

THE LAW OF THE ITERATED LOGARITHM FOR SMOOTH FUNCTIONS

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ABSTRACT. A version of the Law of the Iterated Logarithm for smooth functions in the upper-half space is proved. As a consequence, we show that certain size conditions on the gradient and the gradient of the laplacian of a smooth function, lead to self-improvement growth properties. The results are applied in situations where harmonicity is not present.

1. INTRODUCTION

In this paper, we study self-improving growth properties of smooth functions in the upper half-space $\mathbb{R}_+^{d+1} = \{(x, y) : x \in \mathbb{R}^d, y > 0\}$ under size restrictions on their derivatives. A function $u \in C^1(\mathbb{R}_+^{d+1})$ is called a *Bloch function* if

$$B = \sup\{y|\nabla u(x, y)| : (x, y) \in \mathbb{R}_+^{d+1}\} < \infty$$

and we will refer to B as the Bloch constant of u . Analytic Bloch functions in a half-plane play a crucial role in several topics as conformal mappings, harmonic measure, Bergman spaces, trigonometric series and Zygmund functions among others (see [13, 1, 5, 8, 3, 11, 12] and references therein). Significant examples of harmonic Bloch functions are those of the form $u = \log |f'|$ where $f : \mathbb{R}_2^+ \rightarrow \mathbb{C}$ is a conformal mapping as well as harmonic extensions of lacunary trigonometric series with bounded coefficients. Observe that if u is a Bloch function in $\mathbb{R}_+^{d+1}$ with Bloch constant B , then the following global estimate holds

$$\limsup_{y \rightarrow 0} \frac{|u(x, y)|}{\log(1/y)} \leq B, \quad x \in \mathbb{R}^d.$$

The celebrated Makarov's *Law of the Iterated Logarithm* (LIL) for Bloch harmonic functions says that, in the presence of harmonicity, the global bound $\log(1/y)$ can be substantially improved for almost every point $x \in \mathbb{R}^d$. The following theorem is the half-space version of Makarov's original result in the unit disk ([11, 12]). See also [2, 10] for further results, still in the harmonic case.

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Theorem A. *Let u be a harmonic Bloch function in $\mathbb{R}_+^{d+1}$ with Bloch constant B . Then there exists a constant $C > 0$ only depending on d and B such that*

$$(1.1) \quad \limsup_{y \rightarrow 0} \frac{|u(x, y)|}{\sqrt{\log(1/y) \log \log \log(1/y)}} \leq C$$

for almost every $x \in \mathbb{R}^d$. Moreover (1.1) is sharp in the sense that there exist harmonic Bloch functions u for which the limsup in (1.1) is bounded below by a positive constant for almost every $x \in \mathbb{R}^d$.

Our main purpose is to understand to what extent harmonicity can be relaxed in this result. This problem has already been considered in [7, 4] but the results there do not provide the right analog of the LIL. Note that the function $u(x, y) = \log(1/y)$ is Bloch and exhibits the maximal vertical growth allowed by the Bloch condition at every $x \in \mathbb{R}^d$ so no self-improvement occurs. Since $\Delta u(x, y) = y^{-2}$, this example suggests that in order to obtain meaningful self-improvement results for non-harmonic Bloch functions, the behavior of their laplacian may play an important role.

We introduce two classes of smooth functions in $\mathbb{R}_+^{d+1}$ whose gradient and the gradient of the laplacian have controlled growth in terms of certain gauge functions. Let $\psi, \varepsilon : (0, 1] \rightarrow (0, +\infty)$ and define

$$\mathcal{B}_\psi \equiv \{u \in C^1(\mathbb{R}_+^{d+1}) : y|\nabla u(x, y)| \leq \psi(y), \ x \in \mathbb{R}^d, \ 0 < y \leq 1\},$$

$$\mathcal{B}_{\psi, \varepsilon} \equiv \mathcal{B}_\psi \cap \{u \in C^3(\mathbb{R}_+^{d+1}) : y^3|\nabla \Delta u(x, y)| \leq \varepsilon(y), \ x \in \mathbb{R}^d, \ 0 < y \leq 1\}.$$

We will eventually require the following assumptions on ψ and ε :

$$(1.2) \quad \psi \text{ is non-increasing,}$$

$$(1.3) \quad \varepsilon \leq \psi,$$

$$(1.4) \quad \frac{1}{y} \int_0^y \psi(t) dt \leq A \psi(y), \quad 0 < y \leq 1,$$

for some positive constant A . Observe that (1.7) and (1.9) easily imply the following doubling property:

$$(1.5) \quad \psi(y/2) \leq 2A\psi(y), \quad 0 < y \leq 1.$$

Associated to ψ , define the *square function* Ψ as

$$(1.6) \quad \Psi(y) = \int_y^1 \frac{\psi^2(t)}{t} dt, \quad 0 < y \leq 1.$$

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$$(1.10) \quad \psi(y/2) \leq 2A\psi(y), \quad 0 < y \leq 1.$$

Associated to ψ , define the *square function* Ψ as

$$(1.11) \quad \Psi(y) = \int_y^1 \frac{\psi^2(t)}{t} dt, \quad 0 < y \leq 1.$$

The choice of the constant function $\psi \equiv B > 0$ gives $\Psi(y) = B^2 \log(1/y)$ and $\mathcal{B}_\psi$ becomes then the class of Bloch functions with Bloch constant not greater than B . We now state the main result of this paper.

Theorem 1.1. *Suppose that $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ, ε satisfy (1.7), (1.8), (1.9) and let Ψ be the square function associated to ψ defined by (1.11). Then there exists a constant $C > 0$ only depending on d, ψ, ε such that*

$$(1.12) \quad \limsup_{y \rightarrow 0} \frac{\left| u(x, y) - \int_y^1 t \Delta u(x, t) dt \right|}{\sqrt{\Psi(y) \log \log \Psi(y)}} \leq C,$$

for almost every $x \in \mathbb{R}^d$.

One of the remarkable ideas of the work of Makarov was that the boundary behavior of a harmonic Bloch function in a half space can be related to the asymptotic behavior of a dyadic martingale (see [11, 12]). Actually the mean value property of harmonic functions can be understood as an analog to the cancellation property of dyadic martingales. When the function $u \in C^3(\mathbb{R}_+^{d+1})$ is not harmonic, it turns out that instead of u , the following expression

$$T(x, y) = u(x, y) - y \frac{\partial u}{\partial y}(x, y) - \int_y^1 h \Delta u(x, h) dh, \quad 0 < y < 1, x \in \mathbb{R}^d,$$

can be compared with a dyadic martingale. Since one can estimate the quadratic variation of this dyadic martingale by the functions ψ and ε , the proof of Theorem 1.1 can be transferred to the context of dyadic martingales.

The assumptions on the function ε in the next two corollaries guarantee that the integral in the numerator of (1.12) can be absorbed by the denominator of (1.12).

Corollary 1.1. *Let ψ, ε and $u \in \mathcal{B}_{\psi, \varepsilon}$ be as in Theorem 1.1. Suppose, in addition, that*

$$(1.13) \quad \limsup_{y \rightarrow 0} \frac{\int_y^1 \varepsilon(t) t^{-1} dt}{\sqrt{\Psi(y) \log \log \Psi(y)}} = M < \infty.$$

Then there exists a constant $C > 0$ only depending on d , ψ and M , such that

$$\limsup_{y \rightarrow 0} \frac{|u(x, y)|}{\sqrt{\Psi(y) \log \log \Psi(y)}} \leq C,$$

for almost every $x \in \mathbb{R}^d$.

In the case of Bloch functions, one gets the following result.

Corollary 1.2. *Let $u \in C^3(\mathbb{R}_+^{d+1})$. Assume that there exists a constant $B > 0$ such that*

$$y|\nabla u(x, y)| \leq B, \quad y^3|\nabla \Delta u(x, y)| \leq \varepsilon(y) \leq B$$

for $0 < y \leq 1$, $x \in \mathbb{R}^d$. Suppose, in addition, that

$$(1.14) \quad \limsup_{y \rightarrow 0} \varepsilon(y) \sqrt{\frac{\log(1/y)}{\log \log \log(1/y)}} = D < \infty.$$

Then there exists a constant $C > 0$ only depending on d , B and D , such that (1.1) holds for almost every $x \in \mathbb{R}^d$.

Hence a version of the LIL for smooth (non-harmonic) Bloch functions u holds under the restriction

$$(1.15) \quad y^3|\nabla \Delta(u)(x, y)| \leq C(\log(1/y))^{-1/2}$$

if $0 < y < 1$ and $x \in \mathbb{R}^d$. It is worth mentioning that the exponent $1/2$ in (1.15) is sharp. Actually for $0 < \alpha < 1$, the function $u(x, y) = (\log(1/y))^\alpha$ is in $\mathcal{B}_{\psi, \varepsilon}$ where $\psi(y) = \varepsilon(y) = (\log(1/y))^{\alpha-1}$ while the LIL (1.1) can only hold for $\alpha \leq 1/2$.

We now make two remarks.

1. Given two functions $\psi, \eta : (0, 1] \rightarrow (0, +\infty)$ consider the class $\mathcal{K}_{\psi, \eta}$ of functions $u \in C^2(\mathbb{R}_+^{d+1})$ such that $y|\nabla u(x, y)| \leq \psi(y)$ and $y^2|\Delta u(x, y)| \leq \eta(y)$ for any $x \in \mathbb{R}^d$ and $0 < y \leq 1$. Assume that ψ satisfies (1.7) and (1.9) and in addition, that there exists a constant $A' > 0$ such that

$$(1.16) \quad \int_y^1 \frac{\eta(t)}{t} dt \leq A' \psi(y), \quad 0 < y \leq 1.$$

The proof of Theorem 1.1 applies with minor modifications and (1.12) holds for any function $u \in \mathcal{K}_{\psi, \eta}$. Since the assumption (1.16) implies that the integral in the numerator of (1.12) is bounded by the denominator of (1.12), it follows that (1.1) holds for any function $u \in \mathcal{K}_{\psi, \eta}$. If ψ is a constant function, condition (1.16) reduces to the integrability of $\eta(t)/t$. In this case, one can also prove (1.1) by decomposing u as the sum of a Green potential and a harmonic Bloch function and applying Theorem A. It is worth mentioning that the functions in $\mathcal{B}_{\psi, \varepsilon}$ considered in Theorem 1.1 may not satisfy (1.16) and its laplacian may not have a convergent Green potential.

2. Regarding the growth of the gradient, one may ask until what extent the Bloch condition can be relaxed and still obtain self-improvement results.

We will show that there exists a growth threshold condition on the gradient which provides self-improvement results as the LIL in Theorem 1.1 (see section 4.3 for details).

Theorem 1.1 can be applied to obtain versions of the LIL in two natural situations where harmonicity is not available. The first one concerns logarithms of positive harmonic functions. Let v be a positive harmonic function in the upper half-space $\mathbb{R}_+^{d+1}$. Harnack's inequality gives that there exists a constant $C = C(d) > 0$ such that

$$|\log v(x, y) - \log v(x, 1)| \leq C \log(1/y)$$

for $0 < y \leq 1/2$ and $x \in \mathbb{R}^d$. Note that if v is the harmonic extension to $\mathbb{R}_+^{d+1}$ of a finite singular measure in $\mathbb{R}^d$, then $\log v(x, y) \rightarrow -\infty$, as $y \rightarrow 0$ for almost every $x \in \mathbb{R}^d$. The next result shows that the right growth of $|\log v|$ is given by a square function $A(v)$ and its corresponding deviation $|\log v(x, y) + A^2(v)(x, y)|$ is governed by a LIL.

Corollary 1.3. *Let v be a positive harmonic function in the upper half-space $\mathbb{R}_+^{d+1}$. Consider the square function*

$$A^2(v)(x, y) = \int_y^1 \frac{t |\nabla v(x, t)|^2}{v^2(x, t)} dt, \quad 0 < y < 1, x \in \mathbb{R}^d.$$

Then there exists a constant $C > 0$ only depending on d , such that

$$\limsup_{y \rightarrow 0} \frac{|\log v(x, y) + A^2(v)(x, y)|}{\sqrt{\log(1/y) \log \log \log(1/y)}} \leq C,$$

for almost every $x \in \mathbb{R}^d$.

We will also show that there are positive harmonic functions v in $\mathbb{R}_+^{d+1}$ such that $A^2(v)(x, y) \geq C \log(1/y)$, for any $x \in \mathbb{R}^d$ and $0 < y \leq 1/2$. In particular, this will eventually show that the integral in (1.12) (Theorem 1.1) can be larger than the denominator.

The second application of Theorem 1.1 concerns analytic selfmappings of the unit disc $\mathbb{D}$ of the complex plane. Let $d_h(z, w)$ denote the hyperbolic distance in $\mathbb{D}$ between the points $z, w \in \mathbb{D}$. The Schwarz-Pick Lemma says that any analytic mapping $f : \mathbb{D} \rightarrow \mathbb{D}$ is a contraction in the hyperbolic metric, that is, $d_h(f(z), f(w)) \leq d_h(z, w)$ for any $z, w \in \mathbb{D}$, or equivalently, the hyperbolic derivative defined as

$$D_h(f)(z) = \lim_{w \rightarrow z} \frac{d_h(f(w), f(z))}{d_h(w, z)} = \frac{(1 - |z|^2) |f'(z)|}{1 - |f(z)|^2}$$

is bounded by 1 for any $z \in \mathbb{D}$. Moreover, equality at a single point implies equality at every point in the unit disc and that f is an automorphism of $\mathbb{D}$. Our last result can be understood as a holomorphic version of Corollary 1.3.

Corollary 1.4. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic mapping. Consider the square function*

$$A^2(f)(\xi, r) = \int_0^r \frac{4 \log(1/t) |f'(t\xi)|^2}{(1 - |f(t\xi)|^2)^2} dt, \quad 0 < r < 1, \xi \in \partial\mathbb{D}.$$

Then there exists a constant $C > 0$ such that

$$\limsup_{r \rightarrow 1} \frac{|d_h(f(r\xi), 0) - A^2(f)(\xi, r)|}{\sqrt{\log(1/(1-r)) \log \log \log(1/(1-r))}} \leq C,$$

for almost every $\xi \in \partial\mathbb{D}$.

It is worth mentioning that a conical analog of the square function $A^2(f)$ was considered in [6]. In this conical analog, the radius and length appearing in the definition of $A^2(f)$ are replaced by a Stolz angle and hyperbolic area.

The paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the LIL for dyadic martingales. Theorem 1.1 and Corollaries 1.1 and 1.2 are proved in section 4, which also includes the discussion of the threshold condition on the gradient giving self-improving. Finally, Corollaries 1.3 and 1.4 are proved in section 5.

2. PRELIMINARY RESULTS

Our first auxiliary result collects some elementary estimates about the function Ψ .

Lemma 2.1. *Suppose that ψ satisfies conditions (1.7) and (1.9) and let Ψ be its square function defined by (1.11). Then*

- a) $\Psi(y) \geq \psi^2(1) \log\left(\frac{1}{y}\right)$ for $0 < y \leq 1$. In particular, $\Psi(y) \rightarrow +\infty$ as $y \rightarrow 0^+$.
- b) There exists a constant $C = C(\psi) > 0$ such that $\Psi(y) \geq C\psi^2(y)$ for any $0 < y \leq 1/2$.
- c) There exists a constant $C = C(\psi) > 0$ such that the function Ψ satisfies the following doubling property:

$$\Psi(y/2) \leq C\Psi(y)$$

whenever $0 < y \leq 1/2$.

Proof. Part a) easily follows from the assumption that ψ is non-increasing.

Now take $0 < y \leq 1/2$. Then, using (1.10), one has

$$\Psi(y) \geq \int_y^{2y} \frac{\psi^2(t)}{t} dt \geq (\log 2) \psi^2(2y) \geq \frac{\log 2}{4A^2} \psi^2(y),$$

which proves b). Finally, if $0 < y \leq 1/2$ then

$$\begin{aligned} \int_{y/2}^y \frac{\psi^2(t)}{t} dt &\leq (\log 2) \psi^2(y/2) \leq (\log 2) 16A^4 \psi^2(2y) \\ &\leq 16A^4 \int_y^{2y} \frac{\psi^2(t)}{t} dt \leq 16A^4 \Psi(y). \end{aligned}$$

Therefore,

$$\Psi(y/2) = \int_{y/2}^y \frac{\psi^2(t)}{t} dt + \int_y^1 \frac{\psi^2(t)}{t} dt \leq (1 + 16A^4)\Psi(y).$$

□

A cube Q in $\mathbb{R}^d$ is a set of the form $Q = \prod_{j=1}^d [a_j, b_j]$, where $a_j < b_j$, $j = 1, \dots, d$ and $b_1 - a_1 = \dots = b_d - a_d = l = l(Q) > 0$, the side length of Q . Let $Q = \prod_{j=1}^d [a_j, b_j]$ be a fixed cube in $\mathbb{R}^d$ with side length $l = l(Q) \leq 1$. Given $u \in \mathcal{C}^2(\mathbb{R}_+^{d+1})$, denote, for $x \in Q$ and $0 < y \leq 1$,

$$(2.1) \quad T(x, y) = u(x, y) - y \frac{\partial u}{\partial y}(x, y) - \int_y^1 h \Delta u(x, h) dh.$$

Now, let $0 < s < t \leq l$ and consider the block in $\mathbb{R}_+^{d+1}$ given by

$$(2.2) \quad R_{s,t} = \{(x, y) \in \mathbb{R}_+^{d+1} : x \in Q, s < y < t\}.$$

Observe that $\partial R_{s,t}$ consists of two horizontal faces parallel to Q at heights s and t and $2d$ lateral faces $L_j = Q_j \times [s, t]$, $j = 1, \dots, 2^d$, where Q_j is a $(d-1)$ -dimensional cube of the form

$$Q_j = \{(x_1, \dots, x_{j-1}, c_j, x_{j+1}, \dots, x_d) : x_i \in [a_i, b_i], c_j \in \{a_j, b_j\}\}.$$

Given a cube $Q \subset \mathbb{R}^d$ and $0 < t < l(Q)$, we use the notation

$$\oint_Q T(x, t) dx = \frac{1}{|Q|} \int_Q T(x, t) dx.$$

The following lemma controls the vertical variation of T on cubes.

Lemma 2.2. *Let $u \in \mathcal{B}_\psi$. Then for any cube $Q \subset \mathbb{R}^d$ of sidelength $l \leq 1$ and any $0 < s \leq t \leq l$ we have*

$$(2.3) \quad \oint_Q T(x, t) dx = \oint_Q T(x, s) dx + \frac{1}{|Q|} \sum_j \int_{L_j} y \frac{\partial u}{\partial n},$$

where L_j are the lateral faces of $\partial R_{s,t}$ and n is the unit outward normal. In particular,

$$(2.4) \quad \left| \oint_Q (T(x, t) - T(x, s)) dx \right| \leq \frac{2d}{l} \int_s^t \psi(y) dy.$$

Proof. By Green's first identity applied to the functions y and u in the domain $R_{s,t}$ we get

$$(2.5) \quad \int_{R_{s,t}} y \Delta u = \int_{\partial R_{s,t}} \left(y \frac{\partial u}{\partial n} - u \frac{\partial y}{\partial n} \right),$$

where, as usual, n denotes unit outward normal. Since $\frac{\partial y}{\partial n} = 0$ on the lateral faces of $\partial R_{s,t}$ and $\frac{\partial y}{\partial n} = \pm 1$ on the horizontal faces of $\partial R_{s,t}$, (2.5) reduces to

$$\begin{aligned}
\int_Q (u(x, t) - u(x, s)) dx &= \int_Q \left(t \frac{\partial u}{\partial y}(x, t) - s \frac{\partial u}{\partial y}(x, s) \right) dx \\
&+ \sum_j \int_{L_j} y \frac{\partial u}{\partial n} - \int_Q \int_s^t y \Delta u(x, y) dy dx
\end{aligned}$$

which, according to (2.1), is exactly (2.3). Note that, if $L_j = Q_j \times [s, t]$ is one of the lateral faces of $\partial R_{s,t}$ then

$$\left| \frac{1}{|Q|} \int_{L_j} y \frac{\partial u}{\partial n} \right| \leq \frac{|Q_j|}{|Q|} \int_s^t \psi(y) dy \leq \frac{1}{l} \int_s^t \psi(y) dy.$$

Adding over j , one obtains (2.4). $\square$

Corollary 2.1. *Let $u \in \mathcal{B}_\psi$ where ψ is an integrable function in $(0, 1)$. Then for any cube $Q \subset \mathbb{R}^d$, the limit*

$$(2.6) \quad \lim_{y \rightarrow 0} \oint_Q T(x, y) dx \equiv T_Q$$

exists.

Proof. The corollary is a direct consequence of (2.4) and the fact that ψ is integrable. $\square$

Corollary 2.2. *Let $u \in \mathcal{B}_\psi$ where ψ satisfies (1.7) and (1.9). Then there exists a constant $C > 0$ only depending on d and the constant A in inequality (1.9), such that for any cube $Q \subset \mathbb{R}^d$ of side length $l \leq 1$ and any $0 < s \leq t \leq l$ we have*

$$(2.7) \quad \left| \oint_Q (T(x, t) - T(x, s)) dx \right| \leq C\psi(l).$$

In particular,

$$(2.8) \quad \left| \oint_Q T(x, t) dx - T_Q \right| \leq C\psi(l)$$

whenever $0 < t \leq l$.

Proof. From (2.4) and (1.9), it follows

$$\left| \oint_Q (T(x, t) - T(x, s)) dx \right| \leq \frac{2d}{l} \int_0^l \psi(y) dy \leq 2dA\psi(l).$$

This gives (2.7). Estimate (2.8) follows from Corollary 2.1 letting $s \rightarrow 0$. $\square$

Our next auxiliary result collects estimates on the horizontal oscillation of the function T .

Lemma 2.3. *Given $u \in \mathcal{C}^2(\mathbb{R}_+^{d+1})$ let $T(x, y)$ be the function defined in (2.1). Let $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ, ε satisfy (1.7) and (1.8). Then*

$$(2.9) \quad |T(z, y) - T(x, y)| \leq 2 \left(\frac{|z - x|}{y} + 1 \right) \psi(y),$$

for $x, z \in \mathbb{R}^d$ and $0 < y \leq 1$. In particular, if $|z - x| \leq My$, then

$$(2.10) \quad |T(z, y) - T(x, y)| \leq 2(M + 1)\psi(y).$$

Proof. Observe that $T(z, y) - T(x, y) = I - II - III$, where

$$\begin{aligned} I &= u(z, y) - u(x, y), \\ II &= y \left(\frac{\partial u}{\partial y}(z, y) - \frac{\partial u}{\partial y}(x, y) \right), \\ III &= \int_y^1 t (\Delta u(z, t) - \Delta u(x, t)) dt. \end{aligned}$$

Then $|I| \leq |z - x|\psi(y)/y$ and $|II| \leq 2\psi(y)$. Note that $|\Delta u(z, t) - \Delta u(x, t)| \leq |z - x|\varepsilon(t)t^{-3}$. Using (1.7) and (1.8), we deduce

$$|III| \leq |z - x| \int_y^1 \frac{\varepsilon(t)}{t^2} dt \leq |z - x| \int_y^1 \frac{\psi(t)}{t^2} dt \leq \frac{|z - x|}{y} \psi(y).$$

So (2.9) follows. This finishes the proof. $\square$

Lemma 2.4. *Let $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ, ε satisfy (1.7), (1.8) and (1.9). Let T be the function defined in (2.1). Then there exists a constant $C > 0$ only depending on d, ψ and ε , such that if $Q \subset \mathbb{R}^d$ is a cube with $l = l(Q) \leq 1$, $x \in Q$ and $0 < t \leq l/2 \leq y \leq l$, then*

$$(2.11) \quad \left| \oint_Q T(z, t) dz - T(x, y) \right| \leq C \psi(l).$$

In particular,

$$(2.12) \quad |T_Q - T(x, y)| \leq C \psi(l)$$

whenever $x \in Q$ and $l/2 \leq y \leq l$.

Proof. By (2.7) in Corollary 2.2,

$$(2.13) \quad \left| \oint_Q (T(z, t) - T(z, y)) dz \right| \leq C \psi(l)$$

where C depends only on d and ψ . On the other hand, from (2.10) in Lemma 2.3, for any $z \in Q$ we have

$$(2.14) \quad |T(z, y) - T(x, y)| \leq 2(2\sqrt{d} + 1)\psi(y) \leq 2(2\sqrt{d} + 1)\psi(l/2).$$

Now (2.11) follows from (2.13), (2.14) and the doubling property (1.10). Letting $t \rightarrow 0$ in (2.11) we obtain (2.12). $\square$

Corollary 2.3. *Let $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ, ε satisfy (1.7), (1.8) and (1.9). Then there exists a constant $C > 0$ only depending on d, ψ and ε , such that if $Q, Q' \subset \mathbb{R}^d$ are (closed) cubes such that $l(Q) = l(Q') = l \leq 1$ and $Q \cap Q' \neq \emptyset$ then*

$$(2.15) \quad |T_Q - T_{Q'}| \leq C\psi(l).$$

Proof. Choose $x \in Q \cap Q'$. Then, from (2.12)

$$|T_Q - T_{Q'}| \leq |T_Q - T(x, l)| + |T(x, l) - T_{Q'}| \leq 2C\psi(l).$$

□

3. DYADIC MARTINGALES

For simplicity, we will restrict our attention to the unit cube $Q_0 = [0, 1]^d$. By $\mathcal{D}_n$ we denote the family of all *dyadic* cubes in Q_0 of *generation* n , that is, those cubes of the form

$$\prod_{j=1}^d [(k_j - 1)2^{-n}, k_j 2^{-n})$$

where $k_j \in \{1, 2, \dots, 2^n\}$ for $j = 1, \dots, d$. A *dyadic martingale* in Q_0 is a sequence $\{T_n\}_{n=0}^\infty$ of step functions $T_n : Q_0 \rightarrow \mathbb{R}$ satisfying the following two conditions:

- (1) T_n is constant on each dyadic cube of generation n . (Hereafter $T_n(Q)$ denotes the value of T_n on Q , whenever $Q \in \mathcal{D}_n$).
- (2) If $Q \in \mathcal{D}_{n-1}$ and $Q = Q^1 \cup Q^2 \cup \dots \cup Q^{2^d}$ is the decomposition of Q into its 2^d descendants $Q^j \in \mathcal{D}_n$, then

$$T_{n-1}(Q) = \frac{1}{2^d} \sum_{j=1}^{2^d} T_n(Q^j),$$

that is, the value of T_{n-1} on Q is the arithmetic mean of the values of T_n on the 2^d dyadic descendants of Q of generation n .

The first condition means that T_n is measurable with respect to the σ -algebra $\mathcal{D}_{n-1}^*$ generated by the dyadic cubes of generation $n-1$ and the second one means that the conditional expectation of T_n with respect to $\mathcal{D}_{n-1}^*$ is T_{n-1} . The *increments* of the martingale $\{T_n\}$ are defined as $X_0 \equiv T_0$ and $X_k = T_k - T_{k-1}$, $k = 1, 2, \dots$. Note that

$$T_n = \sum_{k=1}^n X_k, \quad n = 1, 2, \dots$$

Given $x \in Q_0$ and $n \in \mathbb{N}$ there is a unique dyadic “tower” $Q_n \subset Q_{n-1} \subset \dots \subset Q_1 \subset Q_0$ such that $Q_k \in \mathcal{D}_k$ and $x \in Q_k$ for $k = 0, 1, \dots, n$.

The asymptotic behaviour of a dyadic martingale $\{T_n\}$ is governed by the so called *quadratic variation* $\langle T \rangle_n$ defined as follows. Fix $n \in \mathbb{N}$ and, for each $x \in Q_0$, consider the dyadic tower $Q_n \subset Q_{n-1} \subset \dots \subset Q_1 \subset Q_0$

containing x . Each member of the tower, say Q_{k-1} , has a dyadic decomposition $Q_{k-1} = Q_k^1 \cup \dots \cup Q_k^{2^d}$ where $Q_k^j \in \mathcal{D}_k$ (and there exists a unique $j = 1, \dots, 2^d$ such that $x \in Q_k^j$). Observe that $X_k = T_k - T_{k-1}$ is constant on each Q_k^j . The quadratic variation $\langle T \rangle_n$ of $\{T_n\}$ is defined as

$$\langle T \rangle_n(x) = \sum_{k=1}^n \frac{1}{2^d} \sum_{j=1}^{2^d} (T_k(Q_k^j) - T_{k-1}(x))^2, \quad x \in Q_0.$$

Note that $\langle T \rangle_n$ is constant on dyadic cubes of generation $n-1$ and for any $x \in Q_0$, the sequence $\langle T \rangle_n(x)$ is non decreasing. We denote

$$\langle T \rangle_\infty(x) = \lim_{n \rightarrow \infty} \langle T \rangle_n(x), \quad x \in Q_0,$$

regardless the limit is finite or infinite. By well known results in martingale theory (see [14]), $\{T_n(x)\}$ converges to a finite limit for almost every point $x \in \{x \in Q_0 : \langle T \rangle_\infty(x) < \infty\}$. Therefore, it is natural to ask about the asymptotic behaviour of $\{T_n\}$ on the set $\{x \in Q_0 : \langle T \rangle_\infty(x) = \infty\}$. One of the most beautiful results in Probability, the so called *Law of the Iterated Logarithm* (LIL for short), gives an astonishingly precise answer to this question. In our specific setting, it says that there exists a constant $C = C(d) > 0$ such that

$$(3.1) \quad \limsup_{n \rightarrow \infty} \frac{|T_n(x)|}{\sqrt{\langle T \rangle_n(x) \log \log \langle T \rangle_n(x)}} \leq C,$$

for almost every $x \in \{x \in Q_0 : \langle T \rangle_\infty(x) = \infty\}$. Moreover (3.1) is sharp in the sense that, under certain size restrictions on the increments $|X_k|$, the limsup is also bounded below for almost every $x \in Q_0$ (see [2, 15]). Note that the LIL (3.1) typically gives more information than the Law of Large Numbers. If, for instance, the increments of the martingale are uniformly bounded, that is, $|X_k| = |T_k - T_{k-1}| \leq C$ for any $k = 1, 2, \dots$, then the trivial uniform estimate is $\sup\{|T_n(x)| : x \in Q_0\} \leq Cn$, while the Law of Large Numbers would give $|T_n(x)| = o(n)$ a.e. $x \in Q_0$. However, since $\langle T \rangle_n(x) \leq C^2 n$, the LIL (3.1) gives that $|T_n(x)| = O(\sqrt{n \log \log n})$ for a.e. $x \in Q_0$, which is a substantial improvement of the previous estimates.

4. PROOFS OF MAIN RESULTS

4.1. Reduction to the martingale setting. Let $u \in \mathcal{B}_\psi$, where ψ satisfies (1.7) and (1.9). In this section we will introduce a dyadic martingale which captures the asymptotic behaviour of $u(x, y)$ as $y \rightarrow 0$. Since the problem is local we will assume hereafter that $x \in Q_0 = [0, 1]^d$ and we will define a dyadic martingale in Q_0 . Note that (1.9) implies in particular the integrability of ψ so, by Corollary 2.1, the limit

$$(4.1) \quad T_n(Q) \equiv T_Q = \lim_{y \rightarrow 0} \frac{1}{|Q|} \int_Q T(x, y) dx$$

exists for any $Q \in \mathcal{D}_n$, where $T(x, y)$ is given by (2.1). This assignment clearly defines a dyadic martingale $\{T_n\}$ in Q_0 whose main properties are collected in the following auxiliary results.

Lemma 4.1. *Let $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ and ε satisfy (1.7), (1.8) and (1.9) and let $\{T_n\}$ be the dyadic martingale in Q_0 associated to u as in (4.1). Then there exists a constant $C > 0$ only depending on d , ψ and ε , such that*

$$(4.2) \quad \left| u(x, y) - \int_y^1 t \Delta u(x, t) dt - T_n(x) \right| \leq C \psi(2^{-n}),$$

for any $x \in Q_0$ and $2^{-(n+1)} \leq y \leq 2^{-n}$, $n = 1, 2, \dots$

Proof. Using the function $T(x, y)$ defined in (2.1), note that

$$u(x, y) - \int_y^1 h \Delta u(x, t) dt - T_n(x) = T(x, y) - T_n(x) + y \frac{\partial u}{\partial y}(x, y).$$

From (2.12) in Lemma 2.4 we get

$$|T(x, y) - T_n(x)| \leq C \psi(2^{-n}), \quad x \in Q_0, n \geq 1.$$

On the other hand, using the doubling property (1.10),

$$\left| y \frac{\partial u}{\partial y}(x, y) \right| \leq \psi(y) \leq 2A \psi(2^{-n})$$

and (4.2) follows. $\square$

Lemma 4.2. *Let $u \in \mathcal{B}_{\psi, \varepsilon}$ where ψ, ε satisfy (1.7), (1.8) and (1.9) and let $\{T_n\}$ be the dyadic martingale in Q_0 associated to u as in (4.1). Then there exists a constant $C > 0$ only depending on d , ψ and ε , such that*

$$(4.3) \quad |T_k(x) - T_{k-1}(x)| \leq C \psi(2^{-k}), \quad k = 1, 2, \dots,$$

$$(4.4) \quad \langle T \rangle_n(x) \leq C \int_{2^{-n}}^1 \frac{\psi^2(t)}{t} dt, \quad n = 1, 2, \dots,$$

for any $x \in Q_0$.

Proof. Note that (4.3) is a direct consequence of (2.15) in Corollary 2.3 and the definition of martingale. As for (4.4), observe that, from (4.3),

$$\langle T \rangle_n(x) \leq C^2 \sum_{k=1}^n \psi^2(2^{-k}), \quad x \in Q_0,$$

and, since ψ is non-increasing and doubling, (4.4) follows in an elementary way. $\square$

4.2. Proofs of Theorem 1.1 and corollaries 1.1, 1.2.

Proof of Theorem 1.1. Let $u \in \mathcal{B}_{\psi, \varepsilon}$. We can restrict our attention to the unit cube Q_0 . Let C denote a positive constant only depending on d, ψ and ε whose value may change from line to line. Let $\{T_n\}$ be the dyadic martingale in Q_0 associated to u as in (4.1). If $2^{-(n+1)} \leq y \leq 2^{-n}$ then by (4.4)

$$(4.5) \quad \Psi(y) \geq \int_{2^{-n}}^1 \frac{\psi^2(t)}{t} dt \geq C^{-1} \langle T \rangle_n(x), \quad x \in Q_0.$$

Then, by (4.2) and (4.5),

$$\begin{aligned} & \frac{\left| u(x, y) - \int_y^1 h \Delta u(x, h) dh \right|}{\sqrt{\Psi(y) \log \log \Psi(y)}} \leq \\ & \leq \frac{C |T_n(x)|}{\sqrt{\langle T \rangle_n(x) \log \log \langle T \rangle_n(x)}} + \frac{C \psi(2^{-n})}{\sqrt{\Psi(y) \log \log \Psi(y)}}. \end{aligned}$$

Now, from parts a) and b) of Lemma 2.1,

$$\frac{\psi(2^{-n})}{\sqrt{\Psi(y) \log \log \Psi(y)}} \leq \frac{C}{\sqrt{\log \log \Psi(y)}} \rightarrow 0 \quad \text{as } y \rightarrow 0^+,$$

so the result follows from the LIL (3.1) applied to the martingale $\{T_n\}$. $\square$

Proof of Corollary 1.1. Fix $x \in Q_0$. From the definition of $\varepsilon(t)$ and Fubini's theorem we get

$$\begin{aligned} \left| \int_y^1 h \Delta u(x, h) dh \right| & \leq \left| \int_y^1 h (\Delta u(x, h) - \Delta u(x, 1)) dh \right| + |\Delta u(x, 1)| \int_y^1 h dh \\ & \leq \int_y^1 h \int_h^1 \frac{\varepsilon(t)}{t^3} dt dh + \frac{1}{2} |\Delta u(x, 1)| \\ & \leq \frac{1}{2} \int_y^1 \frac{\varepsilon(t)}{t} dt + \frac{1}{2} |\Delta u(x, 1)|. \end{aligned}$$

So the corollary follows from (1.12) and assumption (1.13). $\square$

Proof of Corollary 1.2. Since $\Psi(y) \leq B^2 \log(1/y)$, it is sufficient to check, by direct computation, that hypothesis (1.14) implies

$$\limsup_{y \rightarrow 0} \frac{\int_y^1 \frac{\varepsilon(t)}{t} dt}{\sqrt{\log(1/y) \log \log \log(1/y)}} < \infty.$$

Then Corollary 1.2 follows from Corollary 1.1. $\square$

4.3. Threshold condition for self-improving. The goal of this subsection is to understand which growth conditions on the gradient give self-improving results as Theorem 1.1. Standard examples of lacunary series provide a negative answer in this direction: for any $0 < \delta < 1$ there exists u harmonic in $\mathbb{R}_+^2$, satisfying $y|\nabla u(x, y)| \leq y^{-\delta}$ for any $(x, y) \in \mathbb{R}_+^2$ and such that

$$\limsup_{y \rightarrow 0} y^\delta |u(x, y)| > 0,$$

for almost every $x \in \mathbb{R}$. Since the trivial global bound for $|u(x, y)|$ is $O(y^{-\delta})$ as $y \rightarrow 0$, no self-improvement occurs. On the other hand, if $u \in \mathcal{B}_\psi$ then

$$(4.6) \quad |u(x, y)| \leq |u(x, 1)| + \int_y^1 \frac{\psi(t)}{t} dt, \quad x \in \mathbb{R}^d,$$

for $0 < y \leq 1$, therefore the vertical growth of $|u|$ is governed by the integral at the right hand side of (4.6). In view of Theorem 1.1, self-improvement will occur provided that

$$(4.7) \quad \sqrt{\Psi(y) \log \log \Psi(y)} = o\left(\int_y^1 \frac{\psi(t)}{t} dt\right) \quad \text{as } y \rightarrow 0.$$

As usual, the notation $f(y) = o(g(y))$ means that $f(y)/g(y)$ tends to 0 as $y \rightarrow 0$. It is then natural to ask if there exists a “threshold” growth condition on ψ implying (4.7). This is the content of the next result.

Theorem 4.1. *Suppose that $\psi : (0, 1] \rightarrow (0, +\infty)$ satisfies (1.7), (1.10) and the following concavity condition*

$$(4.8) \quad \psi(y/2) \psi(2y) \leq \psi^2(y), \quad 0 < y \leq 1/2.$$

If, in addition,

$$(4.9) \quad \log \psi(y) = o\left(\frac{\log\left(\frac{1}{y}\right)}{\log \log\left(\frac{1}{y}\right)}\right) \quad \text{as } y \rightarrow 0,$$

then (4.7) holds.

Remark 4.1. The specific logarithmic expression at the right hand side of (4.9) comes up, in a natural way, when solving the ODE derived from the identity

$$\sqrt{\Psi(y) \log \log \Psi(y)} = \int_y^1 \frac{\psi(t)}{t} dt$$

together with standard asymptotic estimates. Condition (4.9) can be seen as a threshold assumption on ψ that guarantees self-improvement. Some concavity property like (4.8) seems to be necessary for the argument behind Theorem 4.1 to work. For instance one can take $\psi(y) = (\log(1/y))^\alpha$ where $\alpha > 0$.

Note that the doubling property (1.10) provides constants $C_1, C_2 > 0$ such that, if $2^{-(n+1)} \leq y \leq 2^{-n}$ for some positive integer n , then

$$\begin{aligned} C_1 \sum_{k=1}^n \psi(2^{-k}) &\leq \int_y^1 \frac{\psi(t)}{t} dt \leq C_2 \sum_{k=1}^n \psi(2^{-k}), \\ C_1 \sum_{k=1}^n \psi^2(2^{-k}) &\leq \Psi(y) \leq C_2 \sum_{k=1}^n \psi^2(2^{-k}). \end{aligned}$$

Then, taking $a_k = \psi(2^{-k})$, $k = 1, 2, \dots$, Theorem 4.1 can be rephrased in discrete terms as follows.

Lemma 4.3. *Let $\{a_n\}$ be a sequence of positive numbers satisfying*

$$(4.10) \quad a_n \leq a_{n+1} \leq C a_n, \quad n = 1, 2, \dots,$$

$$(4.11) \quad a_{n+1} a_{n-1} \leq a_n^2, \quad n = 1, 2, \dots,$$

$$(4.12) \quad \log a_n = o\left(\frac{n}{\log n}\right),$$

for some constant $C \geq 1$. Then

$$(4.13) \quad \sqrt{\left(\sum_{k=1}^n a_k^2\right) \log \log \left(\sum_{k=1}^n a_k^2\right)} = o\left(\sum_{k=1}^n a_k\right) \quad \text{as } n \rightarrow \infty.$$

Proof of Lemma 4.3. Let us perform some preliminary reductions. Considering the function $F(x) = x \log \log x$, $x > 1$, (4.13) reads

$$F\left(\sum_{k=1}^n a_k^2\right) = o\left(\left(\sum_{k=1}^n a_k\right)^2\right),$$

which is, in turn, equivalent to

$$\sum_{k=1}^n a_k^2 = o\left(G\left(\sum_{k=1}^n a_k\right)\right),$$

where $G(x) = x^2 / \log \log x$. By Stolz's lemma and (4.10), it is sufficient to show that

$$(4.14) \quad \lim_{n \rightarrow \infty} \frac{a_n^2}{a_n G'\left(\sum_{k=1}^n a_k\right)} = 0.$$

Since $G'(x)$ is comparable to $H(x) = x / \log \log x$ as $x \rightarrow \infty$, (4.14) is equivalent to

$$(4.15) \quad \lim_{n \rightarrow \infty} \frac{a_n}{H\left(\sum_{k=1}^n a_k\right)} = 0.$$

Now, another application of Stolz's lemma says that (4.15) follows from

$$(4.16) \quad \lim_{n \rightarrow \infty} \frac{(a_n - a_{n-1}) \log \log \left(\sum_{k=1}^n a_k\right)}{a_n} = 0.$$

We can assume $a_1 = 1$. It is now convenient to translate (4.16) into multiplicative form. For that, put

$$a_n = \prod_{k=1}^n (1 + \lambda_k), \quad n = 1, 2, \dots$$

Then it is easy to check that (4.10), (4.11) and (4.12) translate, respectively, into

$$(4.17) \quad 0 \leq \lambda_n \leq C - 1, \quad n = 1, 2, \dots,$$

$$(4.18) \quad \lambda_n \geq \lambda_{n+1}, \quad n = 1, 2, \dots,$$

$$(4.19) \quad \sum_{k=1}^n \lambda_k = o\left(\frac{n}{\log n}\right).$$

On the other hand, since $a_k \leq C^k$, $k = 1, 2, \dots$, we deduce

$$\log \log \left(\sum_{k=1}^n a_k \right) \leq 2 \log n$$

if n is sufficiently large. Therefore,

$$\frac{a_n - a_{n-1}}{a_n} \log \log \left(\sum_{k=1}^n a_k \right) \leq 2 \lambda_n \log n$$

and (4.16) would be deduced from the estimate

$$(4.20) \quad \lambda_n = o\left(\frac{1}{\log n}\right) \quad \text{as } n \rightarrow \infty.$$

Finally, to show (4.20), observe that, from (4.18) and (4.19), we have

$$n \lambda_n \leq \sum_{k=1}^n \lambda_k = o\left(\frac{n}{\log n}\right)$$

so (4.20) follows. This proves Lemma 4.3 and, consequently, Theorem 4.1. $\square$

5. PROOFS OF COROLLARIES 1.3 AND 1.4

Proof of Corollary 1.3. Consider $u = \log v$. Harnack's inequality provides a constant $B = B(d) > 0$ such that

$$y |\nabla u(x, y)| = \frac{y |\nabla v(x, y)|}{v(x, y)} \leq B, \quad (x, y) \in \mathbb{R}_+^{d+1}.$$

A calculation shows that $\Delta u = -|\nabla v|^2/v^2$ and we deduce that there exists a constant $C = C(d) > 0$ such that

$$y^3 |\nabla(\Delta u)(x, y)| \leq C, \quad (x, y) \in \mathbb{R}_+^{d+1}.$$

So, $u \in \mathcal{B}_{\psi, \varepsilon}$ by choosing constant functions ψ and ε . Since

$$\int_y^1 t \Delta u(x, t) dt = - \int_y^1 \frac{t |\nabla v(x, t)|^2}{v^2(x, t)} dt = -A^2(v)(x, y), \quad 0 < y < 1, x \in \mathbb{R}^d,$$

part (a) of Theorem 1.1 finishes the proof. $\square$

Harnack's inequality shows that there exists a constant $B = B(d) > 0$ such that $A^2(v)(x, y) \leq B \log(1/y)$ for any $x \in \mathbb{R}^d$ and $0 < y \leq 1/2$. We will now show that this inequality is sharp. Actually, we will construct positive harmonic functions v in $\mathbb{R}_+^{d+1}$ verifying the converse inequality: $A^2(v)(x, y) \geq C \log(1/y)$ for any $x \in \mathbb{R}^d$, $0 < y \leq 1/2$ and a certain constant $C > 0$. Functions v with this property will be constructed as harmonic extensions of certain positive measures μ in $\mathbb{R}^d$ that will be defined inductively, declaring its mass on any dyadic cube. Let $\mu(Q) = 1$ for any dyadic cube $Q \subset \mathbb{R}^d$ of side-length 1. Fix positive real numbers $\{p_i : i = 1, \dots, 2^d\}$ such that

$$\sum_{i=1}^{2^d} p_i = 1.$$

Fix an integer $n \geq 1$. Let $Q \in \mathcal{D}_{n-1}$ be a dyadic cube of generation $n-1$ and assume, by induction, that $\mu(Q)$ has already been defined. Write

$$Q = \bigcup_{j=1}^{2^d} Q^j,$$

where $Q^j \in \mathcal{D}_n$. Then set $\mu(Q^j) = p_j \mu(Q)$, $j = 1, \dots, 2^d$. This defines the measure μ . Let $v = P[\mu]$ denote its harmonic extension to $\mathbb{R}_+^{d+1}$. Assume that not all the weights p_i are the same, that is, $\max\{p_1, \dots, p_{2^d}\} > 2^{-d}$. Then there exist constants $C_i > 1$, $i = 1, 2$, such that for any $x \in \mathbb{R}^d$ and any $0 < y \leq 1/2$ we have

$$\sup\{v(x, t) : y < t < C_1 y\} \geq C_2 v(x, y).$$

Hence

$$\begin{aligned} \int_y^{C_1 y} \frac{|\nabla v(x, t)|}{v(x, t)} dt &\geq \sup\{|\log v(x, t) - \log v(x, y)| : y < t < C_1 y\} \\ &\geq \log C_2. \end{aligned}$$

Cauchy-Schwarz inequality gives

$$\int_y^{C_1 y} \frac{|\nabla v(x, t)|}{v(x, t)} dt \leq (\log C_1)^{1/2} \left(\int_y^{C_1 y} \frac{t |\nabla v(x, t)|^2}{v^2(x, t)} dt \right)^{1/2}$$

and one deduces

$$\int_y^{C_1 y} \frac{t |\nabla v(x, t)|^2}{v^2(x, t)} dt \geq \frac{(\log C_2)^2}{\log C_1}.$$

Given $0 < y \leq 1/2$, let N be the largest non negative integer such that $C_1^N y < 1$. Then

$$A^2(v)(x, y) \geq \sum_{k=1}^N \int_{C_1^{k-1}y}^{C_1^k y} \frac{t|\nabla v(x, t)|^2}{v^2(x, t)} dt \geq \frac{N \log^2(C_2)}{\log C_1}.$$

Hence there exists a constant $C > 0$ such that $A^2(v)(x, y) \geq C \log(1/y)$, $0 < y \leq 1/2$.

Proof of Corollary 1.4. Consider $u(z) = -\log(1 - |f(z)|^2)$, $z \in \mathbb{D}$. Note that $\sup\{|u(z) - d_h(f(z), 0)| : z \in \mathbb{D}\} < \infty$. By Schwarz-Pick Lemma, there exists a constant $B > 0$ such that

$$(1 - |z|^2)|\nabla u(z)| \leq B, \quad z \in \mathbb{D}.$$

A calculation gives that $\Delta u(z) = 4|f'(z)|^2(1 - |f(z)|^2)^{-2}$, $z \in \mathbb{D}$. Hence there exists a constant $C > 0$ such that

$$(1 - |z|^2)^3 |\nabla(\Delta u)(z)| \leq C, \quad z \in \mathbb{D}.$$

So, $u \in \mathcal{B}_{\psi, \varepsilon}$ by choosing constant functions ψ and ε . Since

$$\int_0^r \log(1/t) \Delta u(t\xi) dt = 4 \int_0^r \frac{\log(1/t) |f'(t\xi)|^2}{(1 - |f(t\xi)|^2)^2} dt, \quad 0 < r < 1, \xi \in \partial\mathbb{D},$$

part (a) of Theorem 1.1 in the setting of the unit disc, finishes the proof. $\square$

Remark 5.1. Note that Schwarz-Pick Lemma gives that $A^2(f)(\xi, r) \leq \log(1 - r)^{-1}$, $0 < r < 1$, $\xi \in \partial\mathbb{D}$. This estimate is sharp. For instance if f is a finite Blaschke product, there exists a constant $C = C(f) > 0$ such that the converse estimate $A^2(f)(\xi, r) > C \log(1 - r)^{-1}$ holds for any $1/2 < r < 1$ and $\xi \in \partial\mathbb{D}$. There are also infinite Blaschke products verifying the converse estimate. A Blaschke product f is called of bounded compression if there exists a constant $c = c(f) > 0$ such that the hyperbolic diameter of $f(B)$ is bigger than c , for any hyperbolic disc B of hyperbolic radius 1. It turns out that f is of bounded compression if and only if there exists a constant $c_1 = c_1(f) > 0$ such that

$$\sup\{|D_h(f)(z)| : z \in B\} \geq c_1,$$

for any hyperbolic disc B of hyperbolic radius 1 (see [9]). This condition implies that there exists a constant $c_2 = c_2(f) > 0$ such that

$$\int_{(1-r)/2}^{1-r} \frac{(1 - t^2) |f'(t\xi)|^2}{(1 - |f(t\xi)|^2)^2} dt > c_2, \quad 0 < r < 1, \xi \in \partial\mathbb{D}.$$

We deduce that there exists a constant $c_3 = c_3(f) > 0$ such that $A^2(f)(\xi, r) \geq c_3 \log(1 - r)^{-1}$, $1/2 < r < 1$, $\xi \in \partial\mathbb{D}$.

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