BLASCHKE PRODUCTS WITH PRESCRIBED RADIAL LIMITS

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1. *Introduction*

The purpose of this paper is to give a result on the existence of Blaschke products with prescribed radial limits at certain subsets of the unit circle in the complex plane.

Let E be a finite subset of the unit circle T and let ϕ be a function defined on E with $\sup\{|\phi(e^{it})| : e^{it} \in E\} \leqslant 1$. G. Cargo [4] proved that there exists a Blaschke product I such that

$$\lim_{r\to 1} I(re^{it}) = \phi(e^{it}) \quad \text{for } e^{it} \in E.$$

If f is an arbitrary function defined in the open unit disc D, e^{it} is a point of the unit circle and γ is the radius from 0 to e^{it}, the radial cluster set of f at e^{it} is the set of points $\alpha \in \mathbb{C}$ such that there exists a sequence $\{z_n\}$ in γ with $\lim_{n\to\infty} z_n = e^{it}$, such that $\lim_{n\to\infty} f(z_n) = \alpha$. C. Belna, P. Colwell and G. Piranian [1] have proved the following more general result. Let $E = \{e^{it_m}\}$ be a countable subset of the unit circle and let $\{K_m\}$ be a sequence of nonempty, closed and connected subsets of the closed unit disc. Then, there exists a Blaschke product such that its radial cluster set at e^{it_m} is K_m, $m = 1, 2, \ldots$.

Our aim is to extend these results to more general sets E, in the case of dealing with radial limits.

By the F. and M. Riesz theorem, a bounded analytic function in the unit disc is determined by its radial limits at a set of positive measure of the circle. So, if we try to interpolate *general* functions by radial limits of Blaschke products, it is natural to restrict ourselves to subsets E of the unit circle of zero Lebesgue measure.

A set is called of type F_σ if it is a countable union of closed sets, and it is called of type G_δ if it is a countable intersection of open sets. Observe that a closed subset of the unit circle is of type F_σ and G_δ. The closure of a set E will be denoted by $\bar{E}$.

Our result is the following.

THEOREM. *Let E be a subset of the unit circle of zero Lebesgue measure and of type F_σ and G_δ. Let ϕ be a function defined on E with* $\sup\{|\phi(e^{it})| : e^{it} \in E\} \leqslant 1$ *and such that for each open set $\mathcal{U}$ of the complex plane, $\phi^{-1}(\mathcal{U})$ is of type F_σ and G_δ. Then there exists a Blaschke product I extending analytically to $T\backslash\bar{E}$ such that*

$$\lim_{r\to 1} I(re^{it}) = \phi(e^{it}) \quad \textit{for } e^{it} \in E.$$

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We consider sets E of type F_σ and G_δ because in this situation the cases $\phi \equiv 0$ and $\phi \equiv 1$ of the theorem have been considered by R. Berman (see Theorems 4.4 and 4.9 of [2]).

THEOREM (R. Berman). *Let E be a subset of the unit circle of zero Lebesgue measure and of type F_σ and G_δ. Then there exist Blaschke products B_0 and B_1 such that:*
(i) *B_0 extends analytically to $T\backslash\bar{E}$ and $\lim_{r\to 1} B_0(re^{it}) = 0$ if and only if $e^{it}\in E$;*
(ii) *$\lim_{r\to 1} B_1(re^{it}) = 1$ if and only if $e^{it}\in E$.*

A posteriori, in our result, ϕ has to be a pointwise limit of continuous functions. This turns to be equivalent [7, p. 141] to $\phi^{-1}(\mathscr{U})$ being of type F_σ for all open sets $\mathscr{U}$ of the complex plane. Then, for $|\alpha| < 1$, the set $E_\alpha = \{e^{it}: \phi(e^{it}) = \alpha\}$ has to be of type G_δ. Nevertheless, in [3] it is proved that E_α is meagre, that is, a countable union of sets such that its closure has no interior. So, in the general case, the hypothesis $\phi^{-1}(\mathscr{U})$ is of type F_σ for $\mathscr{U}$ open, cannot be sufficient.

Our theorem does not cover the result of C. Belna, P. Colwell and G. Piranian, even when the compact sets are points, because a countable set has not to be of type G_δ. For example, by Baire's Category theorem, the set $\{e^{it}: t\in\mathbb{Q}\}$ is not of type G_δ. Nevertheless, one can show that the proof of the theorem can be adapted to recover their result.

Let H^∞ be the Banach space of all bounded analytic functions in the open unit disc D with the norm

$$\|f\|_\infty = \sup\{|f(z)|: z\in D\}.$$

The main idea of our proof is to use a result of A. Stray [10] on the Pick–Nevanlinna interpolation problem in order to show that from the existence of functions in the unit ball of H^∞ with some radial limits at points of E, one can obtain Blaschke products with the same radial limits at E. This is done in Section 2. Section 3 is devoted to the proof of the theorem.

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2. *The Pick–Nevanlinna interpolation problem*

Given two sequences of numbers $\{z_n\}, \{w_n\}$ in D, the classical Pick–Nevanlinna interpolation problem consists in finding all analytic functions $f\in H^\infty$ satisfying $\|f\|_\infty \leqslant 1$ and $f(z_n) = w_n, n = 1, 2, \ldots$. We shall denote it by

$$(*) \qquad \text{Find } f\in H^\infty, \quad \|f\|_\infty \leqslant 1,\ f(z_n) = w_n,\ n = 1, 2, \ldots.$$

Pick and Nevanlinna found necessary and sufficient conditions in order that the problem (∗) has a solution. Let $\mathscr{G}$ be the set of all solutions of the problem (∗). Nevanlinna showed that if $\mathscr{G}$ consists of more than one element, there is a parametrization of the form

$$\mathscr{G} = \left\{f\in H^\infty: f = \frac{p\phi+q}{r\phi+s}: \phi\in H^\infty, \|\phi\|_\infty \leqslant 1\right\},$$

where p, q, r, s are certain analytic functions in D, depending on $\{z_n\}, \{w_n\}$ and satisfying $ps - qr = B$, the Blaschke product with zeros $\{z_n\}$.

Later, Nevanlinna showed that for all $e^{i\theta} \in T$, the function

$$I_\theta = \frac{pe^{i\theta}+q}{re^{i\theta}+s}$$

is inner. Therefore, if the problem (∗) has more than one solution, there are inner functions solving it. See [5, pp. 6, 165] for the proofs of these results.

Recently, A. Stray [10] has proved that, in fact, for all $e^{i\theta} \in T$ except possibly a set of zero logarithmic capacity, the function I_θ is a Blaschke product. So, if the problem (∗) has more than one solution, there are Blaschke products solving it. Also [9], denoting by $\{z_n\}'$ the set of accumulation points of the sequence $\{z_n\}$, the functions I_θ extend analytically to $T \setminus \{z_n\}'$.

The connection of these results with our theorem is given in the following proposition.

PROPOSITION. *Let E be a subset of the unit circle. Assume that there exist a Blaschke product B_0 that extends analytically to $T\setminus\bar{E}$ with $\lim_{r\to 1} B_0(re^{it}) = 0$ for $e^{it} \in E$, and an analytic function f_1 in the unit ball of H^∞, $f_1 \not\equiv 1$, such that $\lim_{r \to 1} f_1(re^{it}) = 1$ for $e^{it} \in E$. Then for each analytic function g in the unit ball of H^∞, there exists a Blaschke product I that extends analytically to $T\setminus\bar{E}$, such that*

$$\lim_{r\to 1}(I(re^{it})-g(re^{it})) = 0 \quad \textit{for} \quad e^{it} \in E.$$

Because of the result of R. Berman cited in the introduction, the hypotheses of the Proposition are satisfied if E is a subset of the unit circle of zero Lebesgue measure and of type F_σ and G_δ.

Proof of the Proposition. Let $\{z_n\}$ be the zeros of B_0 and $w_n = 2^{-1}g(z_n)(1+f_1(z_n))$ for $n = 1, 2, \ldots$. Consider the Pick–Nevanlinna interpolation problem,

$$(*) \qquad \text{Find } f \in H^\infty, \quad \|f\|_\infty \leqslant 1, \; f(z_n) = w_n, \; n = 1, 2, \ldots.$$

Since $2^{-1}g(1+f_1)$ is a solution of (∗) and it is a nonextremal point of the unit ball of H^∞, the problem (∗) has more than one solution. Actually, if $f_0 = 2^{-1}g(1+f_1)$, since

$$\int_0^{2\pi} \log(1-|f_0(e^{i\theta})|)\, d\theta > -\infty,$$

one can consider the function

$$E(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log(1-|f_0(e^{it})|)\, dt\right), \quad z \in D.$$

Now $f_0 + B_0 E$ is a solution of (∗) different from f_0.

Now, by the theorem of A. Stray, there exists a Blaschke product I extending analytically to $T\setminus\bar{E}$, solving (∗). Therefore,

$$I = g\frac{1+f_1}{2} + B_0 h$$

for some $h \in H^\infty$. Then, since $\lim_{r\to 1} f_1(re^{it}) = 1$ and $\lim_{r\to 1} B_0(re^{it}) = 0$ for $e^{it} \in E$, one has

$$\lim_{r\to 1}(I(re^{it})-g(re^{it})) = 0 \quad \text{for} \quad e^{it} \in E,$$

and this proves the Proposition.

3. *Proof of the Theorem*

First, let us assume that ϕ is a simple function. Because of the topological hypothesis on ϕ, one has

$$\phi = \sum_{k=1}^{n} \alpha_k \chi_{E_k},$$

where χ_{E_t} is the characteristic function of the set E_t, $\{E_k\}$ are subsets pairwise disjoint of the unit circle of type F_σ and G_δ, and $\sup\{|\alpha_k| : k = 1, \ldots, n\} \leqslant 1$.

According to the Proposition of Section 2 and the theorem of R. Berman cited in the introduction, in order to prove the theorem when ϕ is simple, it is sufficient to show the following result.

LEMMA. *Let E be a subset of the unit circle of zero Lebesgue measure and of type F_σ and G_δ. Assume $E = \bigcup_{k=1}^{n} E_k$, where $\{E_k\}$ are sets of type F_σ and G_δ pairwise disjoint. Let $\phi = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$, where $\sup\{|\alpha_k| : k = 1, \ldots, n\} \leqslant 1$. Then there exists an analytic function f of the unit ball of H^∞ such that*

$$\lim_{r \to 1} f(re^{it}) = \phi(e^{it}) \quad \textit{for } e^{it} \in E.$$

Proof of the Lemma. Considering a conformal mapping from the unit disc to the right half plane $\Pi = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$, one can assume $\alpha_k \in \bar{\Pi}, k = 1, \ldots, n$, and the problem is to find an analytic function f on D such that

$$\operatorname{Re} f(z) \geqslant 0 \text{ for } z \in D \quad \text{and} \quad \lim_{r \to 1} f(re^{it}) = \alpha_k \text{ for } e^{it} \in E_k, \quad k = 1, \ldots, n. \tag{1}$$

(1) has the advantage that the sum of functions taking values in Π also takes values in Π. Therefore, for $1 \leqslant k \leqslant n$, one has to construct an analytic function f_k in D such that

$$\begin{aligned} &\operatorname{Re} f_k(z) \geqslant 0 \quad \text{for } z \in D, \\ &\lim_{r \to 1} f_k(re^{it}) = \alpha_k \quad \text{for } e^{it} \in E_k, \\ &\lim_{r \to 1} f_k(re^{it}) = 0 \quad \text{for } e^{it} \in E \backslash E_k, \end{aligned} \tag{2}$$

because then $f = f_1 + \ldots + f_n$ will satisfy (1).

Fix $1 \leqslant k \leqslant n$. One can assume $\alpha_k \neq 0$. Since k is fixed, in the following construction the subindex k will be omitted.

Since E_k and $E \backslash E_k$ are sets of zero Lebesgue measure and of type F_σ and G_δ, there exist (see the proof of Theorem 3 of [6]) two positive measures μ and μ^* on T such that

$$\lim_{h \to 0} \frac{\mu\{e^{is} : t-h < s < t+h\}}{h} = +\infty \quad \text{if } e^{it} \in E_k \tag{3}$$

$$\text{and } \mu'(e^{it}) = 0 \quad \text{if } e^{it} \in T \backslash E_k,$$

$$\lim_{h \to 0} \frac{\mu^*\{e^{is} : t-h < s < t+h\}}{h} = +\infty \quad \text{if } e^{it} \in E \backslash E_k \tag{4}$$

$$\text{and } \mu^{*\prime}(e^{it}) = 0 \quad \text{if } e^{it} \in T \backslash (E \backslash E_k).$$

In the same proof, the authors consider u, u^*, the Poisson integrals of the measures μ, μ^*, and v, v^*, the harmonic conjugates of u, u^*. Taking $g = u+iv$ and $g^* = u^*+iv^*$ and using (3) and (4), they prove

$$\begin{gathered}\lim_{r\to 1} g(re^{it}) \quad \text{exists and is finite for } e^{it}\in T\backslash E_k,\\ \lim_{r\to 1} \operatorname{Re} g(re^{it}) = +\infty \quad \text{for } e^{it}\in E_k,\\ \lim_{r\to 1} g^*(re^{it}) \quad \text{exists and is finite for } e^{it}\in T\backslash(E\backslash E_k),\\ \lim_{r\to 1} \operatorname{Re} g^*(re^{it}) = +\infty \quad \text{for } e^{it}\in E\backslash E_k.\end{gathered} \tag{5}$$

We use now an idea of W. Rudin [8]. Since $\operatorname{Re} g(z) \geqslant 0$ and $\operatorname{Re} g^*(z) \geqslant 0$ for $z\in D$, the function

$$q(z) = \frac{g(z)^{\frac{1}{2}}}{g(z)^{\frac{1}{2}}+g^*(z)^{\frac{1}{2}}}$$

is analytic in D. Furthermore, from (5) one obtains

$$\begin{gathered}\lim_{r\to 1} q(re^{it}) = 1 \quad \text{if } e^{it}\in E_k,\\ \lim_{r\to 1} q(re^{it}) = 0 \quad \text{if } e^{it}\in E\backslash E_k.\end{gathered} \tag{6}$$

One has

$$\operatorname{Re} q(z) = \frac{|g(z)| + \operatorname{Re}(g(z)^{\frac{1}{2}}\overline{g^*(z)^{\frac{1}{2}}})}{|g(z)|+|g^*(z)|+2\operatorname{Re}(g(z)^{\frac{1}{2}}\overline{g^*(z)^{\frac{1}{2}}})}. \tag{7}$$

Since $|\operatorname{Arg}(g(z)^{\frac{1}{2}})| \leqslant \pi/4$ and $|\operatorname{Arg}(g^*(z)^{\frac{1}{2}})| \leqslant \pi/4$ for $z\in D$, one obtains

$$\operatorname{Re}(g(z)^{\frac{1}{2}}\overline{g^*(z)^{\frac{1}{2}}}) \geqslant 0,$$

and from (7) one can deduce

$$0 \leqslant \operatorname{Re} q(z) \leqslant 1 \quad \text{for } z\in D. \tag{8}$$

Now, take M to be a rectangle contained in the right half plane such that $0, \alpha_k \in \partial M$. Let Φ be the conformal mapping from the strip $\{z\in\mathbb{C}: 0 < \operatorname{Re}(z) < 1\}$ into M such that $\Phi(0) = 0$ and $\Phi(1) = \alpha_k$, and consider the function $f = \Phi\circ q$. Since M is contained in the right half plane, $\operatorname{Re} f(z) \geqslant 0$ for $z\in D$. Moreover, from (6) one obtains

$$\begin{gathered}\lim_{r\to 1} f(re^{it}) = \Phi(1) = \alpha_k \quad \text{for } e^{it}\in E_k,\\ \lim_{r\to 1} f(re^{it}) = \Phi(0) = 0 \quad \text{for } e^{it}\in E\backslash E_k.\end{gathered}$$

This gives (2), and the Lemma is proved.

Let us consider now the general case. Applying the Proposition of Section 2 and the result of R. Berman cited in the introduction, in order to prove the theorem one has only to construct an analytic function f of the unit ball of H^∞ such that

$$\lim_{r\to 1} f(re^{it}) = \phi(e^{it}) \quad \text{for } e^{it}\in E.$$

Consider a conformal mapping S from the unit disc into the square $Q=(-1,1)\times(-1,1)$. Since ∂Q is a Jordan curve, S extends homeomorphically to $\bar{D}$. Considering $S\circ\phi$, one can assume that the function ϕ takes values in $\bar{Q}$, and the problem is to find an analytic function f in D, taking its values in Q, such that

$$\lim_{r\to 1} f(re^{it})=\phi(e^{it}) \quad \text{for } e^{it}\in E. \tag{9}$$

Consider the squares $Q_1=[0,1)\times[0,1)$, $Q_2=(-1,0)\times[0,1)$, $Q_3=(-1,0]\times(-1,0)$, $Q_4=(0,1)\times(-1,0)$, and let α_i be the centre of Q_i. The squares Q_i are pairwise disjoint, and

$$Q=\bigcup_{i=1}^{4} Q_i.$$

By hypothesis, the sets $E_i=\{e^{it}\in E:\phi(e^{it})\in Q_i\}$ are of type F_σ and G_δ. Take

$$\phi_1=\sum_{i=1}^{n}\alpha_i\chi_{E_i},$$

where χ_{E_i} is the characteristic function of the set E_i. Since ϕ takes its values in the square Q, the choice of $\{\alpha_i\}$ and $\{E_i\}$ gives

$$\sup_{e^{it}\in E}\max\{|\mathrm{Re}(\phi(e^{it})-\phi_1(e^{it}))|,|\mathrm{Im}(\phi(e^{it})-\phi_1(e^{it}))|\}\leqslant\tfrac{1}{2} \tag{10}$$

and

$$\sup_{e^{it}\in E}\max\{|\mathrm{Re}\,\phi_1(e^{it})|,|\mathrm{Im}\,\phi_1(e^{it})|\}\leqslant\tfrac{1}{2}. \tag{11}$$

Applying the Lemma and a conformal mapping, one obtains an analytic function f_1 on D, taking its values in the square $Q/2=(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})$ such that

$$\lim_{r\to 1} f_1(re^{it})=\phi_1(e^{it}) \quad \text{for } e^{it}\in E. \tag{12}$$

Furthermore, using the fact that finite unions and intersections of sets of type F_σ and G_δ are also of type F_σ and G_δ, one can check that if $\mathscr{U}\subset\mathbb{C}$ is open, the set

$$\{e^{it}\in E:\phi(e^{it})-f_1(e^{it})\in\mathscr{U}\} \quad \text{is of type } F_\sigma \text{ and } G_\delta. \tag{13}$$

Now, using (10), (12) and (13), one can repeat these arguments, changing ϕ to $(\phi-f_1)/\frac{1}{2}$. Then one obtains an analytic function f_2 in the unit disc taking its values in $Q/2$, such that

$$\sup_{e^{it}\in E}\max\left\{\left|\mathrm{Re}\left(\frac{\phi-f_1}{\frac{1}{2}}-f_2\right)(e^{it})\right|,\left|\mathrm{Im}\left(\frac{\phi-f_1}{\frac{1}{2}}-f_2\right)(e^{it})\right|\right\}\leqslant\tfrac{1}{2}.$$

Therefore

$$\sup_{e^{it}\in E}\max\left\{\left|\mathrm{Re}\left(\phi-f_1-\frac{f_2}{2}\right)(e^{it})\right|,\left|\mathrm{Im}\left(\phi-f_1-\frac{f_2}{2}\right)(e^{it})\right|\right\}\leqslant\frac{1}{2^2}.$$

Also, if $\mathscr{U}$ is an open set of the complex plane,

$$\left\{e^{it}\in E:\left(\frac{\phi-f_1}{\frac{1}{2}}-f_2\right)(e^{it})\in\mathscr{U}\right\} \quad \text{is of type } F_\sigma \text{ and } G_\delta.$$

Repeating this process, one obtains analytic functions f_j on the unit disc, taking values in the square $Q/2$, such that

$$\sup_{e^{it}\in E} \max\left\{\left|\operatorname{Re}\left(\phi-\sum_{j=0}^{n}\frac{1}{2^j}f_{j+1}\right)(e^{it})\right|, \left|\operatorname{Im}\left(\phi-\sum_{j=0}^{n}\frac{1}{2^j}f_{j+1}\right)(e^{it})\right|\right\} \leqslant \frac{1}{2^n}. \tag{14}$$

Now, for $z\in D$, let us consider

$$f(z) = \sum_{j=0}^{\infty}\frac{1}{2^j}f_{j+1}(z).$$

Since f_j takes its values in the square $Q/2$, one has

$$\max\{|\operatorname{Re} f(z)|, |\operatorname{Im} f(z)|\} \leqslant \sum_{j=0}^{\infty}\frac{1}{2}\frac{1}{2^j} = 1.$$

So f takes its values in the square Q.

Now let us check that $\lim_{r\to 1} f(re^{it}) = \phi(e^{it})$ for $e^{it}\in E$.

Fix $\varepsilon > 0$ and take a natural number n such that $32^{-n} \leqslant \varepsilon$. Applying (14) and using the fact that the functions f_j take values in the square $Q/2$, one has, for $1-r$ small enough,

$$\left|\phi(e^{it})-\sum_{j=0}^{\infty}\frac{1}{2^j}f_{j+1}(re^{it})\right| \leqslant \left|\phi(e^{it})-\sum_{j=0}^{n}\frac{1}{2^j}f_{j+1}(re^{it})\right| + \sum_{j=n+1}^{\infty}\frac{1}{2^j}\frac{1}{\sqrt{2}}$$
$$\leqslant \frac{2}{2^n}+\frac{1/\sqrt{2}}{2^n} \leqslant 32^{-n} \leqslant \varepsilon.$$

Therefore

$$\lim_{r\to 1} f(re^{it}) = \phi(e^{it}) \quad \text{for } e^{it}\in E,$$

and this gives the proof of the theorem.

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