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A CHARACTERIZATION OF THE LEADING COEFFICIENT OF NEVANLINNA'S PARAMETRIZATION

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1. Introduction

Let H^∞ be the Banach space of all bounded analytic functions in the open unit disc D, with the norm $\|f\|_\infty = \sup\{|f(z)|: z \in D\}$. Given two sequences of points $\{z_n\}, \{w_n\}$ in D, the classical Pick-Nevanlinna problem consists on finding analytic functions $f \in H^\infty$ satisfying $\|f\|_\infty \leq 1$ and $f(z_n) = w_n$, $n = 1, 2, \dots$. We will denote it as follows:

$$(*) \quad \text{Find } f \in H^\infty,\ \|f\|_\infty \leq 1,\ f(z_n) = w_n, \qquad n = 1, 2, \dots .$$

Pick and Nevanlinna found necessary and sufficient conditions in order that such an analytic function exists. Let E be the set of all solutions of the problem $(*)$. Nevanlinna showed that if E has more than one element, there exist analytic functions p, q, r, s in D such that

$$E = \left\{ f \in H^\infty : f = \frac{p\varphi + q}{r\varphi + s},\ \varphi \in H^\infty, \|\varphi\|_\infty \leq 1 \right\} \tag{1.1}$$

$$ps - qr = B \tag{1.2}$$

where B is the Blaschke product with zeros $\{z_n\}$. See [2, p. 165] for the proof. Let us remark that there is no explicit formula for the coefficients p, q, r, s in terms of the sequences $\{z_n\}, \{w_n\}$.

We will say that a Pick-Nevanlinna problem $(*)$ with more than one solution has the function s as leading coefficient if s is analytic in D and there exist analytic functions p, q, r in D such that if E is the set of all solutions of $(*)$, the functions p, q, r, s verify (1.1) and (1.2).

In this note, fixed a Blaschke sequence $\{z_n\}$ in D, we get a characterization of the functions that can appear as leading coefficients of Pick-Nevanlinna problems $(*)$.

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In order to exclude trivial situations, we will not consider the Pick-Nevanlinna problem with $w_n = 0$, $n = 1, 2, \dots$. Observe that in this case, one can take $p \equiv B$, $s \equiv 1$, $q \equiv r \equiv 0$. If h is a function defined in the unit circle such that $\log |h|$ is integrable, let $E(h)$ be the function defined by

$$E(h)(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|h(e^{it})\right|dt\right), \qquad z \in D.$$

A point in the unit ball of a Banach space is called an exposed point of the unit ball if there is a continuous real functional on the space that equals 1 at the point but takes values less than 1 elsewhere in the unit ball. Our main result is the following.

THEOREM. *Let $\{z_n\}$ be a Blaschke sequence of points of the unit disc and let B be the Blaschke product with zeros $\{z_n\}$. Let s be an analytic function in D. Then, the following are equivalent:*

(i) *s belongs to the Smirnov class $N^+(D)$, s^{-1} is a non-extreme point of the unit ball of H^∞ and the function $F(e^{it}) = (s(e^{it}) + B(e^{it})\overline{E(|s|^2-1)^{1/2}}(e^{it}))^{-2}$ is an exposed point of the Hardy space H^1.*

(ii) *There exists a sequence of complex numbers $\{w_n\}$ such that the Pick-Nevanlinna problem in $(*)$ has the function s as leading coefficient.*

The notions of extreme and exposed points are discussed in Section 2, where we also introduce some results that are needed for the proof of the theorem that is given in Section 3. Finally in Section 4, we deal with Pick-Nevanlinna problems with a finite number of points.

If B is a finite Blaschke product with zeros $z_1, \dots, z_N$, let M_B be the space of complex linear combinations of the functions $\{(1-\overline{z_n}z)^{-1}: n = 1, 2, \dots, N\}$ and let $\mathbf{C}B$ be the complex multiples of B. The result for finite Pick-Nevanlinna problems is the following.

COROLLARY 1. *Let $z_1, \dots, z_N$ be points in the unit disc and let B be the Blaschke product with zeros $z_1, \dots, z_N$. Let s be an analytic function in D. Then, the following are equivalent:*

(i) *s^{-1} is a non-extreme point of the unit ball of H^∞ and $s \in M_B + \mathbf{C}B$.*

(ii) *There exist complex numbers $w_1, \dots, w_N$ such that the Pick-Nevanlinna problem in $(*)$ has the function s as leading coefficient.*

Let us observe that in this situation the result is more satisfactory because no condition on exposed points is needed.

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2. Exposed points in H^1

A point in the unit ball of a Banach space is called an extreme point of the unit ball if it cannot be written as a proper convex combination of two points of the unit ball. In [1, p. 484] it is shown that a function $f \in H^\infty$, $\|f\|_\infty \leq 1$ is an extreme point of the unit ball of H^∞ if and only if

$$\int_0^{2\pi} \log(1 - |f(e^{it})|)\, dt = -\infty.$$

Thus, an exposed point, as it has been defined before, must be an extreme point. For $0 < p < \infty$, let H^p be the usual Hardy spaces of analytic functions in D. Since the extreme points of the unit ball of H^1 are the outer functions of unit norm (see [1, p. 470]), the exposed points of H^1 must be outer functions. A function $F \in H^1$ will be called an exposed point of H^1 if $F\|F\|_1^{-1}$ is an exposed point of the unit ball of H^1.

A function in H^1 is called rigid if no other functions, except for positive multiples of it, have the same argument as it almost everywhere on ∂D. For instance, a function in H^1 whose reciprocal also is in H^1, is rigid. It is well known that a function $F \in H^1$ is exposed if and only if it is rigid. See [4, p. 486].

The following result of J. Garnett connects the exposed points of H^1 with the Adamyan-Arov-Krein parametrization. See [2, p. 157 and p. 179] or [4, p. 493].

THEOREM (J. GARNETT). *Let $F \in H^1$, $\|F\|_1 = 1$, be an exposed point of the unit ball of H^1. Then, the coset*

$$K = \left\{ \frac{F}{|F|} + h : h \in H^\infty, \left\| \frac{F}{|F|} + h \right\|_\infty \leq 1 \right\}$$

has more than one element and defining $\chi \in H^\infty$ by

$$\frac{1 + \chi(z)}{1 - \chi(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} |F(e^{it})|\, dt, \qquad z \in D,$$

one has

$$K = \left\{ \frac{F}{|F|} - \frac{F(1-\chi)(1-\varphi)}{1 - \chi\varphi} : \varphi \in H^\infty, \|\varphi\|_\infty \leq 1 \right\}.$$

Now, let us outline a result of D. Sarason that is needed in the proof of the theorem. Let $F \in H^2$, $\|F\|_2 = 1$, be an outer function, and consider

$$H(F)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\left|F(e^{it})\right|^2 dt, \qquad z \in D,$$

$$a = \frac{2F}{H(F)+1}, \quad b = \frac{H(F)-1}{H(F)+1}.$$

One can check that $F = a(1-b)^{-1}$, $b(0) = 0$ and $|a|^2 + |b|^2 = 1$ almost everywhere on ∂D. Actually, one can prove that this decomposition is unique. In [4], D. Sarason conjectured that $F^2 \in H^1$ is exposed if and only if $\|a^2(1-be^{i\theta})^{-2}\|_1 = 1$ for all $e^{i\theta} \in \partial D$ and proved the following implication.

THEOREM (D. SARASON). *Let $F \in H^2$, $\|F\|_2 = 1$ be an outer function such that F^2 is an exposed point of H^1. Then, for every inner function I, $a^2(1-bI)^{-2}$ is an exposed point of H^1 and $\|a^2(1-bI)^{-2}\|_1 = 1$.*

Finally, let us mention some well known properties of the coefficients of Nevanlinna's parametrization that are also needed in the proof of the theorem. Given a Pick-Nevanlinna problem $(*)$ with more than one solution, the four functions p, q, r, s are not determined by conditions (1.1) and (1.2). In fact, changing in (1.1) φ by

$$e^{i\gamma}\frac{\varphi+\alpha}{1+\bar{\alpha}D}$$

where $\gamma \in \mathbf{R}$ and $\alpha \in D$, and taking $c = e^{-i\gamma/2}(1-|\alpha|^2)^{-1/2}$, one can get other functions

$$\begin{aligned} p_\alpha &= c\left(pe^{i\gamma} + \bar{\alpha}q\right), & q_\alpha &= c\left(pe^{i\gamma}\alpha + q\right), \\ r_\alpha &= c\left(re^{i\gamma} + \bar{\alpha}s\right), & s_\alpha &= c\left(re^{i\gamma}\alpha + s\right), \end{aligned} \tag{2.1}$$

that also satisfy (1.1) and (1.2). Indeed, this is the extent of the arbitrariness of these four functions (see [3, p. 299]). It is well known that these systems of four functions have some common properties. In particular, if four analytic functions p, q, r, s in D satisfy (1.1) and (1.2), they belong to the Smirnov class $N^+(D)$ and they also satisfy

$$p = B\bar{s}, \quad q = B\bar{r}, \quad |s|^2 - |q|^2 = 1 \text{ a.e. on } \partial D, \tag{2.2}$$

$$\max\{|p(z)|, |q(z)|, |r(z)|\} \le |s(z)|, \ z \in D. \tag{2.3}$$

Furthermore, s^{-1} is an outer function of the unit ball of H^∞. See [3, p. 491] for the proof.

3. Proof of the theorem

Let us begin with the following result.

LEMMA 1. *Let s be the leading coefficient of the Pick-Nevanlinna problem in $(*)$. Then the following problem also has leading coefficient s:*

$(*)'$ *Find* $f \in H^\infty$, $\|f\|_\infty \leq 1$, $f(z_n) = E(1-|s|^{-2})^{1/2}(z_n)$, $\qquad n = 1, 2, \ldots$

Proof of Lemma 1.
Let B be the Blaschke product with zeros $\{z_n\}$. There exist analytic functions p, q, r in D satisfying $ps - qr = B$ and

$$(3.1) \qquad \{f \in H^\infty : f \text{ solves } (*)\} = \left\{ \frac{p\varphi + q}{r\varphi + s} : \varphi \in H^\infty, \|\varphi\|_\infty \leq 1 \right\}.$$

First, let us assume $w_n \neq 0$ for $n = 1, 2, \ldots$. By (2.2), one has $|s|^2 - |q|^2 = 1$ almost everywhere on ∂D. Therefore, there exists an inner function I such that

$$(3.2) \qquad q = IE\left(|s|^2 - 1\right)^{1/2}.$$

It is clear that in order to prove Lemma 1 it is sufficient to show

$$(3.3) \quad \{f \in H^\infty : f \text{ solves } (*)'\} = \left\{ \frac{p\varphi + q/I}{(rI)\varphi + s} : \varphi \in H^\infty, \|\varphi\|_\infty \leq 1 \right\}.$$

If f is a solution of $(*)'$, If solves $(*)$ and applying (3.1) there exists a function φ in the unit ball of H^∞ such that

$$If = \frac{p\varphi + q}{r\varphi + s}.$$

Since s is outer, applying (3.2) and $ps - qr = B$, one gets

$$(3.4) \qquad If - IE\left(1 - |s|^{-2}\right)^{1/2} = \frac{p\varphi + q}{r\varphi + s} - \frac{q}{s} = \frac{B\varphi}{s(r\varphi + s)}.$$

Since $w_n \neq 0$ for $n = 1, 2, \ldots, I$ and B have no common zeros. Then, I divides φ in H^∞ and one has

$$If = \frac{pI\varphi_1 + q}{rI\varphi_1 + s}$$

for some function φ_1 of the unit ball of H^∞. Therefore,

$$f = \frac{p\varphi_1 + q/I}{rI\varphi_1 + s}$$

and this gives one of the inclusions of (3.3). From $ps - qr = B$, it follows that

$$\frac{p\varphi + q/I}{rI\varphi + s} - \frac{q}{Is} = \frac{B\varphi}{s(rI\varphi + s)}$$

for each function φ in the unit ball of H^∞. Now, since q/Is solves $(*)'$, one gets the other inclusion of (3.3).

Now, let us consider the general case where $w_n = 0$ for some n. Take $A = \{n: w_n = 0\}$ and let B_1 be the Blaschke product with zeros $\{z_n: n \in A\}$ and $B_2 = B/B_1$. Thus, f solves $(*)$ if and only if $f = B_1 f_1$ where f_1 is a solution of the following Pick-Nevanlinna problem:

$$(*)_1 \quad \text{Find } f \in H^\infty,\ \|f\|_\infty \le 1,\ f(z_n) = \frac{w_n}{B_1(z_n)} \quad \text{for } n \notin A.$$

Therefore, from (3.1), one gets

$$\{f \in H^\infty: f \text{ solves } (*)_1\} = \left\{\frac{p\varphi/B_1 + q/B_1}{r\varphi + s} : \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}.$$

Now, the first part of the proof shows that s is the leading coefficient of the Pick-Nevanlinna problem:

$$(*)_2 \quad \text{Find } f \in H^\infty,\ \|f\|_\infty \le 1,\ f(z_n) = E(1 - |s|^{-2})^{1/2}(z_n) \quad \text{for } n \notin A.$$

Therefore, there exist analytic functions p^*, q^*, r^* in D such that

$$\{f \in H^\infty: f \text{ solves } (*)_2\} = \left\{\frac{p^*\varphi + q^*}{r^*\varphi + s} : \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}$$

and $p^*s - q^*r^* = B_2$. Moreover $q^*/s = E(1 - |s|^{-2})^{1/2}$. As in the first part of the proof, one only has to show

$$(3.5) \quad \{f \in H^\infty: f \text{ solves } (*)'\} = \left\{\frac{(p^*B_1)\varphi + q^*}{(r^*B_1)\varphi + s} : \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}.$$

Let f be a solution of $(*)'$. Since f solves $(*)_2$, there exists a function φ in the unit ball of H^∞ such that

$$f = \frac{p^*\varphi + q^*}{r^*\varphi + s}.$$

Using $p^*s - q^*r^* = B_2$, one has

$$f - \frac{q^*}{s} = \frac{B_2\varphi}{s(r^*\varphi + s)}. \tag{3.6}$$

Since $f(z_n) = (q^*/s)(z_n) = E(1 - |s|^{-2})^{1/2}(z_n)$ for $n = 1, 2, \ldots,$ from (3.6) one gets $\varphi(z_n) = 0$ for $n \in A$. Therefore $\varphi = B_1\varphi_1$ for some function φ_1 of the unit ball of H^∞. Then,

$$f = \frac{(p^*B_1)\varphi_1 + q^*}{(r^*B_1)\varphi_1 + s}$$

and this gives one of the inclusions of (3.5). The other one is an easy consequence of the fact that q^*/s solves $(*)'$ and $(p^*B_1)s - q^*(r^*B_1) = B$. □

Let us now go into the proof of the theorem.

(ii) ⇒ (i). Applying Lemma 1, one can assume $w_n = E(1 - |s|^{-2})^{1/2}(z_n)$ for $n = 1, 2, \ldots$. Since s is the leading coefficient of $(*)$, there exist analytic functions p, q, r in D such that $ps - qr = B$ and

$$\{f \in H^\infty: f \text{ solves } (*)\} = \left\{\frac{p\varphi + q}{r\varphi + s}: \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}.$$

Applying (2.2) and using $q \ne 0$, it is easy to check that s^{-1} is a non-extreme point of the unit ball of H^∞. So, one only has to prove that the function

$$F(e^{it}) = \left(s(e^{it}) + B(e^{it})\overline{E\left(|s|^2 - 1\right)^{1/2}(e^{it})}\right)^{-2}$$

is an exposed point of H^1.

Claim 1. $(s + r)^{-2}$ is an exposed point of H^1.

Assume the claim holds and let us finish the proof. Applying (2.2), one gets an inner function I such that

$$q = IE\left(|s|^2 - 1\right)^{1/2}.$$

Since $q/s = IE(1 - |s|^{-2})^{1/2}$ *solves* $(*)$, *one gets* $I \equiv 1$. *By* (2.2),

$$r = B\bar{q} = B\overline{E(|s|^2 - 1)^{1/2}} \quad \text{a.e.}(dm) \text{ on } T.$$

Therefore, the claim gives that the function

$$(s + r)^{-2} = \left(s + B\overline{E(|s|^2 - 1)^{1/2}}\right)^{-2}$$

is an exposed point of H^1 *and this finishes the proof of* (*ii*) $\Rightarrow$ (*i*).

Proof of Claim 1. (1) Assume $r(0) = 0$. Using $ps - qr = B$, one gets

$$\frac{p + q}{r + s} - \frac{p\varphi + q}{r\varphi + s} = \frac{B(1 - \varphi)}{(r + s)(r\varphi + s)}.$$

Therefore

$$(3.7) \quad \frac{p + q}{r + s}\bar{B} - \frac{p\varphi + q}{r\varphi + s}\bar{B} = \frac{1 - \varphi}{(r + s)(r\varphi + s)} \quad \text{a.e.}(dm) \text{ on } T.$$

Now, let us consider the coset

$$K = \{f\bar{B}: f \in H^\infty \text{ solves } (*)\} = \left\{\frac{p\varphi + q}{r\varphi + s}\bar{B}: \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}$$
$$= \left\{\frac{p + q}{r + s}\bar{B} + h: h \in H^\infty, \left\|\frac{p + q}{r + s}\bar{B} + h\right\|_\infty \le 1\right\}$$

and choose $\gamma = -\operatorname{Arg}(s(0)^{-2})$. From (3.7), using $r(0) = 0$, it follows that

$$(3.8) \quad \operatorname{Re}\int_0^{2\pi} e^{i\gamma}\frac{p + q}{r + s}(e^{it})\overline{B(e^{it})}\,dt = \sup\left\{\operatorname{Re}\int_0^{2\pi} e^{i\gamma}g(e^{it})\,dt: g \in K\right\}.$$

Now, following [2, p. 160], one can deduce that there exists a unique function $F \in H^1$, $\|F\|_1 = 1$, such that

$$(3.9) \quad \frac{p + q}{r + s}\bar{B} = \frac{F}{|F|} \quad \text{a.e.}(dm) \text{ on } \partial D.$$

Therefore F is an exposed point of H^1. On the other hand, A. Stray ([5, p. 491]) has observed that the function $(s + r)^{-2}$ is in H^1. Using (2.2), one gets

$$F = (s + r)^{-2}\|(s + r)^{-2}\|_1^{-1}$$

and this gives the claim in the case $r(0) = 0$.

(2) Assume $r(0) \neq 0$. From (2.3), one has $|r(z)| < |s(z)|$, $z \in D$. Now, taking $\alpha = \overline{-(r/s)(0)}$ and $\gamma = 0$ in (2.1), one gets analytic functions $p_\alpha, q_\alpha, r_\alpha, s_\alpha$ in D such that

$$\{f \in H^\infty : f \text{ solves } (*)\} = \left\{ \frac{p_\alpha \varphi + q_\alpha}{r_\alpha \varphi + s_\alpha} : \varphi \in H^\infty, \|\varphi\|_\infty \leq 1 \right\}$$

and $r_\alpha(0) = 0$, $p_\alpha s_\alpha - q_\alpha r_\alpha = B$.

The first case of the proof shows that the function $(s_\alpha + r_\alpha)^{-2}$ is an exposed point of H^1. Now, formulas (2.1) give

$$\begin{aligned}(s+r)^{-2} &= c^{-1}((1-\bar{\alpha})s_1 + (1-\alpha)r_1)^{-2} \\ &= c^{-2}(1-\bar{\alpha})^{-2}\left(s_1 + \frac{1-\alpha}{1-\bar{\alpha}} r_1\right)^{-2},\end{aligned}$$

and applying Sarason's result cited in section 2, one gets that $(s+r)^{-2}$ is an exposed point of H^1 and this finishes the proof of the claim. □

(i) ⇒ (ii).

Consider $F_1 = F\|F\|_1^{-1}$. Since $s \in N^+(D)$ and $F \in H^1$ is outer, the function $\overline{B(e^{it}) E(|s|^2 - 1)^{1/2}}(e^{it})$ has an analytic extension to D that belongs to the Smirnov class $N^+(D)$. An easy computation gives

$$\frac{F_1}{|F_1|} = \frac{B\bar{s} + E(|s|^2-1)^{1/2}}{\overline{BE(|s|^2-1)^{1/2}} + s} \bar{B} \quad \text{a.e.}(dm) \text{ on } \partial D, \tag{3.10}$$

and then

$$\frac{F_1}{|F_1|} B - \frac{E(|s|^2-1)^{1/2}}{s} = \frac{B}{s\left(s + \overline{BE(|s|^2-1)^{1/2}}\right)} \quad \text{a.e.}(dm) \text{ on } \partial D. \tag{3.11}$$

The right hand term in (3.11) has a bounded analytic extension to D because the denominator is in the Smirnov class and

$$\begin{aligned}&|s(e^{it})|^2 \left| 1 + \frac{\overline{BE(|s|^2-1)^{1/2}}}{s}(e^{it}) \right| \\ &\quad \geq |s(e^{it})|^2 \left(1 - \left(1 - \frac{1}{|s(e^{it})|^2} \right)^{1/2} \right) \geq \frac{1}{2} \quad \text{a.e}(dm) e^{it} \in \partial D.\end{aligned}$$

From (3.11) it follows that the function $F_1(e^{it})B(e^{it})/|F_1(e^{it})|$ has a bounded analytic extension to D, that we will still call $F_1B/|F_1|$. Moreover, since s is outer, (3.11) also gives

$$\frac{F_1}{|F_1|}B(z_n) = E\left(1 - |s|^{-2}\right)^{1/2}(z_n), \qquad n = 1,2,\dots.$$

Now, take $w_n = E(1 - |s|^{-2})^{1/2}(z_n)$, $n = 1,2,\dots$ and consider the Pick-Nevanlinna problem in $(*)$. Since $E(1 - |s|^{-2})^{1/2}$ is a non-extreme point of the unit ball of H^∞ that solves $(*)$, the problem $(*)$ has more than one solution. Actually, the function

$$E\left(1 - |s|^{-2}\right)^{1/2} + BE\left(1 - \left(1 - |s|^{-2}\right)^{1/2}\right)$$

also solves $(*)$. Now, let us show that s is a leading coefficient of $(*)$.

Claim 2. Let φ be a function of the unit ball of H^∞. Then, the function

$$\frac{B(e^{it})\overline{s(e^{it})}\varphi(e^{it}) + E\left(|s|^2 - 1\right)^{1/2}(e^{it})}{B(e^{it})\overline{E\left(|s|^2 - 1\right)^{1/2}(e^{it})}\varphi(e^{it}) + s(e^{it})}$$

has an analytic extension to D that solves $(*)$.

Proof of Claim 2. Applying (3.10), one gets that $B\bar{s}$ has an analytic extension to D belonging to the Smirnov class. Also,

$$B\bar{s}s - E\left(|s|^2 - 1\right)^{1/2}B\overline{E\left(|s|^2 - 1\right)^{1/2}} = B \qquad \text{a.e.}(dm) \text{ on } \partial D.$$

Now, it is easy to check that for each function φ of the unit ball of H^∞,

$$\frac{B\bar{s}\varphi + E\left(|s|^2 - 1\right)^{1/2}}{B\overline{E\left(|s|^2 - 1\right)^{1/2}}\varphi + s} - \frac{E\left(|s|^2 - 1\right)^{1/2}}{s} = \frac{B\varphi}{s\left(B\overline{E\left(|s|^2 - 1\right)^{1/2}}\varphi + s\right)}$$

a.e. on ∂D

and, since s is outer,

$$\frac{B\bar{s}\varphi + E\left(|s|^2 - 1\right)^{1/2}}{B\overline{E\left(|s|^2 - 1\right)^{1/2}}\varphi + s}(z_n) = E\left(1 - |s|^{-2}\right)^{1/2}(z_n), \qquad n = 1,2,\dots.$$

Moreover,

$$\frac{B\bar{s}\varphi + E(|s|^2-1)^{1/2}}{B\overline{E(|s|^2-1)^{1/2}}\varphi + s} \in N^+(D)$$

because the denominator is an outer function. Now

$$\left|\frac{B(e^{it})\overline{s(e^{it})}\varphi(e^{it}) + E(|s|^2-1)^{1/2}(e^{it})}{B(e^{it})\overline{E(|s|^2-1)^{1/2}}(e^{it})\varphi(e^{it}) + s(e^{it})}\right| \le 1 \quad \text{a.e.}(dm) e^{it} \in \partial D$$

gives Claim 2. □

Let us continue the proof of the theorem. Since F_1 is an exposed point of H^1, the result of J. Garnett cited in Section 2, gives

$$\begin{aligned}\{f \in H^\infty: f \text{ solves } (*)\} &= \left\{\frac{F_1}{|F_1|}B + Bh: h \in H^\infty, \left\|\frac{F_1}{|F_1|}B + Bh\right\|_\infty \le 1\right\} \\ (3.12) \qquad &= B\left\{\frac{F_1}{|F_1|} + h: h \in H^\infty, \left\|\frac{F_1}{|F_1|} + h\right\|_\infty \le 1\right\} \\ &= B\left\{\frac{F_1}{|F_1|} - \frac{F_1(1-\chi)(1-\varphi)}{1-\chi\varphi}: \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}\end{aligned}$$

where

$$\frac{1+\chi(z)}{1-\chi(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}|F_1(e^{it})|\,dt.$$

Taking

$$p_1 = BF_1^{1/2} - \frac{F_1}{|F_1|}B\frac{\chi}{F_1^{1/2}(1-\chi)}, \quad q_1 = \frac{F_1}{|F_1|}B\frac{1}{F_1^{1/2}(1-\chi)} - BF_1^{1/2},$$

$$r_1 = \frac{-\chi}{(1-\chi)F_1^{1/2}}, \qquad s_1 = \frac{1}{F_1^{1/2}(1-\chi)},$$

one can check

$$(3.13) \quad \{f \in H^\infty: f \text{ solves } (*)\} = \left\{\frac{p_1\varphi + q_1}{r_1\varphi + s_1}: \varphi \in H^\infty, \|\varphi\|_\infty \le 1\right\}$$

and

$$p_1 s_1 - q_1 r_2 = B, \quad r_1(0) = 0. \tag{3.14}$$

Now, since

$$|F| = \frac{1/|s|^2}{\left|1 + \overline{BE(|s|^2-1)^{1/2}}/s\right|^2} = \frac{1 - \left|\overline{BE(|s|^2-1)^{1/2}}/s\right|^2}{\left|1 + \overline{BE(|s|^2-1)^{1/2}}/s\right|^2}$$
$$= \operatorname{Re}\left(\frac{1 - \overline{BE(|s|^2-1)^{1/2}}/s}{1 + \overline{BE(|s|^2-1)^{1/2}}/s}\right) \quad \text{a.e.}(dm) \text{ on } \partial D,$$

one gets

$$\|F\|_1 = \operatorname{Re}\left(\frac{1 - \overline{BE(|s|^2-1)^{1/2}}(0)/s(0)}{1 + \overline{BE(|s|^2-1)^{1/2}}(0)/s(0)}\right) \tag{3.15}$$
$$= \frac{1 - \left|\overline{BE(|s|^2-1)^{1/2}}(0)/s(0)\right|^2}{\left|1 + \overline{BE(|s|^2-1)^{1/2}}(0)/s(0)\right|^2}.$$

Therefore, applying (3.15),

(3.16)

$$|s_1(0)|^2 = |F_1(0)|^{-1} = |F(0)|^{-1}\|F\|_1 = |s(0)|^2 - \left|\overline{BE(|s|^2-1)^{1/2}}(0)\right|^2.$$

Now, looking at (2.1) it is natural to consider

$$\bar{\alpha} = -\frac{\overline{BE(|s|^2-1)^{1/2}}}{s}(0).$$

Since $\alpha \in D$, Claim 2 and (3.13) show that there exists a function φ in the unit ball of H^∞ such that

$$\frac{B\bar{s}\alpha + E(|s|^2-1)^{1/2}}{\overline{BE(|s|^2-1)^{1/2}}\alpha + s} = \frac{p_1\varphi + q_1}{r_1\varphi + s_1}. \tag{3.17}$$

Applying (2.2), from (3.17) one gets

$$\text{(3.18)}\qquad \frac{1-|\alpha|^2}{\left|s+\overline{BE(|s|^2-1)^{1/2}\alpha}\right|^2}=\frac{1-|\varphi|^2}{|r_1\varphi+s_1|^2}\quad \text{a.e.}(dm) \text{ on } \partial D.$$

Now, since $s+\overline{BE(|s|^2-1)^{1/2}}\alpha$ is an outer function and $r_1(0)=0$, (3.18) gives

$$\frac{1}{|s(0)|^2\left(1-\left|\overline{BE(|s|^2-1)^{1/2}}(0)/s(0)\right|^2\right)}=\frac{\left|E(1-|\varphi|^2)(0)\right|}{|s_1(0)|^2}.$$

Thus, (3.16) shows that $|E(1-|\varphi|^2)(0)|=1$ and therefore $\varphi\equiv 0$. So, (3.17) and (3.18) give

$$\text{(3.19)}\qquad \frac{B\bar{s}\alpha+E(|s|^2-1)^{1/2}}{\overline{BE(|s|^2-1)^{1/2}}\alpha+s}=\frac{q_1}{s_1}\quad \text{a.e.}(dm) \text{ on } \partial D$$

and

$$\text{(3.20)}\qquad \frac{1-|\alpha|^2}{\left|s+\overline{BE(|s|^2-1)^{1/2}}\alpha\right|^2}=\frac{1}{|s_1|^2}\quad \text{a.e.}(dm) \text{ on } \partial D.$$

Since $s+\overline{BE(|s|^2-1)^{1/2}}\alpha$ and s_1 are outer functions, there exists $e^{i\gamma}\in\partial D$ such that

$$\frac{1-|\alpha|^2}{\left(s(z)+\overline{BE(|s|^2-1)^{1/2}}(z)\alpha\right)^2}=\frac{e^{-i\gamma}}{s_1^2(z)},\qquad z\in D,$$

that is to say,

$$\text{(3.21)}\qquad s_1(z)=e^{-i\gamma/2}\,\frac{s(z)+\overline{BE(|s|^2-1)^{1/2}}(z)\alpha}{(1-|\alpha|^2)^{1/2}},\qquad z\in D.$$

Now, (3.19) and (3.21) give

$$q_1 = \frac{q_1}{s_1} s_1 = \frac{e^{-i\gamma/2}}{\left(1 - |\alpha|^2\right)^{1/2}} \left(B\bar{s}\alpha + E\left(|s|^2 - 1\right)^{1/2} \right) \quad \text{a.e.}(dm) \text{ on } \partial D$$

and applying (2.2),

(3.22)

$$r_1 = B\overline{q_1} = \frac{e^{i\gamma/2}}{\left(1 - |\alpha|^2\right)^{1/2}} \left(s\bar{\alpha} + \overline{BE\left(|s|^2 - 1\right)^{1/2}} \right) \quad \text{a.e.}(dm) \text{ on } \partial D.$$

Since p_1, q_1, r_1, s_1 are the coefficients of a parametrization of the solutions of $(*)$, from formulas (2.1) it follows that the function

$$\begin{aligned} &\frac{e^{i\gamma/2}}{\left(1 - |\alpha|^2\right)^{1/2}} \left(s_1 - r_1 \alpha e^{-i\gamma} \right) \\ &= e^{i\gamma/2} \Bigg(\frac{e^{-i\gamma/2}}{\left(1 - |\alpha|^2\right)} \left(s + \overline{BE\left(|s|^2 - 1\right)^{1/2}} \alpha \right) \\ &\qquad - \alpha e^{-i\gamma} \frac{e^{i\gamma/2}}{\left(1 - |\alpha|^2\right)} \left(s\bar{\alpha} + \overline{BE\left(|s|^2 - 1\right)^{1/2}} \right) \Bigg) \\ &= \left(1 - |\alpha|^2\right)^{-1} \left(s + \overline{BE\left(|s|^2 - 1\right)^{1/2}} \alpha - |\alpha|^2 s - \alpha \overline{BE\left(|s|^2 - 1\right)^{1/2}} \right) = s \end{aligned}$$

is the leading coefficient of $(*)$ and this proves the theorem.

Remark 1. By an argument similar to this proof, one can show that if a function s satisfies the conditions in (i), a sequence $\{w_n\}$ will produce a Pick-Nevanlinna problem with leading coefficient s if and only if there exists an inner function I such that the following two conditions are satisfied,

1. $w_n = I(z_n)E(1 - |s|^{-2})^{1/2}(z_n)$, $n = 1, 2, \ldots,$
2. The function $F(e^{it}) = \left(s(e^{it}) + \overline{B(e^{it}) I(e^{it}) E\left(1 - |s|^{-2}\right)^{1/2}(e^{it})} \right)^{-2}$ is an exposed point of H^1.

4. Proof of Corollary 1

If A is a subspace of the Hardy space H^2, we will denote by $A^\perp$ the orthogonal complement of A. We will use the following result.

LEMMA 2. *If* $s \in (BH^2)^\perp + \mathbf{C}B$, *the function*

$$B(e^{it})\overline{E(|s|^2 - 1)^{1/2}(e^{it})}$$

has an analytic extension to D *that belongs to the Hardy space* H^2.

Proof of Lemma 2. First, let us observe that it is sufficient to check

$$E(|s|^2 - 1)^{1/2} \in (BH^2)^\perp + \mathbf{C}B. \tag{4.1}$$

Indeed if (4.1) holds, since $(BH^2)^\perp$ is the closure of the linear combinations of the functions

$$\left\{(1 - \overline{z_n}z)^{-1} : n = 1, 2, \ldots\right\},$$

there exist complex numbers λ_0, λ_{kN} such that

$$\lim_{N\to\infty} \sum_{k=1}^{N} \lambda_{kN}(1 - \overline{z_k}z)^{-1} + \lambda_0 B(z) = E(|s|^2 - 1)^{1/2}(z),$$

where the convergence is in H^2. Therefore,

$$\lim_{N\to\infty} \sum_{k=1}^{N} \overline{\lambda_{kN}} \frac{e^{it}}{e^{it} - z_k} + \overline{\lambda_0 B(e^{it})} = \overline{E(|s|^2 - 1)^{1/2}(e^{it})}$$

in $L^2(\partial D)$ and then

$$\lim_{N\to\infty} \sum_{k=1}^{N} \overline{\lambda_{kN}} \frac{e^{it}B(e^{it})}{e^{it} - z_k} + \overline{\lambda_0} = B(e^{it})\overline{E(|s|^2 - 1)^{1/2}(e^{it})} \tag{4.2}$$

in $L^2(\partial D)$. Now, since $B(z_k) = 0$, the functions appearing in the left side of (4.2) extend analytically to a neighbourhood of $\overline{D}$ and this gives Lemma 2.

In order to prove (4.1) let us check that there exists a constant $a \in \mathbf{C}$ such that for all $\varphi \in H^2$,

$$\left\langle E(|s|^2 - 1)^{1/2}, B\varphi \right\rangle = a\overline{\varphi(0)}, \tag{4.3}$$

where $\langle , \rangle$ is the scalar product in H^2. Then (4.1) will follow by duality.

Since $E(|s|^2 - 1)^{1/2}$ is an outer function, Beurling's theorem (see [2, p. 84]) give that the subspace $\{E(|s|^2 - 1)^{1/2}P$: P polynomial$\}$ is dense in H^2. So, there exist functions $\varphi_n \in H^2$ such that

$$\varphi = \lim_{n\to\infty} E\left(|s|^2 - 1\right)^{1/2} \varphi_n \tag{4.4}$$

in H^2. Therefore,

$$\begin{aligned}\left\langle E\left(|s|^2-1\right)^{1/2}, B\varphi\right\rangle &= \lim_{n\to\infty}\left\langle E\left(|s|^2-1\right)^{1/2}, BE\left(|s|^2-1\right)^{1/2}\varphi_n\right\rangle \\ &= \lim_{n\to\infty}\int_0^{2\pi}\left(\left|s(e^{it})\right|^2-1\right)\overline{B(e^{it})\varphi_n(e^{it})}\,dt.\end{aligned} \tag{4.5}$$

Since, $s \in (BH^2)^\perp + \mathbf{C}B$, there exist complex numbers μ_0, μ_{kN} such that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\mu_{kN}\left(1-\overline{z_k}e^{it}\right)^{-1} + \mu_0 B(e^{it}) = s(e^{it})$$

in H^2. Then, for a fixed n, Cauchy's theorem gives

(4.6)

$$\begin{aligned}&\int_0^{2\pi}\left(\left|s(e^{it})\right|^2-1\right)B(e^{it})\varphi_n(e^{it})\,dt \\ &= \lim_{N\to\infty}\sum_{k,m=1}^{n}\mu_{kN}\overline{\mu_{mN}}\int_0^{2\pi}\frac{1}{1-\overline{z_k}e^{it}}\,\frac{e^{it}}{e^{it}-z_m}B(e^{it})\varphi_n(e^{it})\,dt \\ &\quad+\sum_{k=1}^{n}\mu_{kN}\int_0^{2\pi}\frac{1}{1-\overline{z_k}e^{it}}\overline{\mu_0 B(e^{it})}B(e^{it})\varphi_n(e^{it})\,dt \\ &\quad+\sum_{k=1}^{n}\overline{\mu_{kN}}\int_0^{2\pi}\frac{e^{it}}{e^{it}-z_k}\mu_0 B(e^{it})B(e^{it})\varphi_n(e^{it})\,dt \\ &\quad+\left(|\mu_0|^2-1\right)\int_0^{2\pi}B(e^{it})\varphi_n(e^{it})\,dt \\ &= \lim_{N\to\infty}\sum_{k=1}^{N}\mu_{kN}\overline{\mu_0}\varphi_n(0) + \left(|\mu_0|^2-1\right)B(0)\varphi_n(0) \\ &= \varphi_n(0)\left(\lim_{N\to\infty}\overline{\mu_0}\sum_{k=1}^{N}\mu_{kN} + \left(|\mu_0|^2-1\right)B(0)\right) = c\varphi_n(0),\end{aligned}$$

where

$$c = \lim_{N\to\infty} \overline{\mu_0} \sum_{k=1}^{N} \mu_{kN} + \left(|\mu_0|^2 - 1\right)B(0)$$

$$= \overline{\mu_0}(s(0) - \mu_0 B(0)) + \left(|\mu_0|^2 - 1\right)B(0).$$

Now, applying (4.5) and (4.6), one gets

$$\left\langle E\left(|s|^2 - 1\right)^{1/2}, B\varphi \right\rangle = \lim_{n\to\infty} \overline{c\varphi_n(0)}$$

and then

$$\left\langle E\left(|s|^2 - 1\right)^{1/2}, B\varphi \right\rangle = \overline{\left(\frac{c\varphi(0)}{E\left(|s|^2 - 1\right)^{1/2}(0)}\right)}.$$

This proves (4.3) and finishes the proof of Lemma 2. □

Let us go now into the proof of Corollary 1.

(ii) ⇒ (i). Applying (2.2) one can easily check that s^{-1} is a non-extreme point of the unit ball of H^∞. The fact that $s \in M_B + \mathbf{C}B$ can be found in [3, p. 287].

(i) ⇒ (ii). Since B is a finite Blaschke product, the function s extends analytically to a neighbourhood of $\overline{D}$. Now, applying the theorem, one only has to check that the function

$$\left(s(e^{it}) + B(e^{it})\overline{E\left(|s|^2 - 1\right)^{1/2}(e^{it})}\right)^{-2}$$

is an exposed point of H^1. Let $r(z)$ be the function in H^2 having boundary values $B(e^{it})\overline{E\left(|s|^2 - 1\right)^{1/2}}(e^{it})$ given by Lemma 2. Consider

$$F(z) = s(z)^{-2}\left(1 + \frac{r}{s}(z)\right)^{-2}, \qquad z \in D.$$

Since s is outer and $\operatorname{Re}(1 + r/s(z)) > 0$ for $z \in D$, the function F also is outer. Furthermore,

$$|F(e^{it})| \le |s(e^{it})|^{-2}\left(1 - \left(1 - |s(e^{it})|^{-2}\right)^{1/2}\right)^{-2} \le 4|s(e^{it})|^2$$

$$\text{a.e}(dm)e^{it} \in \partial D$$

and then $F \in H^1$. Moreover, $F^{-1} = (s + r)^2 \in H^1$ and one gets that F is an exposed point of H^1 (see [4, p. 486]). This proves Corollary 1. □

The same arguments of this proof give the following result.

COROLLARY 2. *Let $\{z_n\}$ be a Blaschke sequence of points of D and let B be the Blaschke product with zeros $\{z_n\}$. Let s be an analytic function in D such that $s \in (BH^2)^{\perp} + \mathbf{C}B$ and s^{-1} is a non-extreme point of the unit ball of H^∞. Then, taking $w_n = E(1 - |s|^{-2})^{1/2}(z_n)$ for $n = 1, 2, \ldots,$ the Pick-Nevanlinna problem in $(*)$ has the function s as leading coefficient.*

Remark 2. By an argument similar to the proof of Corollary 1, one can show that if a function s satisfies the conditions (i) of Corollary 1, a finite sequence $\{w_n \quad n = 1, 2, \ldots, N\}$ produce a Pick-Nevanlinna problem with leading coefficient s if and only if there exist an inner function I such that the following two conditions are satisfied,

1. $w_n = E(1 - |s|^{-2})^{1/2}(z_n)$, $n = 1, 2, \ldots, N$,
2. $IE(1 - |s|^{-2})^{1/2} \in M_B + \mathbf{C}B$.

Remark 3. If B is a Blaschke product with infinite zeros $\{z_n\}$, let M_B be the space of analytic functions in D that can be obtained as uniform limits on compacts of D of finite linear combinations of the functions $\{(1 - \overline{z_n}z)^{-1}$: $n = 1, 2, \ldots\}$. From Corollary 1, one could ask if the condition in the theorem imposing that the function F is an exposed point of H^1, could be replaced by $s \in M_B + \mathbf{C}B$. The answer is negative and, in fact, one can show that there exists an analytic function $s \in N^+(D)$, $s \in M_B + \mathbf{C}B$ and s^{-1} is a non-extreme point of the unit ball of H^∞, in such a way that there exists no sequence $\{w_n\}$ such that the Pick-Nevanlinna problem in $(*)$ has the function s as leading coefficient.

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