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Boundary behaviour of inner functions and holomorphic mappings

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Let f be a holomorphic function in the unit disk omitting a set A of values of the complex plane. If A has positive logarithmic capacity, R. Nevanlinna proved that f has a radial limit at almost every point of the unit circle. If A is any infinite set, we show that f has a radial limit at every point of a set of Hausdorff dimension 1. A localization technique reduces this result to the following theorem on inner functions. If I is an inner function omitting a set of values B in the unit disk, then for any accumulation point b of B in the disk, there exists a set of Hausdorff dimension 1 of points in the circle where I has radial limit b.

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0 Introduction and description of results

We are interested in describing the radial boundary values of holomorphic mappings f from the unit disk $\mathbf{D}$ into a given domain Ω of the complex plane $\mathbf{C}$.

Let us denote by $F(f)$ the set $F(f) = \{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta})$ exists, (possibly ∞)$\}$. This is the so called *Fatou set* of f. If Ω is bounded, then a classical theorem of Fatou asserts the existence of radial boundary values *a.e.*, or, with the notation above that $|F(f)| = 2\pi$. (If $A \subset [0, 2\pi]$ then $|A|$ denotes its Lebesgue measure.) More generally, if the logarithmic capacity of the complement of Ω is not zero, then f belongs to the Nevanlinna class and in this case we also have $|F(f)| = 2\pi$.

On the other hand, if the complement of Ω is just a finite set then any holomorphic covering map from $\mathbf{D}$ onto Ω has a countable Fatou set.

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The classical results that we have just recalled tell us what the size of the Fatou set of f is in terms of the size of the complement of the range of f, Ω, in two extreme cases: the complement is very large (*i.e.* of positive capacity) or very small (*i.e.* just countable). In this context, it remains to understand what happens in the intermediate situation when the complement is an infinite set but of zero logarithmic capacity. It turns out, and this is the main result of this paper, that there is no scale of possibilities: the Fatou set in this intermediate situation is always of Hausdorff dimension 1.

Theorem 1. *Let Ω be a hyperbolic domain in* $\mathbf{C}$, *let f be a holomorphic function from* $\mathbf{D}$ *into* Ω. *Then*

(a) *If* $cap(\mathbf{C} \setminus \Omega) > 0$, *then* $|F(f)| = 2\pi$.
(b) *If* $cap(\mathbf{C} \setminus \Omega) = 0$ *and* $\sharp(\mathbf{C} \setminus \Omega) = +\infty$ *then* $\dim(F(f)) = 1$.
(c) *If* $\sharp(\mathbf{C} \setminus \Omega) < +\infty$, *then $F(f)$ is at least countable.*

Here and hereafter, "cap" means logarithmic capacity, "dim" means Hausdorff dimension, and "#" means number of points. Also, a planar domain Ω is termed *hyperbolic* if its complement in $\mathbf{C}$ contains at least 2 points, or in other terms if it is holomorphically and regularly covered by the unit disk.

As we mentioned above (a) is classical (due to Nevanlinna, see [10, p. 180]), (c) requires a little argument which we shall give later on, and, finally, (b) is the main result of this paper. Also, in the remaining cases where Ω is the whole complex plane or the complement of Ω is a single point, it may happen $F(f) = \emptyset$ (see [3, p. 44]).

We should remark that since the holomorphic functions we are dealing with omit at least two points in $\mathbf{C}$, they are normal (in the Lehto-Virtanen sense) and, consequently, if a radial limit exists at a point $e^{i\theta}$ then the angular limit also exists at $e^{i\theta}$.

The conditions on Ω, although given in terms of the complement of Ω in $\mathbf{C}$, have intrinsic meaning; recall that $\text{cap}(\mathbf{C} \setminus \Omega) > 0$ means that Ω has Green's function, or, equivalently that Brownian motion on Ω is transient, while $\sharp(\mathbf{C} \setminus \Omega) < +\infty$ means that Ω has finite Poincaré area.

The result above is sharp, in the following sense: given any infinite closed set E of zero capacity, there is a holomorphic function g from $\mathbf{D}$ onto $\mathbf{C} \setminus E$ (namely, a holomorphic covering map) such that $|F(g)| = 0$ (see [10, p. 211]) and, of course, $\dim(F(g)) = 1$, while, similarly, the covering maps from $\mathbf{D}$ onto the plane with finitely many points removed have radial boundary values for, exactly, a countable set of θ's. The corresponding results for covering maps, in the general framework of Riemann surfaces, are proved in [5].

0.1

Let f be a holomorphic function from $\mathbf{D}$ into Ω and let g be a holomorphic covering map from $\mathbf{D}$ onto Ω. Then we can factorize $f = g \circ b$ where b is a holomorphic mapping from $\mathbf{D}$ into $\mathbf{D}$. Recall that a holomorphic mapping b from

$\mathbf{D}$ into $\mathbf{D}$ is called *inner* if almost all of its radial boundary values have modulus one. If the function b above is not inner then, no matter what Ω is, one has that $|F(f)| > 0$. Thus in the context of this paper only the case b inner is interesting.

One possible approach to the proof of Theorem 1 consists simply in pulling back the known boundary behaviour of g through b to f. We shall discuss this approach in Sect. 1.1.

0.2

But inner functions could enter into the picture in another way, namely, through *localization*.

Let $a \in \partial\Omega$, and $\varepsilon > 0$, and consider a connected component G of $\{z \in \mathbf{D} : |f(z) - a| < \varepsilon\}$. G is simply connected. Let φ be a Riemann mapping from $\mathbf{D}$ onto G, then the mapping $h = (1/\varepsilon)\{f \circ \varphi(z) - a\}$ allows us to study the behaviour of f near a, and in the case of interest h is an inner function.

By means of this simple and well known idea the proof of Theorem 1 can be reduced to related results on the geometry of inner functions. We shall discuss this approach in Sect. 1.2.

0.3 The geometry of inner functions

Let f be an inner function and denote by $\mathcal{O}(f)$ the omitted set of f, *i.e.*, $\mathcal{O}(f) = \mathbf{D}\setminus f(\mathbf{D})$. Frostman showed that always $\operatorname{cap}(\mathcal{O}(f)) = 0$, while if E is any relatively closed subset of $\mathbf{D}$ of zero capacity, any holomorphic covering map from $\mathbf{D}$ onto $\mathbf{D} \setminus E$ is an inner function, (see, *e.g.*, [3 p. 37]).

Let $\mathcal{E}(f)$ denote the exceptional set of f, *i.e.*, the set of $e^{i\theta}$'s such that f does not have radial limit of modulus 1 at $e^{i\theta}$. Of course, $|\mathcal{E}(f)| = 0$. The sizes of the sets $\mathcal{E}(f)$ and $\mathcal{O}(f)$ are closely related. This fact was first pointed out by Noshiro and Seidel (see, *e.g.*, [15, p. 328]) by showing that if $\mathcal{O}(f)$ contains two points then $\operatorname{cap}(\mathcal{E}(f)) > 0$. The following known theorem describes that connection in sharp terms. We denote by d *the Poincaré distance in* $\mathbf{D}$.

Theorem A. *If $\mathcal{O}(f)$ has at least two points then*

$$\dim(\mathcal{E}(f)) > 1/2.$$

In fact, if $d(f)$ denotes

$$d(f) = \inf\{d(a, b), a, b \in \mathcal{O}(f), a \neq b\}$$

then

$$\dim(\mathcal{E}(f)) \geq \alpha(d(f))$$

where α is an (absolute) decreasing function defined in $[0, \infty)$, satisfying $\lim_{t\to+\infty} \alpha(t) = 1/2$, $\alpha(0) = 1$.

The function α is only implicitly defined, but the result is sharp, (see [4] and [6]). Observe that if $\mathscr{O}(f)$ contains two points then $\dim(\mathscr{E}(f)) > \frac{1}{2}$, and also that if $f = \exp\left(-\frac{1+z}{1-z}\right)$, then $\mathscr{O}(f) = \{0\}$ while $\mathscr{E}(f) = \{1\}$.

Here we complement Theorem A in a substantial way. We have

Theorem 2. *Let f be inner and let $b \in \mathbf{D}$ be an accumulation point of $\mathscr{O}(f)$ then*

$$\dim\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = b\} = 1\,.$$

This is a corollary of the following quantitative version of Theorem A.

Theorem 3. *Let f be inner and let $b \in \mathscr{O}(f)$. Let*

$$d_f(b) = \inf\{d(z, b) : z \in \mathscr{O}(f) \setminus \{b\}\}\,.$$

Then

$$\dim\left\{\theta \in [0, 2\pi] : \limsup_{r\to 1} d(f(re^{i\theta}), b) < d_f(b)^K\right\} \geq 1 - \frac{M}{\log\log\frac{1}{d_f(b)}}$$

if $d_f(b) \leq m$. The quantities M, K and m are absolute constants.

0.4

It will be convenient to express some of our results in terms of a more general notion of inner function which we now describe.

Definition. *Given a planar domain Ω we say that a holomorphic function f from* **D** *into Ω is* inner into Ω *if*

$$\{e^{i\theta} \in \partial\mathbf{D} : \lim_{r\to 1} f(re^{i\theta}) \text{ exists and belongs to } \Omega\}$$

has measure zero.

The functions which are inner into **D** are the usual inner functions. If F is a holomorphic covering map from **D** onto a domain Ω then F is inner into Ω, and, as a matter of fact, if f is any holomorphic function from **D** into Ω which factorizes as $f = F \circ b$, then f is inner into Ω if and only if b is inner into **D**.

The proof of Theorem 1, case (b), contains the following more precise version.

Theorem 4. *Let Ω be a planar domain and let f be inner into Ω and $a \in \partial\Omega$. Then if a is not isolated in $\partial\Omega$, but for some $\varepsilon > 0$, cap$(\mathbf{D}(a, \varepsilon) \cap \partial\Omega) = 0$ then*

$$\dim(\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = a\}) = 1\,.$$

Of course, an isolated point of $\partial\Omega$ may "attract" just countably many $e^{i\theta}$'s, while if $\operatorname{cap}(\mathbf{D}(a, \varepsilon) \cap \partial\mathbf{D}) > 0$ for all $\varepsilon > 0$ then it could happen that a attracts just one θ, (consider, *e.g.*, a Riemann mapping onto a Jordan domain).

0.5

Holomorphic mappings between hyperbolic planar domains are Lipschitz (with constant 1) with respect to the respective Poincaré metrics. The holomorphic functions from **D** into **C** which are Lipschitz when **D** is endowed with the Poincaré metric and **C** with its euclidean metric are the Bloch functions. For those functions there are related results due to Makarov and Rohde, see [8] and [12]. For instance, Rohde has shown that if f is inner and belongs to the "little" Bloch space, $\mathscr{B}_0$, *i.e.*, $\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0$, then for every $a \in \mathbf{D}$, $\dim\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = a\} = 1$. There are also closely related results concerning the "bounded", as opposed to "divergent", boundary behaviour. For those we refer the reader to [6] and the references therein.

There are several results where, from geometric assumptions on the range of a holomorphic function, one deduces information about its boundary behaviour. We would like to recall here a classical result of Beurling, which we shall use later on, namely: if f is a function holomorphic in the disk whose Riemann surface has finite area then f has radial limits except at most in a set of capacity zero.

0.6

The plan of the paper is as follows. Theorems 1 and 4 are proved in Sect. 1, and Theorem 3 is proved in Sect. 3. Section 2 develops some machinery on hyperbolic derivatives which is needed in the proof of Theorem 3. Finally, Sect. 4 discusses some complements and some questions.

1 Proof of Theorem 1

We shall be giving two proofs of Theorem 1. The first one, which we only sketch, is based on its geometric counterpart, Theorem B, which is proved in [5], and a subordination argument, from [6]. The second proof shows, by means of the localization technique which we have described in Sect. 0.2, that Theorem 1 is a corollary of Theorem 2. This second proof depends only on the results of this paper. Inner functions enter into both arguments, albeit in different ways.

1.1 First proof

Let Ω be a hyperbolic planar domain, and let F be a holomorphic covering map from **D** onto Ω. Assume that the complement of Ω is an infinite set of zero capacity. Then

Theorem B, [5]. *With the notations above, given $\varepsilon > 0$ there exists a closed subset E of $\partial\mathbf{D}$, with Hausdorff dimension $\geq 1-\varepsilon$, and such that if $e^{i\theta} \in E$ then*

$$\lim_{r\to 1} \mathrm{d}_\Omega(F(re^{i\theta}), F(0)) = +\infty\,,$$

uniformly in r*, or, equivalently,*

$$\lim_{r\to 1} \mathrm{dist}(F(re^{i\theta}), \partial\Omega) = 0\,,$$

uniformly in r.

Here and hereafter d_Ω denotes hyperbolic distance in Ω, and *dist* means spherical distance.

Let f be a holomorphic mapping from $\mathbf{D}$ into Ω. We may factorize f as $f = F \circ b$ where b is a holomorphic mapping from $\mathbf{D}$ into $\mathbf{D}$.

Now, if b is not inner, then on a subset A of positive measure of $\partial\mathbf{D}$ we have that $\lim_{r\to 1} b(re^{i\theta})$ exists and belongs to $\mathbf{D}$. Consequently, $|F(f)| > 0$, and we are done.

If b is inner, then we may use the argument of (6, proof of Theorem 2), to transfer the above result on the uniform boundary behaviour of F to the following result on the boundary behaviour of f:

$$\dim\{\theta \in [0, 2\pi] : \lim_{r\to 1} \mathrm{dist}(f(re^{i\theta}), \partial\Omega) = 0\} = 1\,.$$

Since $\mathrm{cap}\,(\partial\Omega) = 0$, $\partial\Omega$ is a totally disconnected set, and so if $\lim_{r\to 1} \mathrm{dist}(f(re^{i\theta}), \partial\Omega) = 0$ it must be the case that $\lim_{r\to 1} f(re^{i\theta})$ exists, allowing ∞ as a limiting value.

1.2 Second proof

Let a be an accumulation point of $\partial\Omega$, with no loss of generality we assume that $a = 0$. Since $\mathrm{cap}\,(\partial\Omega) = 0$ we may find $\varepsilon > 0$, such that $\{w \in \mathbf{C} : |w| = \varepsilon\}$ separates $\partial\Omega$; of course, $\mathrm{cap}\,(\mathbf{D}(0,\varepsilon) \setminus \Omega) = 0$. We shall assume, that $\varepsilon = 1$. Let z_0 be a point of $\mathbf{D}$ with $|f(z_0)| < 1$. (If no such z_0 exists then $1/f$ is a bounded holomorphic function, and Fatou's theorem finishes the argument). Let G be the component of $\{z \in \mathbf{D} : |f(z)| < 1\}$ which contains z_0. The domain G is simply connected, since $0 \notin f(\mathbf{D})$. Also, $\partial G \cap \partial\mathbf{D} \neq \emptyset$ since otherwise f would assume every value in $\{w \in \mathbf{C} : |w| < 1\}$. Let φ be a conformal map from $\mathbf{D}$ into G with $\varphi(0) = z_0$, and consider the function $g = f \circ \varphi$, holomorphic from $\mathbf{D}$ into $\mathbf{D}$.

If g were *not* inner, there would exist a subset E of $\partial\mathbf{D}$ of positive length such that for every $e^{i\theta} \in E$, $\lim_{r\to 1} g(re^{i\theta})$ exists and belongs to $\mathbf{D}$, and $\varphi(e^{i\theta}) = \lim_{r\to 1} \varphi(re^{i\theta})$ exists. Now, for $e^{i\theta} \in E$, one has $\varphi(e^{i\theta}) \in \partial G \cap \partial\mathbf{D}$, since, otherwise, $\lim_{r\to 1} |g(re^{i\theta})| = 1$. Let us denote by $\varphi(E)$ the set $\varphi(E) = \{\varphi(e^{i\theta}) : e^{i\theta} \in E\}$. By Löwner's lemma, $|\varphi(E)| > 0$. Now, if $e^{i\phi} = \varphi(e^{i\theta})$ then, $\lim_{r\to 1} f(\varphi(re^{i\theta}))$ exists, and by the Lehto-Virtanen extension of Lindelöf's theorem to normal functions (f is normal since the complement of Ω contains at least two points), we deduce that the radial limit of f at $e^{i\phi}$ also exists. Consequently, f has radial limits in the set of $\varphi(E)$ of positive length, and we would be done.

It remains to consider the case when g is inner. We now apply Theorem 2 to g to conclude that there exists a subset E of $\partial\mathbf{D}$ of Hausdorff dimension 1, such that if $e^{i\theta} \in E$ then

$$0 = \lim_{r\to 1} g(re^{i\theta}) = \lim_{r\to 1} f(\varphi(re^{i\theta}))\,;$$

we may also assume that if $e^{i\theta} \in E$, $\lim_{r\to 1} \varphi(re^{i\theta}) = \varphi(e^{i\theta})$ exists (since by Beurling's theorem (see Sect. 0.5) this fails to occur at most in a set of capacity zero, and hence of zero dimension). Now, if $e^{i\theta} \in E$, then $\varphi(e^{i\theta}) \in \partial\mathbf{D}$, since otherwise f would vanish at $\varphi(e^{i\theta})$. We may apply the Hamilton-Makarov extension of Löwner's lemma [7], [9], to deduce that $\varphi(E) = \{\varphi(e^{i\theta}) : e^{i\theta} \in E\}$ has dimension 1. And we may argue as in the previous case (g not inner) to conclude that f has radial limit zero in a set, $\varphi(E)$, of Hausdorff dimension 1.

1.3

To complete the proof of Theorem 1 we have to consider the case when $\Omega = \mathbf{C} \setminus \{0, 1\}$ is just a finite set, (with at least 3 points, including ∞ as a possibility).

We follow the lines of the first proof above. Consider a tesselation of $\mathbf{D}$ by hyperbolic triangles, $\mathscr{T}$'s, with vertices at $\infty = \partial\mathbf{D}$; each triangle projects under the covering map F onto a half plane and at the vertices of these triangles F has a radial limit: 0, 1, or ∞. Factorize the given f as $F \circ b$, where b is inner (if b is not inner, we are done). We simply have to observe that each of the connected components of each $b^{-1}(\mathscr{T})$ is a trilateral.

Lemma 1.1. *Let $\{T_j\}_{j=1}^{n}$ be disjoint trilaterals in $\mathbf{D}$ with labeled vertices $a_1(j)$, $a_2(j)$, $a_3(j) \in \partial\mathbf{D}$ and such that if $1 \le j, j' \le n$, $j \ne j'$, and if $1 \le i, i' \le 3$, $i \ne i'$ then $a_i(j) \ne a_{i'}(j')$. Then*

$$\sharp \left\{ \bigcup_{j=1}^{n} \bigcup_{i=1}^{3} \{a_i(j)\} \right\} \ge n + 2\,.$$

We do not assume the trilaterals to be Jordan domains but we assume the vertices to be accesible. To be precise:

Definition. *A trilateral is a simply conected domain with three accesible boundary points, distinct and labeled. These boundary points are its vertices.*

1.4 Proof of Theorem 4

As a matter of fact, this is what is actually proved above in the so called second proof of Theorem 1.

2 Bounds on hyperbolic derivatives

2.1 Nonvanishing bounded holomorphic functions

If Ω is a hyperbolic planar domain, we shall denote by $\lambda_\Omega(w)$, $w \in \Omega$, the density of the Poincaré metric of Ω. That is to say, if $F : \mathbf{D} \to \Omega$ is a holomorphic covering map onto Ω then

$$\lambda_\Omega(F(z))|F'(z)| = \frac{2}{1-|z|^2}\,, \text{ for each } z \in \mathbf{D}\,.$$

For arbitrary holomorphic mappings f between hyperbolic domains Ω_1 and Ω_2, Schwarz's lemma gives

$$\lambda_{\Omega_2}(f(z))|f'(z)| \le \lambda_{\Omega_1}(z)\,.$$

The particular case $\Omega_1 = \mathbf{D}$, $\Omega_2 = \mathbf{D} \setminus \{0\}$ yields the following useful fact:

Lemma 2.1. *If f is a holomorphic function in $\mathbf{D}$ with $0 < |f(z)| < 1$, for every $z \in \mathbf{D}$ then*

$$\frac{|f'(z)|}{|f(z)|} \le \frac{2}{1-|z|^2}\log\frac{1}{|f(z)|} \quad \textit{for each } z \in \mathbf{D}\,. \tag{2.1}$$

For a function f, holomorphic in $\mathbf{D}$ with $f(\mathbf{D}) \subset \mathbf{D} \setminus \{0\}$, we may write

$$\log\frac{1}{|f(z)|} = P(\mu_f)(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}d\mu_f(e^{i\theta})$$

for some positive meassure μ_f on $\partial\mathbf{D}$. It is clear that (2.1) is equivalent to the trivial estimate

$$\left|\int_0^{2\pi} e^{i\theta}d\mu_f(\theta)\right| \le \|\mu_f\| = \log\frac{1}{|f(0)|}\,.$$

Of course when μ_f is a positive multiple of a point mass, *i.e.*, when f is a covering map of $\mathbf{D}\setminus\{0\}$, we have equality. On the other hand, if $\left|\int_0^{2\pi} e^{i\theta}d\mu_f(\theta)\right|$ differs much from $\|\mu_f\|$ then f should be far from being a covering map, *i.e.*, μ_f should be far from being a point mass. To quantify this, the following elementary lemma shall be useful.

Lemma 2.2. *Let α, β be related by $0 < \alpha \le \cos\pi\beta$, $0 < \beta \le 1/2$. Let μ be a positive measure in $\partial\mathbf{D}$, with total mass $\|\mu\| = 1$ and*

$$\left|\int_0^{2\pi} e^{i\theta}d\mu(\theta)\right| \le \alpha\,.$$

For every interval I with $\frac{|I|}{2\pi} \le \beta$ one has

$$\mu(I) \le \sigma = \sigma(\alpha,\beta) = \frac{1+\alpha}{1+\cos\pi\beta} < 1\,.$$

Proof. We may assume that the interval I is centered at -1. Then

$$\begin{aligned} -\alpha \le \int_0^{2\pi} \cos\theta \, d\mu(\theta) &= \int_I \cos\theta \, d\mu(\theta) + \int_{\partial\mathbf{D}\setminus I} \cos\theta \, d\mu(\theta) \\ &\le -(\cos\pi\beta)\mu(I) + \mu(\partial\mathbf{D}\setminus I) \\ &= -(1+\cos\pi\beta)\mu(I) + 1 \,. \end{aligned}$$

In terms of holomorphic functions, and with the same notation, the lemma above reads.

Lemma 2.3. *Let $f : \mathbf{D} \to \mathbf{D}\setminus\{0\}$ be a holomorphic function, suppose that*

$$\frac{|f'(0)|}{|f(0)|\log\frac{1}{|f(0)|}} \le 2\alpha \,,$$

where, $0 \le \alpha \le 1$. *Then for every interval $I \subset \partial\mathbf{D}$, and β,* $0 \le \beta \le \frac{1}{2}$,

$$\frac{|I|}{2\pi} < \beta \Longrightarrow \frac{\mu_f(I)}{\|\mu_f\|} < \sigma(\alpha,\beta) \,.$$

Proof. Let g be the function defined by $g(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\mu_f(\theta)$, so that $\Re g = P\mu_f$. We have

$$\Re g(0) = \|\mu_f\| = \log\frac{1}{|f(0)|} \tag{2.2}$$

and

$$-\frac{f'(0)}{f(0)} = g'(0) = 2\int_0^{2\pi} e^{-i\theta} d\mu_f(\theta). \tag{2.3}$$

Therefore,

$$\left| \int_0^{2\pi} e^{-i\theta} d\mu_f(\theta) \right| \le \alpha \|\mu_f\|$$

and we apply Lemma 2.2.

For the invariant formulation of Lemma 2.3 we need the following definition; we are fixing the parameters α, β of the preceding discussion to be $\alpha = 1/4$, $\beta = 1/4$. Let $J = \{e^{i\theta} : |\theta| < \frac{3}{4}\pi\}$. For $a \in \mathbf{D}$, let $\varphi_a \in \mathscr{M}\ddot{o}b(\mathbf{D})$ be defined by

$$\varphi_a(z) = \frac{a}{|a|}\left(\frac{z+|a|}{1+z|a|}\right) .$$

Observe that

$$\varphi_a(0) = a \,,\ \varphi_a(1) = \frac{a}{|a|} \,,\ \varphi_a(-1) = -\frac{a}{|a|} \,.$$

Define $J_a = \varphi_a(J)$. Observe that $|J_a| \asymp 1 - |a|$.

Lemma 2.4. *Let $f : \mathbf{D} \to \mathbf{D} \setminus \{0\}$ be holomorphic. Assume that*

$$\frac{|f'(z)|}{|f(z)| \log \frac{1}{|f(z)|}}(1-|z|^2) \leq \frac{1}{2}\,, \quad \textit{for some } z \in \mathbf{D}\,,$$

then

$$\frac{\mu_f(J_z)}{|J_z|} > B \log \frac{1}{|f(z)|}\,,$$

where B is an absolute constant.

Proof. Let $\tilde{f} = f \circ \varphi_z$. Then

$$\frac{|\tilde{f}'(0)|}{|\tilde{f}(0)| \log \frac{1}{|\tilde{f}(0)|}} = \frac{|f'(z)|(1-|z|^2)}{|f(z)| \log \frac{1}{|f(z)|}} \leq 2\alpha\,.$$

Also, $\|\mu_{\tilde{f}}\| = \log \frac{1}{|f(z)|}$.

Moreover from $P_{\mu_{\tilde{f}}} = P_{\mu_f} \circ \varphi_z$, we deduce that for a Borel set, A, $A \subset \partial\mathbf{D}$,

$$\mu_{\tilde{f}}(A) = \int_{\varphi_z(A)} |(\varphi_z^{-1})'(e^{i\theta})|\, d\mu_f\,.$$

In particular, if $A = J$ then

$$\mu_{\tilde{f}}(J) = \int_{J_z} |(\varphi_z^{-1})'(e^{i\theta})|\, d\mu_f$$

but on J_z, $|(\varphi_z^{-1})'| \leq C \frac{1}{1-|z|}$, where consequently

$$\mu_{\tilde{f}}(J) \leq C \frac{\mu_f(J_z)}{1-|z|}\,.$$

But, by Lemma 2.3 with $\alpha = 1/4$, $\beta = 1/4$,

$$\frac{\mu_{\tilde{f}}(J)}{\|\mu_{\tilde{f}}\|} \geq 1 - \sigma_0$$

where σ_0 is the σ corresponding to α, β above and therefore

$$\begin{aligned} \frac{\mu_f(J_z)}{1-|z|} &\geq C(1-\sigma_0)\|\mu_{\tilde{f}}\| \\ &= C(1-\sigma_0) \log \frac{1}{|f(z)|} \\ &= B \log \frac{1}{|f(z)|} \end{aligned}$$

with $B = C(1-\sigma_0)$.

2.2 Bounds for the Riesz mass of positive superharmonic functions

Although Lemma 2.4 is what we shall need and use in the following, now that we are at it, it is worthwhile to complete the above analysis.

Let u be a positive superharmonic function in the unit disk $\mathbf{D}$. By Riesz' theorem we may express u as:

$$u(z) = \iint_{\mathbf{D}} G(z,w)\,d\alpha(w) + \int_{\partial\mathbf{D}} P(z,e^{i\theta})\,d\beta(e^{i\theta})$$

where $G(z,w)$ is the Green's function of the unit disk, $G(z,w) = -\log\left|\frac{z-w}{1-z\overline{w}}\right|$, $P(z,e^{i\theta})$ is the Poisson kernel $P(z,e^{i\theta}) = \frac{1-|z|^2}{|z-e^{i\theta}|^2}$, and α and β are positive measures in $\mathbf{D}$ and in $\partial\mathbf{D}$, respectively. Now

$$\partial u(0) = \frac{1}{2}\iint_{\mathbf{D}} \frac{1-|a|^2}{a}\,d\alpha(a) + \int_{\partial\mathbf{D}} e^{-i\theta}\,d\beta(e^{i\theta})$$

and

$$u(0) = \iint_{\mathbf{D}} \log\frac{1}{|a|}\,d\alpha(a) + \int_{\partial\mathbf{D}} d\beta(e^{i\theta})\,.$$

If we write $d\tilde{\alpha} = \frac{1}{2}(1-|a|^2)d\alpha$ then

$$\partial u(0) = \iint_{\mathbf{D}\cup\partial\mathbf{D}} \frac{1}{z}\,d(\tilde{\alpha}+\beta)\,.$$

Observe that if $\alpha(=\Delta u) \equiv 0$ in $\mathbf{D}(0,\varepsilon)$ then

$$|\partial u(0)| \leq \frac{1}{\varepsilon}u(0)\,.$$

Assume that $\|\tilde{\alpha}+\beta\| = 1$, and that $\tilde{\alpha} \equiv 0$ in $\mathbf{D}(0,\varepsilon)$. Let J and I be the arcs $J = \{e^{i\theta} : |\theta| \leq \frac{\pi}{6}\}$, $I = \partial\mathbf{D}\setminus J$, and let $\tilde{J}$, respectively $\tilde{I}$, be the region in $\mathbf{D}$ bounded by J and the geodesic arc connecting $e^{i\frac{\pi}{6}}$ and $e^{-i\frac{\pi}{6}}$. Now, if $|\partial u(0)| \leq \frac{\varepsilon}{4}u(0)$, then, in a manner completely analogous to the argument in 2.1, one deduces that

$$\begin{aligned} |\partial u(0)| &\geq \iint_{\tilde{I}} \Re\frac{1}{\overline{z}}\,d(\tilde{\alpha}+\beta) + \iint_{\tilde{J}} \Re\frac{1}{\overline{z}}\,d(\tilde{\alpha}+\beta) \\ &\geq \frac{1}{2}(\tilde{\alpha}+\beta)(\tilde{I}) - \frac{1}{\varepsilon}\left(1-(\tilde{\alpha}+\beta)(\tilde{I})\right) \\ &= \left(\frac{1}{2}+\frac{1}{\varepsilon}\right)(\tilde{\alpha}+\beta)(\tilde{I}) - \frac{1}{\varepsilon}\,. \end{aligned}$$

Now, for $|z| \geq \varepsilon$ one has that

$$\frac{1}{2}\left(1-|z|^2\right) \le \log\frac{1}{|z|} \le \frac{1}{2\varepsilon}\left(1-|z|^2\right) ,$$

so that

$$\|\tilde{\alpha}+\beta\| \le u(0) \le \frac{1}{\varepsilon}\|\tilde{\alpha}+\beta\| ,$$

and, consequently,

$$(\tilde{\alpha}+\beta)(\tilde{I}) \le \frac{4+\varepsilon}{4+2\varepsilon} .$$

We conclude finally that if $\alpha \equiv 0$ on $\mathbf{D}(0,\varepsilon)$ and $|\partial u(0)| \le \frac{\varepsilon}{4}u(0)$ then

$$(\tilde{\alpha}+\beta)(\tilde{J}) > \frac{\varepsilon}{4+2\varepsilon} \ge \frac{\varepsilon^2}{6}u(0) .$$

2.3 Bounds on hyperbolic derivative for bounded holomorphic functions

Let f be a function holomorphic in the unit disk $\mathbf{D}$ and $f(\mathbf{D}) \subset \mathbf{D}$. The Riesz decomposition of the positive superharmonic function $u = -\log|f|$:

$$u(z) = \iint_{\mathbf{D}} G(z,a)\,d\alpha(a) + \int_{\mathbf{T}} P(z,e^{i\theta})\,d\beta(e^{i\theta})$$

is given by

$$\alpha = \sum_{z\in\mathscr{Z}(f)} \delta_z\,; \quad \beta = \frac{1}{2\pi}\log\frac{1}{|f(e^{i\theta})|} + \sigma$$

where $\mathscr{Z}(f)$ denotes the set of zeros of f, the radial boundary value of f at $e^{i\theta}$ is denoted by $f(e^{i\theta})$, (which exists for *a.e.* $e^{i\theta}$) and σ is some positive singular measure on $\partial\mathbf{D}$.

Let $\tilde{\alpha} = \frac{1}{2}\sum_{z\in\mathscr{Z}(f)}(1-|z|^2)\delta_z$, and denote by μ_f the measure $\mu_f = \tilde{\alpha}+\beta$ (see [1]). Also let us denote $\tilde{J}_z = \varphi_z(\tilde{J})$ and $\tilde{I}_z = \varphi_z(\tilde{I})$ (see 2.1 and 2.2).

If we apply the results of Sect. 2.2 to the case at hand and express the result in a Möbius invariant form as in Sect. 2.1 we obtain:

Lemma 2.5. *With the notations above, if for each $a \in \mathbf{D}$, we let $\varepsilon(a) = \inf_{z\in\mathscr{Z}(f)}\left|\frac{z-a}{1-z\bar{a}}\right|$, then for every $a\in\mathbf{D}$ we have*

$$\frac{|f'(a)|}{|f(a)|\log\frac{1}{|f(a)|}} \le \frac{1}{\varepsilon(a)}\frac{2}{1-|a|^2} \tag{2.4}$$

and

$$\text{if } \frac{|f'(a)|}{|f(a)|\log\frac{1}{|f(a)|}} \le \frac{\varepsilon(a)}{1-|a|^2} \text{ then } \mu_f(\tilde{J}_a) \ge B\varepsilon(a)^2\log\frac{1}{|f(a)|}(1-|a|^2) \tag{2.5}$$

where B is an absolute constant.

2.4 Sharp form of Lemma 2.5, Equation (2.5)

Let f be holomorphic in $\mathbf{D}$ with $f(\mathbf{D}) \subset \mathbf{D}$, and let $a \in \mathbf{D}$ then

$$(2.6) \qquad (1-|a|^2)|f'(a)| \le 2|f(a)| \left(\sum_{b\in\mathscr{Z}(f)} \omega\left(\log\left|\frac{1-a\overline{b}}{a-b}\right|\right) + \log\frac{1}{|f(a)|}\right)$$

where ω is the positive function $\omega(x) = \operatorname{senh} x - x$, $x \ge 0$.

To see this we may assume that $a = 0$, and $f(0) \ne 0$ and write f as the product $f = B \cdot g$ where B is a Blaschke product formed with the zeros of f, and g is a zero-free holomorphic function into $\mathbf{D}$. The inequality follows from observing that $\frac{f'(0)}{f(0)} = \frac{B'(0)}{B(0)} + \frac{g'(0)}{g(0)}$, calculating the first summand explicitly, estimating the second summand by means of Lemma 2.1 and applying the triangle inequality. The inequality (2.6) is sharp.

Observe that if $x \le \log\frac{1}{\varepsilon}$ then $\omega(x) \le \left(\frac{1}{\varepsilon} - 1\right)x$, and also that $|f(0)| \le \prod_{b\in\mathscr{Z}(f)}|b|$; then deduce (2.4) as a corollary of (2.6).

Consider, for $\alpha \in (0,1)$, $\varepsilon \in (0,1)$, the class $\mathscr{F}_{\alpha,\varepsilon}$ of functions holomorphic in $\mathbf{D}$ with $f(\mathbf{D}) \subset \mathbf{D}$, $f(0) = \alpha$ and with no zeros in $\mathbf{D}(0,\varepsilon)$.

Let m be the non negative integer such that $\varepsilon^{m+1} < \alpha \le \varepsilon^m$ then taking into account the elementary Lemma 2.6 below one readily sees that

$$\sup_{f\in\mathscr{F}_{\alpha,\varepsilon}} \frac{|f'(0)|}{|f(0)|} \le 2m\,\omega(\log\frac{1}{\varepsilon}) + 2\omega(\log\frac{1}{\delta}) + 2\log\frac{1}{|f(0)|}$$

where δ is such that $\varepsilon^m\delta = \alpha = |f(0)|$. The bound is actually attained by the function $f(z) = \left(\frac{z+\delta}{1+z\delta}\right)\left(\frac{z+\varepsilon}{1+z\varepsilon}\right)^m$. And therefore

$$\sup_{f\in\mathscr{F}_{\alpha,\varepsilon}} \frac{|f'(0)|}{|f(0)|} = m\left(\frac{1}{\varepsilon} - \varepsilon\right) + \left(\frac{\varepsilon^m}{\alpha} - \frac{\alpha}{\varepsilon^m}\right)$$

This is a result of Royden, [13]. The original proof of Royden is also elementary, but based on a certain recursive scheme.

Lemma 2.6. *Let ω be a positive increasing convex function defined in $[0,\infty)$, with $\omega(0) = 0$. Then given $A \ge 0$, and $B \ge 0$ one has*

$$\begin{aligned}\sup\{\sum_{j=1}^N \omega(x_j) :\ 0 \le x_j \le A,\ \sum_{j=1}^N x_j \le B,\ N \in \mathbf{N}\} \\ = \left\lfloor\frac{B}{A}\right\rfloor\omega(A) + \omega\left(B - A\left\lfloor\frac{B}{A}\right\rfloor\right).\end{aligned}$$

3 Proof of Theorem 3

3.1

The proof of Theorem 3 is based on the following observation. Assume that f omits the origin and another point $b \in \mathbf{D}$. Whenever $f(z)$ is close to b, the hyperbolic derivative of f at z must be small and by Lemma 2.4, it follows that $\mu_f(J_z) > B \log \frac{1}{|b|}|J_z|$. Consider subarcs J_n of J_z such that $\frac{\mu_f(J_n)}{|J_n|}$ is big. An elementary estimate, see Sect. 2.1, shows that $|f(z_{J_n})|$ is small, that is, close to the origin, where z_{J_n} is the point in the unit disc satisfying

$$J_{z_{J_n}} = J_n$$

Now one can repeat the argument interchanging the roles of 0 and b. So, by induction one gets a Cantor type set and evaluates its dimension using the following result (see [11, p. 226]) due to Hungerford and Makarov. This we describe now.

Lemma C. *Let $0 < \varepsilon < C < 1$ be fixed constants. Let $E_k = \bigcup_n J_n^{(k)}$, where $\{J_n^{(k)} : n = 1, 2, \ldots\}$ are pairwise disjoint arcs of the unit circle with the following properties:*

$$E_{k+1} \subset E_k, \; k = 1, 2, \ldots \tag{3.1}$$

$$|J_n^{(k+1)}| < \varepsilon |J_m^{(k)}| \;\; \textit{if} \;\; J_n^{(k+1)} \subset J_m^{(k)} \tag{3.2}$$

$$\sum_{J_n^{(k+1)} \subset J_m^{(k)}} |J_n^{(k+1)}| \geq C\,|J_m^{(k)}| \tag{3.3}$$

Then

$$\dim(\bigcap_k E_k) \geq 1 - \frac{\log C}{\log \varepsilon}\,. \tag{3.4}$$

We shall also need the following direct generalization of the lemma above.

Lemma D. *Let I be an interval. Assume that we are given two sequences of positive real numbers $\{M_j\}_j$ and $\{\delta_j\}_j$ satisfying $M_j \in (0, 1)$ and $\lim_{j\to 0} \delta_j = 0$, and also a fixed constant c, $0 < c < 1$.*

Then there exists a sequence of positive integers $\{t_j\}_j$, with $\lim_{j\to\infty} t_j = +\infty$, which depends only on the sequences $\{M_j\}, \{\delta_j\}$ and the constant c, such that any set E constructed as below has dimension 1:

$$\dim(E) = 1\,.$$

We now describe the rules for the *construction of the sets E*. We need first a notation to denote a certain procedure to associate to an interval H (and parameters δ and t) a collection of disjoint subintervals. It resembles t steps of the construction of a regular Cantor set, but it is much more flexible.

Consider an interval H, a positive integer t and a positive real number δ. The starting generation, 0^{th} generation, consists only of the interval H. The n^{th}

generation consists of subintervals of each of the intervals J of the previous generation, the $(n-1)^{\text{th}}$ generation. The subintervals $\{J_l\}_l$ of J must satisfy only the following two conditions:

$$\begin{aligned} \sum_l |J_l| &\geq c|J| \\ |J_l| &\leq \delta|J|, \text{ for each } l \end{aligned}$$

We stop at the generation t: the collection of the subintervals of H of that generation is denoted by $\mathscr{G}(H, \delta, t)$.

The construction of the admissible subsets E of I proceeds also in a sequence of steps.

– (1) First, one chooses any finite collection of disjoint subintervals J_j^1 of I satisfying

$$\sum_j |J_j^1| \geq M_1|I|\,.$$

Then, to each J_j^1 one assigns a collection of subintervals $\mathscr{G}(J_j^1, \delta_1, t_1)$. This concludes the first step. The union of those intervals obtained in this way is denoted by E_1.

– (2) Once the intervals in E_n have been obtained the intervals of the next step are obtained as follows. First, for any of the intervals I' in E_n one chooses a collection of subintervals J_j' satisfying

$$\sum_j |J_j'| \geq M_{n+1}|I'|\,.$$

Then, to each of these J_j' one assigns a collection of subintervals $\mathscr{G}(J_j', \delta_{n+1}, t_{n+1})$.

This gives the intervals of the $n+1$ step, whose union we denote by E_{n+1}. Clearly, $E_{n+1} \subset E_n$. The set E is

$$E = \cap_n E_n$$

It should be remarked that a naive extension of Lemma C without the t_j's does not hold. It is easy to provide examples.

3.2

Let J be an arc of the unit circle. For $n = 1, 2, \ldots$, consider the collection $\mathscr{D}_n$ of the 2^n disjoint subarcs of J of length $2^{-n}|J|$. A *dyadic subarc* of J is an arc belonging to $\mathscr{D}_n$ for some n. We will use the following elementary covering lemma.

Lemma 3.1 *Let J be an arc of the unit circle. Let $\{J_k\}$ be pairwise disjoint dyadic subarcs of J. Assume*

$$\sum_k |J_k| \geq C|J|$$

Then there exists a subfamily $\{W_k\}$ of $\{J_k\}$ satisfying

$$\sum_k |W_k| \geq \frac{C}{2}|J|$$

and

$$\sum_{J_k \subset L} |J_k| \geq \frac{C}{2}|L|$$

if L is a dyadic subarc of J containing some W_k.

Proof. Let $\mathscr{A}$ be the set of maximal dyadic subarcs L of J satisfying the following properties:

1. L contains some J_k.

2. $\sum_{J_k \subset L} |J_k| < \frac{C}{2}|L|$.

Consider $\mathscr{B} = \{J_k : J_k \subset L \text{ for some } L \in \mathscr{A}\}$ and $\{W_k\} = \{J_k : J_k \notin \mathscr{B}\}$. Then

$$\sum_{J_k \in \mathscr{B}} |J_k| \leq \frac{C}{2} \sum_{L \in \mathscr{A}} |L| \leq \frac{C}{2}|J|\,.$$

Hence,

$$\sum |W_k| \geq \frac{C}{2}|J|\,.$$

If L is a dyadic subarc of J containing some W_k, then $L \notin \mathscr{A}$. So,

$$\sum_{J_k \subset L} |J_k| \geq \frac{C}{2}|L|\,.$$

3.3

The main auxiliary result in the proof of Theorem 3 is the following

Lemma 3.2. *Let f be an inner function omitting the values 0 and b, where $0 < |b| < 1/100$. There exist two positive constants c_1, c_2, $0 < c_1 < 1$, independent of f and b, such that whenever*

$$|f(z) - b| < \frac{1}{2}|b|$$

there exists a collection $\{J_n\}$ of disjoint subarcs of J_z such that

$$c_1|J_z| < \sum_n |J_n| \tag{3.5}$$

$$|J_n| < \frac{1}{|\log|b||^{c_2}}\,|J_z|, \qquad n = 1, 2, \ldots \tag{3.6}$$

$$|f(z_{J_n}) - b| < \frac{|b|}{|\log|b||^{c_2}}, \qquad n = 1, 2, \ldots \tag{3.7}$$

Proof. In this proof C denotes several positive absolute constants. We may assume that $0 < b < 1$.

Since f omits 0 and b, $f = \exp(-F)$ where F is an inner function into the right half plane omitting the values $\log b^{-1} + 2n\pi i$, $n \in \mathbf{Z}$. Let T be the conformal mapping from the right half plane $\mathbf{H}$ onto the unit disc $\mathbf{D}$ given by

$$T(w) = \frac{w - \log b^{-1}}{w + \log b^{-1}}$$

Then $G = T \circ F$ is an inner function omitting the values

$$b_n = T(\log b^{-1} + 2n\pi i) = \frac{n\pi i}{\log|b|^{-1} + n\pi i}\ , \qquad n \in \mathbf{Z}\,.$$

Let

$$G_n = \frac{G - b_n}{1 - \overline{b_n}G}, \qquad \mu_n = \mu_{G_n}\,.$$

Since T preserves the hyperbolic metric, one has

$$\begin{aligned} \tanh \mathrm{d}_{\mathbf{D}}(G(z), 0) &= \tanh \mathrm{d}_{\mathbf{H}}(F(z), \log b^{-1}) \\ &\le \frac{|F(z) - \log b^{-1}|}{|\log|b||} \\ &\le \frac{C}{|\log|b||}\,. \end{aligned} \tag{3.8}$$

(Recall that d_Ω denotes the Poincaré distance of the domain Ω.) Fix $m = \left\lfloor \left(\log|b|^{-1}\right)^{1/2} \right\rfloor$ and observe that

$$\frac{1}{2}(\log|b|^{-1})^{-1/2} \le |b_m| \le (\log|b|^{-1})^{-1/2}\,.$$

Using Lemma 2.1 and (3.8) one deduces

$$\begin{aligned} (1 - |z|^2)|G_m'(z)| &\le 2(1 - |z|^2)|G'(z)| \\ &\le 4|G(z)|\log\frac{1}{|G(z)|} \\ &< \frac{1}{2}|G_m(z)|\log\frac{1}{|G_m(z)|}\,. \end{aligned}$$

Hence by Lemma 2.4

$$\frac{\mu_m(J_z)}{|J_z|} > B\log\log\frac{1}{|b|}\,.$$

Now, let $\{J_n\}$ be the maximal dyadic subarcs of J_z satisfying

$$\frac{\mu_m(J_n)}{|J_n|} > 16 \log \frac{1}{|b_m|} \left(> 8 \log\log \frac{1}{|b|} \right) .$$

It follows from the maximality that

$$\frac{\mu_m(J_n)}{|J_n|} < 32 \log \frac{1}{|b_m|}$$

and thus

$$\sum |J_n| > \frac{1}{32} \left(\frac{1}{|\log |b_m||} \right) \mu_m(J_z) > C |J_z| ,$$

which is (3.5). Also, let $\tilde{z} = z_{J_n}$, then

$$\begin{aligned} \log \frac{1}{|G_m(\tilde{z})|} &= \int_0^{2\pi} \frac{1 - |\tilde{z}|^2}{|e^{i\theta} - \tilde{z}|^2} d\mu_m(e^{i\theta}) \\ &\geq C \frac{\mu_m(J_n)}{|J_n|} > C \log\log \frac{1}{|b|} \end{aligned}$$

and a calculation, similar to (3.8), gives (3.7).

Let $G = \exp(-H)$, where H is an inner function into the right half plane omitting the set of values $\mathscr{L} = \{\log b_n^{-1} + 2k\pi i \, : \, n, k \in \mathbf{Z}\}$.

A calculation shows that

$$\sup_{n>0} \left| \log \frac{1}{|b_n|} - \log \frac{1}{|b_{n+1}|} \right|$$

is bounded by an absolute constant independent of b. Let

$$\Omega = \left\{ w \in \mathbf{C} : 0 < \Re(w) < \frac{1}{2} \log \frac{1}{|b|} \right\} \setminus \mathscr{L} .$$

A simple estimate of the density of the Poincaré metric of Ω gives that if $p, q \in \Omega$ then

$$d_\Omega(p, q) \geq C |\Re(p) - \Re(q)|$$

Since

$$\log \left(\frac{1}{2} \log \frac{1}{|b|} \right) < \Re H(z)$$

$$\left| \Re H(\tilde{z}) - \log \frac{1}{|b_m|} \right| < C \left(\log \frac{1}{|b|} \right)^{-C} ,$$

one deduces that

$$C \log\log \frac{1}{|b|} \leq |\Re H(z) - \Re H(\tilde{z})| \leq d_{\mathbf{D}}(z, \tilde{z})$$

and (3.6) follows.

3.4 Proof of Theorem 3

We shall distinguish two cases. Recall that

$$d_f(0) = \inf\{d(z,0) : z \in \mathscr{O}(f) \setminus \{0\}\}\,.$$

3.4.1. First we assume that $d_f(0) > 0$. Let b, $0 < |b| < 1/100$, another value omitted by f (thus we are taking m, such that $\log \frac{1+m}{1-m} = \frac{1}{100}$). Consider $z \in \mathbf{D}$ such that

$$|f(z) - b| < \frac{1}{2}|b| \tag{3.9}$$

and $J^{(1)} = J_z$. Apply Lemma 3.2 to get subarcs J_n satisfying (3.5), (3.6), (3.7) and consider the subcollection $\{J_n^{(2)}\}$ given by Lemma 3.1 which still satisfies the conditions above (with c_1 replaced by $c_1/2$) and also

$$\sum_{J_n^{(2)} \subset L} |J_n^{(2)}| \geq 2^{-1}c_1|L| \tag{3.10}$$

for any dyadic arc L of $J^{(1)}$ containing some $J_n^{(2)}$. Now using (3.7) instead of (3.9) one can repeat the process in each $J_n^{(2)}$. So by induction, one gets $E = \bigcap_k E_k$, $E_k = \bigcup_n J_n^{(k)}$ satisfying (3.1), (3.2), (3.3) of Lemma C with the constants

$$C = 2^{-1}c_1, \qquad \varepsilon = (\log|b|^{-1})^{-c_2}$$

So, $\dim E \geq 1 - \frac{M}{\log\log\frac{1}{|b|}}$, where M is an absolute constant.

Let $\xi \in E$ and L be a dyadic subarc of J_z which contains ξ. Choose m such that

$$J_{k(m)}^{(m)} \subset L \subset J_{k(m-1)}^{(m-1)}$$

for some $k(m)$, and $k(m-1)$, and one has

$$\sum_{n, J_n^m \subset L} |J_n| \geq 2^{-1}c_1|L|$$

and

$$|f(z_{J_n}) - b| < |b|(\log|b|^{-1})^{-c_2}\,.$$

So, a harmonic measure estimate in the domain $\mathbf{D}\setminus\cup_n\mathbf{D}\left(z_{J_n}, \frac{1}{2}(1-|z_{J_n}|)\right)$ shows that

$$\log\frac{1}{|f(z_L)|} \geq K \log\frac{1}{|b|}\,,$$

where K is an absolute constant. Hence

$$\limsup_{r\to 1} |f(r\xi)| < |b|^K\,.$$

3.4.2. Assume now $d_f(0) = 0$. In other words, f omits a sequence of points b_n tending to the origin. One can assume that b_n tends to 0 as fast as necessary.

We shall use Lemma D with the following specifications for the parameters

$$M_j = \frac{B}{16} \frac{\log |b_{j-1}|^{-1}}{\log |b_j|^{-1}} \qquad j = 2, \dots ,$$

also

$$\delta_j = (\log |b_j|^{-1})^{-c_2}$$

and

$$c = c_1$$

where c_1, c_2 are as in Lemma 3.2. With these data Lemma D provides us with a sequence of positive integers t_j.

Let $f = \exp(-F)$,

$$f_k = \frac{f - \overline{b_k}}{1 - b_k f}, \qquad k = 1, 2, \dots$$

and $\mu_k = \mu_{f_k}$ be the measure corresponding to f_k.

Consider $z \in \mathbf{D}$ such that

$$|f_1(z)| \le |b_1|(\log |b_1|^{-1})^{-c_2} \tag{3.11}$$

and $J^{(1)} = J_z$. As in the proof of Lemma 3.2 one has

$$(1 - |z|^2)|f_2'(z)| < \frac{1}{2}|f_2(z)| \log \frac{1}{|f_2(z)|}$$

and Lemma 2.4 gives

$$\frac{\mu_2(J^{(1)})}{|J^{(1)}|} > B \log \frac{1}{|b_1|}$$

Now choose J_k the maximal dyadic subarcs of $J^{(1)}$ satisfying

$$\frac{\mu_2(J_k)}{|J_k|} > 8 \log |b_2|^{-1} .$$

As in the proof of Lemma 3.2, the maximality implies

$$\frac{\mu_2(J_k)}{|J_k|} < 16 \log |b_2|^{-1}$$

and

$$\sum_k |J_k| > \frac{B}{16} \frac{\log |b_1|^{-1}}{\log |b_2|^{-1}} |J^{(1)}|$$

Moreover, for $w = z_{J_k}$, one has that

$$\log \frac{1}{|f_2(w)|} > 2 \log \frac{1}{|b_2|}$$

and $|f(w) - b_2| < \frac{1}{2}|b_2|$. Now, apply in each J_k Lemma 3.2 t_2 times to get arcs $\{J_n\}$ satisfying

$$\sum |J_n| \geq c_1^{t_2} \sum |J_k| \geq \frac{B}{16} \; \frac{c_1^{t_2} \log |b_1|^{-1}}{\log |b_2|^{-1}} \; |J^{(1)}| \tag{3.12}$$

$$|J_n| < (\log |b_2|^{-1})^{-c_2 t_2} |J^{(1)}|, \qquad n = 1, 2, \dots \tag{3.13}$$

$$|f_2(z_{J_n})| < |b_2| (\log |b_2|^{-1})^{-c_2}, \qquad n = 1, 2, \dots \tag{3.14}$$

Applying Lemma 3.1 one gets a subcollection $\{J_n^{(2)}\}$ satisfying (3.12) with a half of the constant and such that

$$\sum_{J_n \subset L} |J_n| > \frac{B}{32} \frac{c_1^{t_2} \log |b_1|^{-1}}{\log |b_2|^{-1}} |J^{(1)}|,$$

where L is any dyadic subarc of $J^{(1)}$ which contains some $J_n^{(2)}$. So, using (3.14), a harmonic measure argument gives

$$\log \frac{1}{|f_2(z_L)|} \geq C \log \frac{1}{|b_2|}$$

where C is an absolute constant. So, $|f(z_L)|$ is small.

Now, one can repeat the process using (3.14) instead of (3.11) as starting point. By induction, one gets $E_k = \bigcup_n J_n^{(k)}$ satisfying the conditions of Lemma D.

So, $\dim(\bigcap_k E_k) = 1$.

Moreover

$$\log \frac{1}{|f_k(z_L)|} \geq C \log \frac{1}{|b_k|},$$

where C is an absolute constant and L is a dyadic subarc of some $J_n^{(k)}$ which contains some arc from E_{k+1}. Hence, if $\xi \in \bigcap_k E_k$, one has

$$\lim_{r \to 1} f(r\xi) = 0,$$

4 Complements and remarks

4.1

Our results admit local versions. For instance,

Theorem 2′. *Let f be an inner function. Let $b \in \mathbf{D}$ be an accumulation point of $\mathscr{O}(f)$ and let I be an arc with $\mathscr{S}(f) \cap I \neq \emptyset$ then*

$$\dim\{\theta \in I : \lim_{r \to 1} f(re^{i\theta}) = b\} = 1\,.$$

Here $\mathscr{S}(f)$ denotes the singular set: θ belongs to $\mathscr{S}(f)$ if f can not be extended to be analytic in a neighborhood of $e^{i\theta}$.

Observe that if $e^{i\theta} \in \mathscr{S}(f)$, then there exists a sequence $z_n \in \mathbf{D}$ such that $z_n \to e^{i\theta}$ while $f(z_n) \to b$; we just have to start the argument of the proof of Theorem 2 with an appropiate z_n.

4.2

One could expect that if instead of a sequence of omitted values we have a sequence of values which are not taken "too much" then similar results hold, see e.g. [14]. The following result points in that direction. For a non constant holomorphic function f in the unit disk we denote by $\mu_{f,b}$ the measure in $\mathbf{D}$ given by $\mu_{f,b} = \sum_{\{z:f(z)=b\}}(1-|z|^2)\ \delta_z$. The Carleson norm of a measure μ in $\mathbf{D}$ is denoted by $\|\mu\|_{\text{Carleson}}$.

Theorem 5. *Let f be an inner function which omits* 0*. Assume that for a sequence $b_n \in \mathbf{D}$ with $\lim_{n\to\infty} b_n = 0$ we have*

$$\sup_n \|\mu_{f,b_n}\|_{Carleson} < +\infty\,.$$

Then

$$\dim\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = 0\} = 1\,.$$

4.3

There exists a well known parallelism between "small derivative" and "omitted values". Compare, e.g., Theorem 2 with the result of Rohde mentioned in 0.5. A result combining features of both type of hypothesis is the following.

Theorem 6. *Let f be an inner function which omits* 0*. Define*

$$\lambda = \limsup_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{|f(z)|\log\frac{1}{|f(z)|}}$$

Then

$$\dim\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = 0\} = 1 - O(\lambda)$$

The proofs of Theorems 5 and 6 are obtained by an elaboration of the previous proofs and are not presented here.

4.4

There is a beautiful refinement of Fatou's theorem due to Bourgain, [Bo]: if f is a bounded holomorphic function in $\mathbf{D}$, then

$$\dim\{\theta \in [0, 2\pi] : V_f(\theta) < +\infty\} = 1\,,$$

where $V_f(\theta)$ is the variation of f in the radius ending at $e^{i\theta}$, *i.e.*,

$$V_f(\theta) = \int_0^1 |f'(re^{i\theta})| dr .$$

Walter Hayman has asked if there are refinements of Theorem 1 and 2 along the lines of Bourgain's result. In particular, if f, and b are as in Theorem 2, is it true that

$$\dim\{\theta \in [0, 2\pi] : V_f(\theta) < +\infty \text{ and } \lim_{r\to 1} f(re^{i\theta}) = b\} = 1\,?$$

4.5

Our results concern the dimension of the Fatou set, and they are sharp in that respect. But more precise results are possible. Consider, for instance, the situation of Theorem 2. If b is approximated at a <u>certain given rate</u> by the other points of $\mathscr{O}(f)$, then some appropiate h-measure of the set $\{\theta \in [0, 2\pi] : \lim_{r\to 1} f(re^{i\theta}) = b\}$ should be positive.

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