcal{R} - 2\pi| \le 48\pi\delta \le 200\varepsilon. \tag{5.31}$$

Note that this holds for all z such that $\rho(z, z_n) \leqslant m$ and n sufficiently large. Thus, since $\ell(Q_k) = \frac{1}{2}|J_k|$, (5.22), (5.28), (5.29) and (5.31) show

$$\begin{aligned}\left| \int_{\overline{\mathbb{D}}} P_z(w)d\mu(w) - \pi C \right| < &\left| \int_{\overline{\mathbb{D}}} P_z(w)d\mu(w) - \int_{NQ(z)} P_z(w)d\mu(w) \right| \\ &+ \left| \int_{NQ(z)} P_z(w)d\mu(w) - \sum_{k=1}^{N/\delta} P_z(w_k)\mu(Q_k) \right| \\ &+ \left| \sum_{k=1}^{N/\delta} P_z(w_k)\mu(Q_k) - C \sum_{k=1}^{N/\delta} P_z(w_k)\ell(Q_k) \right| \\ &+ \frac{C}{2} \left| \sum_{k=1}^{N/\delta} P_z(w_k)|J_k| - 2\pi \right| \\ \le &\ \varepsilon + \varepsilon + \frac{32}{2}\varepsilon + \frac{C}{2} 200\varepsilon = C^* \varepsilon\end{aligned}$$

for all $z \in D_\rho(z_n, m)$ if n is sufficiently large. This yields (5.21). □

6 Continuous singular inner functions and Blaschke products in $\mathcal{M}$

6.1 Continuous singular inner functions. It is easy to exhibit continuous singular inner functions not belonging to the class $\mathcal{M}$. One way to do this is again to use results of E. Decker [De], who constructed continuous singular measures μ for which $\lim_{r\to 1} S_\mu(r) = 1/2$, for example. Another way is to use a result of K. Stephenson on covering maps. For instance, if $0 \in K$ is a compact subset of the disc of logarithmic capacity zero, the covering map $\Pi : \mathbb{D} \to \mathbb{D} \setminus K$ is a singular inner function. If $\alpha \in K$, $\tau_\alpha \circ \Pi$ is also singular. So if card$K > 1$, $\Pi \notin \mathcal{M}$. Moreover, Π is a continuous singular inner function if 0 is a limit point of K ([St]).

The construction of continuous singular inner functions in the class $\mathcal{M}$ whose support set is the whole circle, thus not porous, uses the singular measures given in the following result. We first introduce the dyadic decomposition of the unit circle.

For $k = 1, 2, \ldots$, let $\mathcal{F}_k$ be the collection of the 2^k pairwise disjoint (half-open) arcs of the unit circle of length $2\pi 2^{-k}$. So, given $J \in \mathcal{F}_k$, there are two arcs J_1, $J_2 \in \mathcal{F}_{k+1}$ such that $J = J_1 \cup J_2$. The arcs in $\mathcal{F}_k$ are called the dyadic arcs of the unit circle of generation k.

Proposition 6.1. *For $2 > \lambda > 1$ let μ be a positive measure on the unit circle such that for any dyadic arcs J, J_1, J_2 in $\partial\mathbb{D}$ with $J = J_1 \cup J_2$ and $\ell(J_i) = \ell(J)/2$, $i = 1, 2$, one has*

$$\frac{\mu(J_i)}{|J_i|} \geqslant \lambda \frac{\mu(J)}{|J|}$$

for $i = 1$ or $i = 2$. Then μ is singular with respect Lebesgue measure, and its corresponding singular inner function S_μ belongs to the class $\mathcal{M}$.

Proof. The fact that μ must be singular is well known. Actually, μ is concentrated on a set of Hausdorff dimension $h(\lambda)/\log 2$, where

$$h(\lambda) = \frac{\lambda}{2}\log\left(\frac{2}{\lambda}\right) + \left(1 - \frac{\lambda}{2}\right)\log\left(\frac{2}{2-\lambda}\right).$$

See [He], where a more general result is proved. To show that $S_\mu \in \mathcal{M}$, observe that by hypothesis, $\mu(J)/|J|$ and $\mu(J_i)/|J_i|$ can never be close simultaneously to any given number, because

$$\big|\mu(J_i)/|J_i| - \mu(J)/|J|\big| > (\lambda - 1)\mu(J)/|J|\,.$$

This contradicts condition (b.1) in Theorem 3. Note that $\rho(z, z_i) \leq \sqrt{1 - 2/81}$ whenever z and z_i are taken such that $J(z_i) = J_i$ and $J(z) = J$. □

Construction of μ. This is well-known; we sketch it for the convenience of the reader. We define the probability measure μ by induction. Put $\mu(\partial\mathbb{D}) = 1$, and assume that the mass of μ has been defined on all of the generation k. If J is such an arc and $J = J_1 \cup J_2$ its decomposition in two disjoint arcs of length $\ell(J)/2$, we define

$$\mu(J_1) = (\lambda/2)\mu(J) \quad \text{and} \quad \mu(J_2) = (1 - \lambda/2)\mu(J).$$

So the mass of μ on arcs of generation $k+1$ is defined. Iterating this construction, we obtain a probability measure μ on $\partial\mathbb{D}$ such that for the dyadic arc $J_n(\theta)$ of the n-th generation containing $e^{i\theta}$,

$$\mu(J_n(\theta)) = (\lambda/2)^{\nu(n,\theta)}(1 - \lambda/2)^{n-\nu(n,\theta)},$$

where $\nu(n,\theta)$ is the number of dyadic arcs of length bigger than 2^{-n} containing $e^{i\theta}$ whose μ-mass is $\lambda/2$ times the μ mass of its predecessor. The Law of Large Numbers tells us that

$$\lim_{n\to\infty} \frac{\nu(n,\theta)}{n} = \lambda/2, \quad \text{for } \mu\text{-a.e. } e^{i\theta} \in \partial\mathbb{D}.$$

So

$$\lim_{n\to\infty} \frac{\log[\mu(J_n(\theta))]^{-1}}{n} = h(\lambda) \quad \text{for } \mu\text{-a.e. } e^{i\theta} \in \partial\mathbb{D},$$

where

$$h(\lambda) = \frac{\lambda}{2}\log\left(\frac{2}{\lambda}\right) + \left(1 - \frac{\lambda}{2}\right)\log\left(\frac{2}{2-\lambda}\right)$$

(see [He]). We deduce that μ is absolutely continuous with respect to Hausdorff measure H_β if $\beta < h(\lambda)/\log 2$ and singular with respect to Hausdorff measure H_γ if $\gamma > h(\lambda)/\log 2$. In particular, μ is singular with respect to linear measure.

6.2 Blaschke products. As mentioned in the Introduction, we now use Theorem 3 to show that any Blaschke product B having its zeros in a Stolz angle Γ belongs to $\mathcal{M}$. Since $\mathcal{M}$ is closed under multiplication, this extends to Blaschke products whose zero sets lie in finitely many Stolz angles. Let us also mention a related result due to D. Marshall and D. Sarason, which states that if $\lambda \in \mathbb{D}$ is not a cluster point of $B|_\Gamma$ and if λ is not in the set $\{B(a) : B'(a) = 0\}$, then $\tau_\lambda \circ B$ is an interpolating Blaschke product (see [L]).

Proof of Corollary 4. Let B be a Blaschke product whose zeros lie in a Stolz angle. We may assume that its vertex is the point 1 and thus

$$\{z \in \mathbb{D} : B(z) = 0\} \subset \{z \in \mathbb{D} : |z| > 1/2, |1 - z| < M(1 - |z|)\} =: \Gamma(M)$$

for some fixed $M > 1$. Let μ be the measure associated to B in Theorem 3. We will check that condition (b.1) in Theorem 3 does not hold. Let $z \in \mathbb{D}$ satisfy

$$\frac{\mu(Q(z))}{\ell(Q(z))} > 0.$$

Then $Q(z) \cap \Gamma(M) \neq \emptyset$. We show that there exists a constant $C = C(M) < 1$ such that for every $z \in \Gamma(M)$, there exists $\tilde{z} \in \mathbb{D}$ with $\rho(z, \tilde{z}) \leqslant C$ such that $Q(\tilde{z}) \cap \Gamma(M) = \emptyset$. Assuming this has been done, it follows immediately that $\mu(Q(\tilde{z})) = 0$, since μ has its support at the zeros of B, but those lie in the cone $\Gamma(M)$. Thus condition (b.1) in Theorem 3 does not hold, and so $B \in \mathcal{M}$.

To show the existence of the constant C, consider the cone

$$\Gamma^* = \left\{ w \in \mathbb{D} : |w| > 1/2, \frac{|\arg w|}{1 - |w|} \leq 2 + \frac{\pi}{2} M \right\}.$$

Since

$$M > \frac{|1 - z|}{1 - |z|} \geq \frac{(2/\pi)|\arg z|}{1 - |z|} \quad \text{whenever } z \in \Gamma(M),$$

we see that $\Gamma(M) \subset \Gamma^*$, the inclusion being strict. Now choose $\tilde{z} \in \mathbb{D}$ satisfying

$$\frac{\arg \tilde{z}}{1 - |\tilde{z}|} = 2 + \frac{\pi}{2} M \quad \text{and} \quad 1 - |z| = 1 - |\tilde{z}|.$$

To show that $Q(\tilde{z})$ does not meet $\Gamma(M)$, it is sufficient to prove that the left corner q of $Q(\tilde{z})$ does not belong to $\Gamma(M)$. Indeed,

$$q = |\tilde{z}| \exp\left(i(\arg \tilde{z} - (1 - |\tilde{z}|))\right).$$

If $\theta = \arg q$, then

$$\frac{|1 - q|}{1 - |q|} \geq \frac{|\sin \theta|}{1 - |q|} \geq \frac{2}{\pi} \left[\frac{\arg \tilde{z}}{1 - |\tilde{z}|} - 1 \right] > \frac{2}{\pi}(1 + \frac{\pi}{2} M) - \frac{2}{\pi} = M.$$

Hence $q \notin \Gamma(M)$. In order to finish the proof, it remains to observe that, by (4.1),

$$1 - \rho^2(\tilde{z}, z) \geq \frac{(1 - |z|)^2}{(4(1 - |z|) + |\arg z - \arg \tilde{z}|)^2} \geq \frac{1}{(4 + 2M\frac{\pi}{2} + 2)^2}.$$

Hence $\rho(z, \tilde{z}) \leq C(M)$. □

We conclude with some observations and open questions.

As a special case of Corollary 4, we mention the following: let B be an interpolating Blaschke product having its zeros in a cone; then $B \in \mathcal{P}$. For example, if $x_n \in]0, 1[$ is any separated sequence (this means that $\rho(x_n, x_m) \geq \delta > 0$ for $n \neq m$), then the associated Blaschke product b is interpolating, hence in $\mathcal{P}$. Taking $x_n = 1 - 2^{-n}$, we get that $\rho(x_n, x_{n+1}) \to 1/3$. Thus, by Schwarz's lemma, $|b| < 1/2$ on the radius $[r_0, 1[$. Hence the pseudohyperbolic diameter of this level set is 1 (although it does not contain arbitrary large pseudohyperbolic discs). Thus $b \notin (Ns)$, a class of functions studied by Nestoridis ([Ne]) and

Tolokonnikov ([To$_2$]) and defined as the set of all inner functions I such that for every $\eta \in]0,1[$ the pseudohyperbolic diameters of the connected components of the set $\{z \in \mathbb{D} : |I(z)| < \eta\}$ are less than $\delta(I,\eta) < 1$. According to [To$_2$], the space (Ns) is a subset of $\mathcal{P}$; therefore, (Ns) is actually a proper subset of $\mathcal{P}$.

We also note that Corollary 4 cannot be generalized to convex domains tangent to the unit circle. In fact, the interpolating Blaschke product

$$b = \frac{S - 1/2}{1 - (1/2)S},$$

where $S(z) = \exp[-(1+z)/(1-z)]$ is the atomic inner function, is, by the definition of $\mathcal{M}$, not in $\mathcal{M}$; but its zeros are located on the horocycle $\frac{1-|z|^2}{|1-z|^2} = 1/2$.

The atomic inner function plays another important role in the representation of functions in $\mathcal{M}$.

Lemma 6.2. *Let S_μ be a singular inner function in $\mathcal{M}$ and let S be the atomic inner function $S(z) = \exp[-(1+z)/(1-z)]$. Then there exists a Blaschke product $b \in \mathcal{P}$ such that*

$$S_\mu = S \circ b.$$

Moreover,

$$\{x \in M(H^\infty) : S_\mu(x) = 0\} \subseteq \{x \in M(H^\infty) : b(x) = 1\}. \tag{6.1}$$

Proof. Following [St] and [GLMR], the function

$$b = \frac{\log S_\mu + 1}{\log S_\mu - 1}$$

is inner and $S_\mu = S \circ b$. Let x be a trivial point. Suppose that $|b(x)| < 1$. Then, by [Ho$_2$], we have $S_\mu(x) = (S \circ b)(x) = S(b(x)) \in \mathbb{D}$. Since S has no zeros in $\mathbb{D}$, we get a contradiction to the fact that S_μ does not take any value in $\mathbb{D} \setminus \{0\}$ on the set of trivial points. Thus $b \in \mathcal{P}$. In particular, b is a Carleson–Newman Blaschke product.

To prove (6.1), take $x \in M(H^\infty)$ with $b(x) \neq 1$. By the Corona Theorem, there exists a net (z_α) in $\mathbb{D}$ with $z_\alpha \to x$, and so $b(z_\alpha) \to b(x)$. Hence $S(b(z_\alpha)) \to S_\mu(x)$ and, by the analyticity of S outside $\{1\}$, we also have $S(b(z_\alpha)) \to S(b(x))$. But $S(b(x)) \neq 0$. □

It would be interesting to give a characterization of those $b \in \mathcal{P}$ for which $S \circ b \in \mathcal{M}$.

We have the following question. Let $\mu = \sum_{n=1}^\infty s_k \delta_{\theta_k}$ be a discrete measure. Assume that $\theta_k \to \theta$ and that $\sum_{k=1}^\infty s_k/|\theta - \theta_k|$ converges. By a result of Cargo

[C], we know that the singular inner function S_μ associated with μ (and each of its subfactors) has a radial limit of modulus one in $e^{i\theta}$. Is $S_\mu \in \mathcal{M}$? This class of singular inner functions is the analogue of the so-called Frostman Blaschke products (see [To$_2$]). For example, it is known that whenever

$$\sup_{\alpha\in\partial\mathbb{D}} \sum_{n=1}^{\infty} \frac{1-|z_n|}{|1-\overline{\alpha} z_n|} < \infty,$$

then the Blaschke product B with zeros (z_n) is in $\mathcal{P}$ (see [To$_2$]).

In general, $\mathcal{M}$ is not closed under taking subfactors, nor under composition. For example, $S \circ S \notin \mathcal{M}$, where S is the atomic inner function. In fact, $S(x) = 0$ for some trivial point x, so by [Ho$_2$], $(S \circ S)(x) = S(0) \in \mathbb{D} \setminus \{0\}$.

Now let $B \in \mathcal{P}$. Suppose that $B(0) \neq 0$. Then, for the same reason as above, $B \circ S \notin \mathcal{M}$. But, as we are going to show,

$$I = \frac{B(0) - B \circ S}{1 - \overline{B(0)} B \circ S} \in \mathcal{M}.$$

In fact, $B \in \mathcal{P}$ implies that $\tilde{B} := (\tau_a \circ \tau_{B(0)}) \circ B$ is a Carleson–Newman Blaschke product for every $a \in \mathbb{D}$. Obviously $\tilde{B}(0) \neq 0$, if $a \neq 0$. By [Mo], we know that $b \circ S$ is an interpolating Blaschke product whenever b is an interpolating Blaschke product which does not vanish at the origin. Hence, for every $a \in \mathbb{D}$, $a \neq 0$,

$$\tau_a \circ I = (\tau_a \circ \tau_{B(0)} \circ B) \circ S$$

is a Carleson–Newman Blaschke product. Thus $I \in \mathcal{M}$. Noticing that $\tau_{B(0)} \circ B$ vanishes at the origin, we can write it in the form $\tau_{B(0)} \circ B = zC$ for some $C \in \mathcal{P}$. Then we get that $I = S(C \circ S)$. Clearly $C \circ S \notin \mathcal{M}$.

In this connection, we may ask whether for every inner function I there exists a second inner function I^* such that $I I^* \in \mathcal{M}$?

Let us also point out that for every inner function $I \in \mathcal{M}$, there exists a Blaschke product $B \in \mathcal{M}$ such that

$$\{x \in M(H^\infty) \setminus \mathbb{D} : I(x) = 0\} = \{x \in M(H^\infty) \setminus \mathbb{D} : B(x) = 0\}$$

and

$$\{x \in M(H^\infty) : |I(x)| = 1\} = \{x \in M(H^\infty) : |B(x)| = 1\}.$$

This is an immediate consequence of a result of Guillory and Sarason [GS] which says that any inner function is codivisible in $H^\infty + C$ with a Blaschke product.

References

[Bi] C. Bishop, *Bounded functions in the little Bloch space*, Pacific J. Math. **142** (1990), 209–225.

[Bu] P. Budde, *Support sets and Gleason parts*, Michigan Math. J. **37** (1990), 367–383.

[C] L. Cargo, *The radial images of Blaschke products*, J. London Math. Soc. **36** (1961), 424–430.

[Ca] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.

[De] E. Decker, *On the boundary behaviour of singular inner functions*, Michigan Math. J. **41** (1994), 547–562.

[Fr] O. Frostman, *Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds. Univ. Mat. Sem. **3** (1935), 1–118.

[Ga] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[GP] J. Garnett and M. Papadimitrakis, *Almost isometric maps of the hyperbolic plane*, J. London Math. Soc. **43** (1991), 269–282.

[GoIz] P. Gorkin and K. Izuchi, *Some counterexamples in subalgebras of $L^\infty(\mathbb{D})$*, Indiana Univ. Math. J. **40** (1991), 1301–1313.

[GLMR] P. Gorkin, L. Laroco, R. Mortini and R. Rupp, *Composition of inner functions*, Results Math. **25** (1994), 252–269.

[GM] P. Gorkin and R. Mortini, *Interpolating Blaschke products and factorization in Douglas algebras*, Michigan Math. J. **38** (1991), 147–160.

[GuIz] C. Guillory and K. Izuchi, *Interpolating Blaschke products of type G*, Complex Variables **31** (1996), 51–64.

[GS] C. Guillory and D. Sarason, *Division* in $H^\infty + C$, Michigan Math. J. **28** (1981), 173–181.

[GIS] C. Guillory, K. Izuchi and D. Sarason, *Interpolating Blaschke products and division in Douglas algebras*, Proc. Roy. Irish Acad. Sci. **84A** (1984), 1–7.

[He] Y. Heurteaux, *Sur la comparaison des mesures avec les mesures de Hausdorff*, C. R. Acad. Sci. Paris **321** (1995), 61–65.

[Ho_1] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[Ho_2] K. Hoffman, *Bounded analytic functions and Gleason parts*, Ann. of Math. **86** (1967), 74–111.

[KL] A. Kerr-Lawson, *Some Lemmas on interpolating Blaschke products*, Canad. J. Math. **21** (1969), 531–534.

[Ko] P. Koosis, *Introduction to H^p-spaces*, second edition, Cambridge Univ. Press, 1998.

[L] Kin Y. Li, *Interpolating Blaschke products and the left spectrum of multiplication operators on the Bergman space*, Hokkaido Math. J. **21** (1992), 295–304.

[McK] P. McKenna, *Discrete Carleson measures and some interpolation problems*, Michigan Math. J. **24** (1977), 311–319.

[MS] G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), 595–611.

[Mo] R. Mortini, *Commuting inner functions*, J. Math. Anal. Appl. **209** (1997), 724–728.

[MT] R. Mortini and V. Tolokonnikov, *Blaschke products of Sundberg–Wolff type*, Complex Variables **30** (1996), 373–384.

[Ne] V. Nestoridis, *Inner functions: noninvariant connected components*, Pacific J. Math. **87** (1980), 199–209.

[Ni] A. Nicolau, *Finite products of interpolating Blaschke products*, J. London Math. Soc. **50** (1994), 520–531.

[Sh] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York–Heidelberg, 1993.

[St] K. Stephenson, *Omitted values of singular inner funtions*, Michigan Math. J. **25** (1978), 91–100.

[To$_1$] V. Tolokonnikov, *Blaschke products with the Carleson–Newman condition, and ideals of the algebra H^∞*, J. Soviet Math. **42** (1988), 1603–1610.

[To$_2$] V. Tolokonnikov, *Blaschke products with Carleson condition, and Douglas algebras*, St. Petersburg Math. J. **3** (1992), 881–892.

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