INNER FUNCTIONS, BLOCH SPACES AND SYMMETRIC MEASURES

A. B. ALEKSANDROV, J. M. ANDERSON *and* A. NICOLAU

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1. *Introduction*

Let H^∞ denote the algebra of bounded analytic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The well-known Schwarz–Pick theorem asserts that if $f \in H^\infty$ with

$$\|f\|_\infty = \sup\{|f(z)|: z \in \mathbb{D}\} \leqslant 1$$

then f decreases hyperbolic distances; that is,

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leqslant \left|\frac{z - a}{1 - \bar{a}z}\right|$$

for all $z, a \in \mathbb{D}$, or, infinitesimally,

$$(1 - |z|^2)|f'(z)| \leqslant 1 - |f(z)|^2 \quad \text{for } z \in \mathbb{D}.$$

A function $I \in H^\infty$ is called *inner* if it has radial limits of modulus 1 at almost every point of the unit circle $\mathbb{T}$. If $E \subset \mathbb{T}$ then $|E|$ denotes its normalized Lebesgue measure. We introduce several measures on $\mathbb{T}$, but the expression 'almost every' always refers to Lebesgue measure. We assume a knowledge of inner functions, such as is to be found in [**9**]. In particular, we may write I as $I = BS$ where

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|}\left(\frac{z_n - z}{1 - \overline{z_n}z}\right)$$

is the Blaschke product associated with the zero set $\{z_n\}$ of I, and

$$S = S[\mu](z) = \exp\left\{-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right\}$$

is the singular inner factor associated with the positive singular measure μ.

The first result of this paper is the construction of an inner function I which, in some sense, decreases hyperbolic distances as much as desired as $|z| \to 1$.

THEOREM 1. *Let* ϕ: $(0,1] \to (0,\infty)$ *be a continuous function with*

$$\lim_{t \to 0} \phi(t) = 0.$$

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Then there exists an inner function I such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{\phi(1-|I(z)|^2)} = 0.$$

We apply this theorem to prove some results on composition operators, Zygmund functions and the existence of certain singular measures.

Recall that a function f, analytic in $\mathbb{D}$, is called a *Bloch function* if the quantity

$$\|f\|_{\mathscr{B}} = \sup\{(1-|z|^2)|f'(z)|: z \in \mathbb{D}\}$$

is finite. The Banach space of all such functions is the Bloch space, denoted by $\mathscr{B}$ with $|f(0)| + \|f\|_{\mathscr{B}}$ as norm. The *little Bloch space* $\mathscr{B}_0$ is the subspace of $\mathscr{B}$ consisting of those $f \in \mathscr{B}$ for which

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

The *Zygmund class* $\Lambda^* = \Lambda^*(\mathbb{T})$ is the space of continuous functions F on $\mathbb{T}$ for which

$$\sup\{|F(e^{i(\theta+h)}) + F(e^{i(\theta-h)}) - 2F(e^{i\theta})|: e^{i\theta} \in \mathbb{T}\} \leqslant K|h|$$

for some constant K. When the quantity above is $o(|h|)$ as $h \to 0$ we say that *F is in the small Zygmund class* $\lambda^*(\mathbb{T})$. Roughly speaking, Zygmund functions are the primitives of functions in the Bloch space, namely an analytic function f is in $\mathscr{B}$ if and only if

$$F(z) = \int_0^z f(t)\,dt$$

belongs to $\Lambda^*(\mathbb{T})$ for $|z| = 1$. Analogous relations hold between $\mathscr{B}_0$ and λ^* (see [**18**] for details).

Some consequences of Theorem 1 are as follows. Given a positive continuous function w: $[0,1) \to (0,+\infty)$ with

$$\lim_{t\to 1^-} w(t) = +\infty,$$

let $H(w)$ denote the Banach space of functions f, analytic in $\mathbb{D}$ such that

$$\|f\|_w = \sup\{|f(z)|w(|z|)^{-1}: z \in \mathbb{D}\} < \infty.$$

COROLLARY 1. *Let w be as above and $\varepsilon > 0$ be given. Then there exists a non-constant inner function I such that the composition operator $C(I)$, defined as*

$$C(I)(f) = f \circ I$$

maps $H(w)$ into $\mathscr{B}_0$. Moreover $C(I)$ is compact with $\|C(I)\| < \varepsilon$.

The argument leading from Theorem 1 to this corollary is very flexible and may be applied to obtain other results of a similar type. One such result is the following.

COROLLARY 2. *Given any sequence $\{f_n\}$ of analytic functions in $\mathbb{D}$, there exists an inner function I such that $f_n \circ I \in \mathscr{B}_0$ for $n = 1, 2, 3, \ldots$.*

Another application of Theorem 1 is as follows.

COROLLARY 3. *Let I be a non-constant inner function satisfying*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{(1-|I(z)|^2)^2} = 0$$

(*that is, as in Theorem* 1 *with* $\phi(t) = t^2$). *Let J be a measurable subset of* $\mathbb{T}$ *and set*

$$E = I^{-1}(J).$$

Then the function

$$F(x) = \int_0^x \chi_E(e^{it})\, dt$$

belongs to $\lambda^*(\mathbb{R})$.

Löwner's lemma asserts, with the above notation, that $|E| = |J|$ whenever $I(0) = 0$ and so, for any inner function I, $0 < |E| < 1$ if $0 < |J| < 1$. The conclusion of Corollary 3 was considered in [**12**] where it was shown that if $F \in \lambda^*(\mathbb{R})$ then $|E| = 0$ or $|E| = 1$ or $\dim(\partial E) = 1$. Thus, if I is as in Corollary 3, the boundary of the pre-image by I of any Borel set of positive measure has Hausdorff dimension 1. In this sense, the inner function I has very wild behaviour.

The proof of Theorem 1 follows from the following two theorems.

THEOREM 2. *Let ϕ: $(0,1] \to (0,\infty)$ be a continuous function, $\phi(0^+) = 0$. Then there exists an interpolating Blaschke product B such that*

$$(1-|z|^2)|B'(z)| \leqslant \phi(1-|B(z)|^2)$$

for all $z \in \mathbb{D}$.

Recall that a Blaschke product is called *interpolating* if

$$\inf_n (1-|z_n|^2)|B'(z_n)| > 0,$$

where $\{z_n\}$ is the zero sequence of B. Such a function cannot belong to $\mathscr{B}_0$ except when it has a finite number of zeros.

The function B in Theorem 2 will in fact be a covering map. Theorem 2 permits us to establish Corollaries 1 and 2 with $\mathscr{B}_0$ replaced by $\mathscr{B}$, but with the extra conclusion that the corresponding inner function is an interpolating Blaschke product.

Functions in $\mathscr{B}_0$ map hyperbolic discs of a fixed diameter into euclidean discs of diameter tending to 0 as one approaches $\mathbb{T} = \partial\mathbb{D}$. The second step of our construction concerns inner functions which map hyperbolic discs of a fixed diameter into hyperbolic discs of diameter tending to 0 as one approaches $\mathbb{T}$.

THEOREM 3. *There exists a non-constant inner function I for which*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0. \tag{1.1}$$

Such an inner function I cannot extend analytically to any point of $\mathbb{T}$. Indeed, if I has an angular derivative at the point $\xi \in \mathbb{T}$, that is, if the quotient

$$\frac{I(z) - I(\xi)}{z - \xi}$$

has a limit when z approaches ξ non-tangentially, then the Julia–Carathéodory lemma asserts that

$$\liminf_{z \to \xi} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} > 0.$$

Moreover, although the inner functions of Theorem 3 are in $\mathscr{B}_0$, they form a strict subclass of $\mathscr{B}_0$, because there exist inner functions in $\mathscr{B}_0$ which can be extended analytically to almost every point of $\mathbb{T}$ (see for example, [**9**]). Inner functions in $\mathscr{B}_0$ have been considered by Bishop in [**3**] and we use some of his ideas.

It is worth mentioning also that the condition (1.1) in Theorem 3 has appeared in [**14**] in connection with composition operators from $\mathscr{B}_0$ into itself. Indeed, Theorem 3 answers a question in [**14**, p. 2686] as to whether there is a function ϕ in $\mathscr{B}_0$ with $C(\phi)$ compact as an operator from $\mathscr{B}_0$ to $\mathscr{B}_0$ such that $\overline{\phi(\mathbb{D})} \cap \mathbb{T}$ is infinite. We may take $\phi(z)$ to be the inner function $I(z)$ of Theorem 3 for which $\overline{\phi(\mathbb{D})} = \overline{\mathbb{D}}$. Also, the completely opposite situation has been considered in [**10**].

Now suppose that $f \in H^\infty$, with $\|f\|_\infty \leqslant 1$. For $\alpha \in \mathbb{T}$ the functions

$$H_\alpha(z) = \frac{\alpha + f(z)}{\alpha - f(z)} \tag{1.2}$$

have positive real part. Hence there exist positive measures σ_α on $\mathbb{T}$ such that the Herglotz representation

$$\operatorname{Re} H_\alpha(z) = \int_{\mathbb{T}} P(z, \xi)\, d\sigma_\alpha(\xi)$$

holds for all $z \in \mathbb{D}$. Here,

$$P(z, \xi) = (1 - |z|^2)|1 - \bar{\xi} z|^{-2}$$

denotes the Poisson kernel. It is well known (and easy to prove) that the measure σ_α is singular for some $\alpha \in \mathbb{T}$ if and only if f is inner. Moreover if f and H_α are related by (1.2) then

$$\lim_{|z| \to 1^-} \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2} = 0$$

if and only if

$$\lim_{|z| \to 1} \frac{(1-|z|^2)|H'_\alpha(z)|}{\operatorname{Re} H_\alpha(z)} = 0. \tag{1.3}$$

So to prove Theorem 3 it is sufficient to construct a singular measure σ such that its Herglotz transform H satisfies (1.3).

To avoid endless repetition, J and J' will henceforth, and throughout the paper, denote adjacent arcs of $\mathbb{T}$ with $|J| = |J'|$.

With this notation we have the following.

THEOREM 4. *Let H be analytic in $\mathbb{D}$ with* $\operatorname{Re} H(z) > 0$ *for $z \in \mathbb{D}$. Let σ be the corresponding measure on $\mathbb{T}$ for which*

$$\operatorname{Re} H(z) = \int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi).$$

The following statements are equivalent:

(a) $$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|H'(z)|}{\operatorname{Re} H(z)} = 0;$$

(b) $$\lim_{|J|\to 0} \frac{\sigma(J)}{\sigma(J')} = 1.$$

Positive measures satisfying (b) are called symmetric (see [**8**]). Thus, to prove Theorem 3 it is sufficient to exhibit a positive singular symmetric measure. In fact, such measures were constructed by L. Carleson in [**5**] in connection with quasiconformal mappings. It is also possible to prove Theorem 3 using a construction of C. Bishop and the following result.

THEOREM 5. *Given an inner function I, consider the positive measure in* $\mathbb{D} \cup \mathbb{T}$,

$$\mu = \sum_{z:\, I(z)=0} (1-|z|^2)\delta_z + 2\sigma,$$

where δ_z *denotes the Dirac mass at z, the sum takes into account the multiplicity of the zeros of I, and* σ *is the measure associated with the singular part of I. The following assertions are equivalent:*

(a) $$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0;$$

(b) *for any* $\varepsilon > 0$ *the following two conditions hold:*

(1.b) $$\lim_{\delta\to 0} \sup_{|Q|<\delta} \left\{ \left| \frac{\mu(Q)}{\mu(Q')} - 1 \right| :\ \frac{\mu(Q)}{|Q|} < \frac{1}{\varepsilon} \right\} = 0;$$

(2.b) $$\lim_{N\to\infty} \sup_{Q} \left\{ \sum_{k=N}^{\infty} \frac{\mu(2^k Q \backslash 2^{k-1} Q)}{2^{2k}\mu(Q)} :\ \frac{\mu(Q)}{|Q|} < \frac{1}{\varepsilon} \right\} = 0.$$

Here Q denotes the Carleson square

$$Q = \{z:\ z = re^{i\theta},\ \theta \in J,\ 1-|J| \leqslant |z| < 1\}$$

corresponding to an interval $J \subset \mathbb{T}$, $|Q| = |J|$ and Q' is the corresponding Carleson square for J'.

As mentioned above, L. Carleson constructed singular symmetric measures. Indeed, let $w(t)$ be a continuous increasing function on $[0, 1]$, with $w(0) = 0$, such that $t^{-1/2}w(t)$ is decreasing. Let σ be a positive measure on $\mathbb{T}$ such that

$$|\sigma(J) - \sigma(J')| \leqslant w(|J|)\sigma(J),$$

for any arc J of the unit circle. L. Carleson showed that the condition

$$\int_0 \frac{w^2(t)}{t}\, dt < \infty$$

implies that σ is absolutely continuous and in fact, its derivative is in L^2. Conversely, if

$$\int_0 \frac{w^2(t)}{t}\, dt = \infty,$$

there exists a positive singular measure on $\mathbb{T}$ such that

$$|\sigma(J) - \sigma(J')| \leqslant w(|J|)\sigma(J),$$

for any arc J of the unit circle.

A similar situation occurs when looking for the best decay one can have in Schwarz's Lemma. Given a positive increasing function w on $(0,1]$, consider

$$\widetilde{w}(t) = t\int_t^1 \frac{w(s)}{s^2}\,ds + tw(1) \quad \text{for } t \in (0,1]. \tag{1.4}$$

Observe that $\widetilde{w}(t) \geqslant w(t)$ for $0 < t < 1$, and $\widetilde{w}(t) \leqslant c(\varepsilon)w(t)$ if $w(t)/t^{1-\varepsilon}$ is decreasing for some $\varepsilon > 0$.

THEOREM 6. *Let w be a positive continuous function on $(0,1]$.*

(a) *Assume that*

$$\int_0 \frac{w^2(t)}{t}\,dt < \infty.$$

Then there is no non-constant inner function I such that

$$(1-|z|^2)\,\frac{|I'(z)|}{1-|I(z)|^2} \leqslant w(1-|z|),$$

for all $z \in \mathbb{D}$.

(b) *Let w be increasing. Assume that there exist constants k and δ such that*

$$\widetilde{w}(t) \leqslant kw(t) \quad \text{if } 0 < t < \delta,$$

and

$$\int_0 \frac{w^2(t)}{t}\,dt = \infty.$$

Then, there exists a non-constant inner function such that

$$(1-|z|^2)\,\frac{|I'(z)|}{1-|I(z)|^2} \leqslant Cw(1-|z|) \quad \text{for } z \in \mathbb{D},$$

where C is an absolute constant.

For instance, the function $w(t) = |\log t|^{-\alpha}$ satisfies (a) when $\alpha > \frac{1}{2}$ and (b) when $\alpha \leqslant \frac{1}{2}$. The construction of the inner function in part (b) of Theorem 6 uses symmetric singular measures. Actually, we need a refinement of the Carleson result, where we assume the integral condition and that $w(t)/t$ decreases. This is done in § 6 by means of Riesz products.

Using Theorem 6, one can prove versions of Corollaries 1 and 2 with $\mathscr{B}_0$ replaced by the space $\mathscr{B}_0(w)$ of holomorphic functions f in the unit disc such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|f'(z)|}{w(1-|z|)} = 0,$$

where w fulfills the conditions in part (b) of Theorem 6.

Corresponding to the Zygmund class and the Bloch space, there are the Zygmund measures, that is, positive measures μ in $\mathbb{T}$ for which

$$|\mu(J) - \mu(J')| = O(|J|) \quad \text{as } |J| \to 0.$$

This condition is equivalent to the fact that the primitive of μ is in the Zygmund class. Piranian [**17**] and Kahane [**13**] constructed finite positive singular measures satisfying

$$|\mu(J) - \mu(J')| = o(|J|) \quad \text{as } |J| \to 0.$$

We call such measures Kahane measures. Using Theorem 1 or Theorem 6 we will construct measures which are simultaneously symmetric and Kahane. In fact, as is to be expected from [**5**] and [**13**], one is able to replace the $o(1)$ condition by a condition of the form $O(w(|J|))$, where w fulfills the conditions in part (b) of Theorem 6. The point is that we do this in a new and uniform way. In private communications, A. Canton [**4**] and F. Nazarov showed us other ways of producing Kahane symmetric measures.

Also, one can establish the following sharp version of Corollary 3.

COROLLARY 4. *Let α be a positive increasing function on $(0,1]$, with $\alpha(0^+) = 0$. Assume that $\alpha(t)/t^{1-\varepsilon}$ is decreasing for some $\varepsilon > 0$. Then, the following assertions are equivalent:*

(a) *there exists a measurable set $E \subset \mathbb{T}$, with $0 < |E| < 1$, such that the measure $\chi_E|d\xi|$ is α-symmetric, that is,*

$$||E \cap J| - |E \cap J'|| \leqslant \alpha(|J|)|E \cap J|,$$

for any arc $J \subset \mathbb{T}$;

(b) *there exists a measurable set $E \subset \mathbb{T}$, with $0 < |E| < 1$, such that the measure $\chi_E|d\xi|$ is α-Zygmund, that is,*

$$||E \cap J| - |E \cap J'|| \leqslant \alpha(|J|)|J|,$$

for any arc $J \subset \mathbb{T}$;

(c)
$$\int_0 \frac{\alpha^2(t)}{t}\, dt = \infty.$$

The hyperbolic metric in $\mathbb{D}$ is the Riemannian metric $\lambda_{\mathbb{D}}(z)\,|dz|$, where $\lambda_{\mathbb{D}}(z) = (1 - |z|^2)^{-1}$. Let Ω be a hyperbolic domain, that is, a domain in the complex plane whose complement has at least two points. Let π: $\mathbb{D} \to \Omega$ be a universal covering map. Then $\lambda_{\mathbb{D}}$ projects to the Poincaré metric $\lambda_{\Omega}(z)\,|dz|$ of Ω, where

$$\lambda_{\Omega}(\pi(z)) \cdot |\pi'(z)| = \lambda_{\mathbb{D}}(z).$$

Schwarz's lemma asserts that holomorphic mappings f from $\mathbb{D}$ into Ω decrease hyperbolic distances, or infinitesimally,

$$(1 - |z|^2)|f'(z)|\lambda_{\Omega}(f(z)) \leqslant 1,$$

for all $z \in \mathbb{D}$.

A holomorphic function f from the unit disc into Ω is called *inner* (into Ω) if

$$\left|\{e^{i\theta}\colon \lim_{r\to 1} f(re^{i\theta}) \text{ exists and belongs to } \Omega\}\right| = 0.$$

If π is a holomorphic covering map from $\mathbb{D}$ into Ω, then π is inner; and as a matter of fact, if f is any holomorphic function from $\mathbb{D}$ into Ω which factorizes $f = \pi \circ b$, where $b: \mathbb{D} \to \mathbb{D}$, then f is inner (into Ω) if and only if b is inner into $\mathbb{D}$ (see [**7**]).

The theorems stated in this introduction have counterparts in this more general setting. For instance, Theorem 6 shows that if Ω is a hyperbolic domain and a positive weight satisfies

$$\int_0 \frac{w^2(t)}{t}\, dt < \infty,$$

then there is no non-constant inner function I into Ω such that

$$(1-|z|^2)|I'(z)|\lambda_\Omega(I(z)) \leqslant w(1-|z|),$$

for all $z \in \mathbb{D}$. On the other hand, if w fulfills the conditions in part (b) of Theorem 6, there exists a non-constant inner function I into Ω such that

$$(1-|z|^2)|I'(z)|\lambda_\Omega(I(z)) \leqslant w(1-|z|) \quad \text{for } z \in \mathbb{D}.$$

The paper is organized as follows. In § 2 we prove Theorem 2 and apply it to establish some results on composition operators. Section 3 contains two proofs of Theorem 3, using Theorems 4 and 5 respectively. Then we use Theorem 3 to establish Theorem 1 and the corollaries mentioned in this introduction, together with other related results. The proof of Theorem 4 is in § 4 and consists of a discretization procedure, which can be adapted to prove Theorem 5. As mentioned, this uses some of the ideas of [**3**]. In § 5 we prove Theorem 6. This uses the existence of singular symmetric measures proved by L. Carleson and a refinement of Theorem 4, whose proof is different from the one in § 4. Also, several ways of constructing singular measures which are both symmetric and Kahane are mentioned. Finally in § 6, we construct singular symmetric measures using Riesz products.

After this paper was completed, we learned that Wayne Smith had previously obtained Theorem 6, and hence Theorem 3, by different methods [**19**].

2. *Interpolating Blaschke products and composition operators*

The proof of Theorem 2 is based on an estimate of the density of the hyperbolic metric on plane domains, due to Beardon and Pommerenke [**2**]. We require only a crude estimate of this type, for which we present a proof.

LEMMA 2.1. *Let Ω be a domain in $\mathbb{D}$ and let f be an analytic function in $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$. Then, for all $z \in \mathbb{D}$,*

$$(1-|z|^2)|f'(z)| \leqslant 6\,\mathrm{dist}(f(z), \partial\Omega) \log \frac{e}{\mathrm{dist}(f(z), \partial\Omega)}.$$

Proof. Let $a \in \partial\Omega$ be such that $\mathrm{dist}(f(z), \partial\Omega) = |f(z) - a|$, and assume first that

$$|f(z) - a| \geqslant \tfrac{1}{4}(1 - |f(z)|^2).$$

Then

$$(1-|z|^2)|f'(z)| \leqslant 1 - |f(z)|^2 \leqslant 4|f(z) - a| \leqslant 6|f(z) - a| \log \frac{e}{|f(z) - a|}.$$

If, on the other hand,

$$|f(z)-a|<\tfrac{1}{4}(1-|f(z)|^2) \tag{2.1}$$

then $a\in\mathbb{D}$, that is, $a\notin\mathbb{T}$. Since

$$S(z)=\exp\left(-\frac{1+z}{1-z}\right)$$

is a universal covering map of the punctured unit disc $\mathbb{D}\setminus\{0\}$, there exists a holomorphic mapping ϕ from $\mathbb{D}$ into $\mathbb{D}$ satisfying

$$\frac{f-a}{1-\overline{a}f}=S\circ\phi.$$

A simple calculation shows that

$$(1-|w|^2)|S'(w)|=2|S(w)|\log|S(w)|^{-1}$$

for $w\in\mathbb{D}$ and hence

$$\begin{aligned}\frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}f(z)|^2}\,|f'(z)| &\leqslant (1-|\phi(z)|^2)|S'(\phi(z))|\\ &=2\left|\frac{f(z)-a}{1-\overline{a}f(z)}\right|\log\left|\frac{f(z)-a}{1-\overline{a}f(z)}\right|^{-1}.\end{aligned}$$

Thus

$$(1-|z|^2)|f'(z)|\leqslant 2\,\frac{|1-\overline{a}f(z)|}{1-|a|^2}\,|f(z)-a|\log\frac{e}{|f(z)-a|}$$

and the result follows from (2.1).

We also use the following elementary result, whose proof is omitted.

LEMMA 2.2. *Let* $h\colon(0,1]\to(0,1]$ *be a continuous function. Then there exists a countable set* $\Lambda\subset\mathbb{D}\setminus\{0\}$ *whose cluster set is contained in* $\mathbb{T}$ *such that, for all* $z\in\mathbb{D}$,

$$\operatorname{dist}(z,\Lambda\cup\mathbb{T})\leqslant h(1-|z|).$$

Proof of Theorem 2. Given $\phi(t)$, consider a continuous function $h\colon(0,1]\to(0,1]$ satisfying

$$6h(t)\log\frac{e}{h(t)}\leqslant\phi(t)$$

for all $t\in(0,1]$. For the set Λ of Lemma 2.2, let B be a holomorphic universal covering of $\mathbb{D}$ onto $\Omega=\mathbb{D}\setminus\Lambda$. Then Lemmas 2.1 and 2.2 show that

$$(1-|z|^2)|B'(z)|\leqslant\phi(1-|B(z)|^2),$$

as required and it remains to show that B is an interpolating Blaschke product. Since $B\in H^\infty$, its radial limit $B(\xi)$ exists for almost every $\xi\in\mathbb{T}$. Moreover, since B is a covering, $B(\xi)\in\Lambda\cup\mathbb{T}$ and hence in fact $B(\xi)\in\mathbb{T}$ for almost every $\xi\in\mathbb{T}$ since Λ is countable. Thus B is inner.

If B had a singular inner factor then there would be at least one value of $\xi\in\mathbb{T}$,

ξ_0 say, with

$$\lim_{r \to 1^-} B(r\xi_0) = 0.$$

We have arranged that $0 \notin \Lambda$ and so this cannot happen. Thus B is a Blaschke product. To prove that it is interpolating it is sufficient to observe that the quantity $(1 - |z|^2)|B'(z)|$ depends only on $B(z)$. Indeed, if $B(a) = B(b)$, then there exists an automorphism ϕ of $\mathbb{D}$ such that $\phi(a) = b$ and $B \circ \phi \equiv B$. Hence

$$(1 - |b|^2)|B'(b)| = (1 - |a|^2)|\phi'(a)|\,|B'(b)| = (1 - |a|^2)|B'(a)|.$$

Thus

$$\inf_n \{(1 - |z_n|^2)|B'(z_n)| \colon B(z_n) = 0\} \geqslant \delta > 0$$

for some suitable δ as required.

Remarks. 1. There exists also a singular inner function satisfying Theorem 2. In fact we may take a universal covering map of $\Omega \cup \{0\}$. Such a function will again not belong to $\mathscr{B}_0$.

2. By taking the set Λ as close to the unit circle as we please, we can have

$$\inf\{(1 - |z|^2)|B'(z)| \colon B(z) = 0\} \geqslant 1 - \delta$$

for any preassigned $\delta > 0$, even though Schwarz's Lemma tells us that

$$(1 - |z|^2)|B'(z)| \leqslant 1$$

for all $z \in \mathbb{D}$. Actually, if the covering map B satisfies $B(\mathbb{D}) \supset r\mathbb{D}$, with $0 < r < 1$, one has $(1 - |z|^2)|B'(z)| \geqslant r$, provided $B(z) = 0$.

Now suppose that $B \in H^\infty$ with $\|B\|_\infty \leqslant 1$. It was shown in **[14]** that the composition operator $C(B)$ is compact in $\mathscr{B}$ if and only if

$$(1 - |z|^2)|B'(z)| = o(1)(1 - |B(z)|^2) \quad \text{as } |B(z)| \to 1.$$

Thus Theorem 2 has the following corollary.

Corollary 2.3. *There exists an interpolating Blaschke product B such that the composition operator*

$$C(B) \colon \mathscr{B} \to \mathscr{B}, \quad C(B)(f) = f \circ B$$

is compact.

Next we consider the space $H(w)$ of analytic functions in the unit disc such that the norm

$$\|f\|_w = \sup\left\{\frac{|f(z)|}{w(|z|)} \colon z \in \mathbb{D}\right\} < \infty.$$

Here w denotes a positive continuous function on $[0, 1)$ with $\lim_{t \to 1^-} w(t) = \infty$.

Corollary 2.4. *For any function w as above and $\varepsilon > 0$, there exists an interpolating Blaschke product B such that the composition operator $C(B)$ maps $H(w)$ into the Bloch space $\mathscr{B}$ and*

$$\|C(B)(f)\|_{\mathscr{B}} \leqslant \varepsilon \|f\|_w$$

for any $f \in H(w)$.

Proof. Replacing w by $\varepsilon^{-1}w$, one can assume that $\varepsilon = 1$. If $f \in H_w$ and $\|f\|_w = 1$ then, from Cauchy's inequality,

$$(1-|z|^2)|f'(z)| \leqslant 4w(|z| + \tfrac{1}{2}(1-|z|)).$$

If we choose $\phi(t)$ so that

$$w(t + \tfrac{1}{2}(1-t))\phi(1-t^2) \leqslant 1$$

for $0 \leqslant t < 1$ then $\phi(t) \to 0$ as $t \to 0$. By Theorem 2, there exists an interpolating Blaschke product B such that

$$(1-|z|^2)|B'(z)| \leqslant \phi(1-|B(z)|^2)$$

for $z \in \mathbb{D}$. Hence for all $z \in \mathbb{D}$,

$$(1-|z|^2)|(f \circ B)'(z)| \leqslant 1.$$

REMARKS. 1. The point about the last result is that one inner function suffices for all the functions in $H(w)$. It is easy to see that for any given analytic function f there is an inner function $I = I(f)$ so that $f \circ I \in \mathscr{B}$. Actually one may take I to be the universal covering map of $\mathbb{D} \setminus \{f^{-1}(m+ni)\colon m, n \in \mathbb{Z}\}$.

2. Elementary considerations enable us to replace the space $H(w)$ by similar spaces defined in terms of the growth of derivatives.

3. In the proof of Corollary 2.4 we may choose a function $\phi(t)$ such that

$$w(t + \tfrac{1}{2}(1-t))\phi(1-t^2) \to 0 \quad \text{as } t \to 1^-.$$

Applying [**14**, Theorem 2] or Corollary 2.3, one can arrange that the composition operator

$$C(B)\colon H(w) \to \mathscr{B}$$

is compact.

COROLLARY 2.5. *Given a sequence $\{f_n\}$ of functions analytic in $\mathbb{D}$, there exists an interpolating Blaschke product B such that $f_n \circ B \in \mathscr{B}$ for $n = 1, 2, 3, \ldots$.*

Proof. It suffices to observe that there is a function $w(r)$ such that $f_n \in H(w)$ for $n = 1, 2, 3, \ldots$. For instance, we may take

$$w(r) = \sum_{n < (1-r)^{-1}} \sup\{|f_n(z)|\colon |z| = r\}.$$

REMARK. In a way similar to the above, we may replace the sequence $\{f_n\}$ by a sequence $\{A_n\}$ of Banach spaces of holomorphic functions in $\mathbb{D}$ and get the corresponding result that $f_n \circ B \in \mathscr{B}$ for any $f_n \in A_n$. Derivatives may be treated similarly, but we omit the details.

Finally in this section, we consider the case $\phi(t) = ct^2$, for $c > 0$, in Theorem 2; that is, let I be an inner function satisfying

$$(1-|z|^2)|I'(z)| \leqslant c(1-|I(z)|^2)^2. \tag{2.2}$$

For any $\alpha \in \mathbb{T}$ consider the holomorphic function

$$F_\alpha = (\alpha + I)/(\alpha - I).$$

Since $\operatorname{Re} F_\alpha > 0$ for $z \in \mathbb{D}$, there exists a positive measure σ_α in $\mathbb{T}$ such that

$$\operatorname{Re} F_\alpha(z) = \int_{\mathbb{T}} P(z, \xi)\, d\sigma_\alpha(\xi)$$

for all $z \in \mathbb{D}$. Since I is inner, the measures σ_α are singular and a simple calculation shows that for all $\alpha \in \mathbb{T}$,

$$\|F_\alpha\|_{\mathscr{B}} \leqslant 8c.$$

Thus the measures σ_α satisfy the Zygmund condition uniformly in α. In other words, there is a constant C_1 such that

$$|\sigma_\alpha(J) - \sigma_\alpha(J')| \leqslant C_1 |J|$$

for all $\alpha \in \mathbb{T}$ and all J, J'.

Denote by $\mathscr{A}(I)$ the σ-algebra generated by the preimages under I of the Lebesgue measurable sets in $\mathbb{T}$ and the sets of measure 0.

THEOREM 2.6. *Let I be an inner function satisfying* (2.2), *and let $h \in L^1(\mathbb{T})$ be measurable with respect to the σ-algebra $\mathscr{A}(I)$. Then the Cauchy transform of h, that is,*

$$F(z) = \int_{\mathbb{T}} \frac{h(\xi)\, d\xi}{1 - \bar{\xi} z} \qquad \text{for } z \in \mathbb{D},$$

is in the Bloch space $\mathscr{B}$.

Proof. We claim that for any $g \in L^1(\mathbb{T})$ and any I inner we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{g(I(e^{i\theta}))}{1 - e^{-i\theta} z}\, d\theta = \sum_{n \leqslant -1} \widehat{g}(n) (\overline{I(0)})^{-n} + \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{i\theta})\, d\theta}{1 - e^{-i\theta} I(z)}.$$

One proves this for $g(\xi) = \xi^n$ with $n \in \mathbb{Z}$, applying Cauchy's formula when $n \geqslant 0$ or the mean value theorem when $n < 0$.

Now there exists $g \in L^1(\mathbb{T})$ such that $h = g \circ I$ and it suffices to show that

$$\int_0^{2\pi} \frac{g(e^{i\theta})}{1 - e^{-i\theta} I(z)}\, d\theta \in \mathscr{B}.$$

We observe that the function

$$f(z) = \int_0^{2\pi} \frac{g(e^{i\theta})}{1 - e^{-i\theta} z}\, d\theta$$

belongs to $H(w)$ where $w(t) = (1 - t)^{-1}$. If I is an inner function satisfying (2.2) then the proof of Corollary 2.4 shows that $f \circ I \in \mathscr{B}$ as required.

The following corollary is now immediate.

COROLLARY 2.7. *Under the assumptions of Theorem* 2.6, *the function*

$$F(x) = \int_0^x h(e^{it})\, dt, \qquad \text{with } h \in L^1,$$

belongs to the Zygmund class $\Lambda^*(\mathbb{R})$.

3. *Inner functions in the small hyperbolic Lipschitz space*

As before we consider the equation

$$\operatorname{Re} H_\alpha(z) = \operatorname{Re} \frac{\alpha + f(z)}{\alpha - f(z)} = \int_{\mathbb{T}} P(z, \xi)\, d\sigma_\alpha(\xi), \tag{3.1}$$

where $\alpha \in \mathbb{T}$, $f \in H^\infty$ with $\|f\|_\infty \leqslant 1$ and $\sigma_\alpha(\xi)$ is the associated positive probability measure on $\mathbb{T}$. The function f is inner if and only if the measure σ_α is singular for some $\alpha \in \mathbb{T}$. In particular, if σ_α is singular for some $\alpha \in \mathbb{T}$ then σ_α is singular for all $\alpha \in \mathbb{T}$. Also, the support of σ_α is a finite set if and only if f is a finite Blaschke product. So this condition is also independent of $\alpha \in \mathbb{T}$. However, the fact that σ_α satisfies some property usually does not imply that σ_β satisfies the same property if $\beta \neq \alpha$. See [**1**], where some examples are considered.

Nevertheless, the fact that f satisfies the conclusion of Theorem 3 can be rephrased in terms of σ_α, with $\alpha \in \mathbb{T}$.

PROPOSITION 3.1. *Suppose that $f \in H^\infty$ with $\|f\|_\infty \leqslant 1$. The following assertions are equivalent:*

(a)
$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|f'(z)|}{1 - |f(z)|^2} = 0,$$

(b)
$$\left| \int_{\mathbb{T}} \frac{\overline{\xi}\, d\sigma_\alpha(\xi)}{(1 - \overline{\xi} z)^2} \right| = o(1) \int_{\mathbb{T}} \frac{d\sigma_\alpha(\xi)}{|1 - \overline{\xi} z|^2} \quad \text{as } |z| \to 1^-,$$

(c)
$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|H_\alpha'(z)|}{\operatorname{Re} H_\alpha(z)} = 0,$$

where f, H_α and σ_α are related by (3.1).

Proof. Fix $\alpha \in \mathbb{T}$. If $H_\alpha = (\alpha + f)(\alpha - f)^{-1}$, then $f = \alpha(H_\alpha - 1)(H_\alpha + 1)^{-1}$ and

$$1 - |f|^2 = \frac{4 \operatorname{Re} H_\alpha}{|1 + H_\alpha|^2}, \quad f' = \frac{2\alpha H_\alpha'}{(H_\alpha + 1)^2}.$$

Thus,

$$\frac{|H_\alpha'|}{\operatorname{Re} H_\alpha} = \frac{2|f'|}{1 - |f|^2}.$$

Thus condition (a) may be written as

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|H_\alpha'(z)|}{\operatorname{Re} H_\alpha(z)} = 0$$

and since

$$H_\alpha'(z) = 2 \int_{\mathbb{T}} \frac{\overline{\xi}\, d\sigma_\alpha(\xi)}{(1 - \overline{\xi} z)^2},$$

and

$$\operatorname{Re} H_\alpha(z) = \int_{\mathbb{T}} \frac{(1 - |z|^2)\, d\sigma_\alpha(\xi)}{|1 - \overline{\xi} z|^2},$$

the result follows.

The proof of Theorem 3 now follows from Proposition 3.1, Theorem 4 and the existence of singular symmetric measures. We may also prove Theorem 3 from the following proposition.

PROPOSITION 3.2. *Let σ be a positive measure on $\mathbb{T}$ and set*

$$S[\sigma](z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi)\right).$$

Then σ is symmetric if and only if

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|S[\sigma]'(z)|}{|S[\sigma](z)| \log(|S[\sigma](z)|^{-1})} = 0.$$

Proof. If

$$H(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi) \quad \text{for } z \in \mathbb{D},$$

then

$$\frac{(1 - |z|^2)|S[\sigma]'(z)|}{|S[\sigma](z)| \log(|S[\sigma](z)|^{-1})} = \frac{(1 - |z|^2)|H'(z)|}{\operatorname{Re} H(z)},$$

and the result follows from Theorem 4.

Note that whenever σ is a singular symmetric measure, then Theorem 3 holds for $I = S[\sigma]$.

There is yet another way of proving Theorem 3. In [**3**], Bishop has constructed a Blaschke product in $\mathscr{B}_0$. In fact, if

$$\mu = \sum_{z, B(z) = 0} (1 - |z|^2)\delta_z,$$

then his construction satisfies

$$\lim_{|Q| \to 0} \frac{\mu(Q)}{\mu(Q')} = 1, \tag{3.2}$$

where, as before, Q and Q' are contiguous Carleson squares of the same size. Applying Theorem 5 one can easily show that (3.2) implies that

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|B'(z)|}{1 - |B(z)|^2} = 0.$$

Observe also that, by Proposition 3.1 and Theorem 4, the corresponding singular measures σ_α, with $\alpha \in \mathbb{T}$, will be symmetric.

The next corollary follows from Theorem 3 and Theorem 1 in [**14**].

COROLLARY 3.3. *There exists an inner function I such that the composition operator $C(I)$ maps $\mathscr{B}$ into $\mathscr{B}_0$ compactly.*

Proof of Theorem 1 *and Corollaries* 1 *and* 2. We set

$$I(z) = B(I_0(z))$$

where B satisfies the hypotheses of Theorem 2 and I_0 the hypotheses of Theorem 3. Then

$$\frac{(1-|z|^2)|I'(z)|}{\phi(1-|I(z)|^2)} = \frac{(1-|z|^2)|B'(I_0(z))|\,|I_0'(z)|}{\phi(1-|B(I_0(z))|^2)} \leqslant \frac{(1-|z|^2)|I_0'(z)|}{1-|I_0(z)|^2} \to 0 \quad \text{as } |z| \to 1^-.$$

Corollaries 1 and 2 then follow also from Corollaries 2.4 and 2.5 by composing with the same inner function I_0. Observe that in any of these results the inner function whose existence is asserted can be chosen to be singular or a Blaschke product. Moreover Remarks 2 and 3 after Corollary 2.4 and the Remark after Corollary 2.5 apply with $\mathscr{B}$ replaced by $\mathscr{B}_0$.

Ideals in the space of inner functions

Let $\mathscr{D}$ be the set of inner functions I for which

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0.$$

We note that $\mathscr{D}$ is an ideal in the space of inner functions with respect to composition from the left. In fact, if $I \in \mathscr{D}$ and $\phi \in H^\infty$ with $\|\phi\|_\infty \leqslant 1$ then it follows from Schwarz's lemma that

$$\frac{(1-|z|^2)|\phi'(I(z))|\,|I'(z)|}{1-|\phi(I(z))|^2} \leqslant \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2}.$$

This shows again that the inner function in Theorem 3 can be taken to be a singular inner function as well as a Blaschke product.

The next result asserts that the only primary ideals (with respect to left composition) of inner functions contained in $\mathscr{B}_0$ are the ones given by functions in $\mathscr{D}$.

Proposition 3.4. *Let I be an inner function such that $\phi \circ I \in \mathscr{B}_0$ for any inner function ϕ. Then $I \in \mathscr{D}$.*

Proof. It is obvious that $I \in \mathscr{B}_0$. If $I \notin \mathscr{D}$ then there exists $\{z_n\} \subset \mathbb{D}$ such that

$$\lim_{n\to\infty} |I(z_n)| = 1$$

and

$$\frac{(1-|z_n|^2)|I'(z_n)|}{1-|I(z_n)|^2} \geqslant \delta > 0$$

for $n = 1, 2, 3, \ldots$. Passing to a subsequence, if necessary, we may assume that $\{I(z_n)\}$ forms an interpolating sequence for H^∞. If ϕ is the corresponding interpolating Blaschke product, then for $n = 1, 2, 3, \ldots$ one has

$$(1-|I(z_n)|^2)|\phi'(I(z_n))| \geqslant C,$$

and

$$(1-|z_n|^2)|I'(z_n)|\,|\phi'(I(z_n))| \geqslant C\,\frac{(1-|z_n|^2)|I'(z_n)|}{1-|I(z_n)|^2} \geqslant C\delta,$$

contradicting the fact that $\phi \circ I \in \mathscr{B}_0$.

It is worth mentioning that there are no ideals with respect to composition from the right contained in $\mathscr{B}_0$. Indeed if one considers the singular inner function

$$\phi(z) = \exp\left[-\left(\frac{1+z}{1-z}\right)\right],$$

then $I \circ \phi$ does not belong to $\mathscr{B}_0$ for any non-constant analytic function I. In fact, if $z \to 1$ along a suitable horocycle then the quantity

$$(1-|z|^2)|I'(\phi(z))|\,|\phi'(z)|$$

cannot tend to zero, no matter what I is.

However, there do exist non-trivial right ideals. For instance, if $\alpha \geqslant 0$ then the set

$$\mathscr{D}_\alpha = \left\{f\colon f \text{ inner}, \ \frac{(1-|z|^2)|f'(z)|}{(1-|f(z)|^2)^{\alpha+1}} = O(1) \text{ as } |z| \to 1\right\}$$

is a bilateral ideal. It is interesting to observe that if $f \in \mathscr{D}_\alpha$ and $g \in \mathscr{D}_\beta$ then $f \circ g \in \mathscr{D}_{\alpha+\beta}$.

Let us next consider $\phi(t) = t^2$ in Theorem 1 so that I is an inner function satisfying

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|I'(z)|}{(1-|I(z)|^2)^2} = 0. \tag{3.3}$$

THEOREM 3.5. *Let I be an inner function satisfying* (3.3) *and let σ_α, for $\alpha \in \mathbb{T}$, be the corresponding singular measures defined by* (3.1). *Then σ_α are* (*uniformly in $\alpha \in \mathbb{T}$*) *Kahane measures, that is,*

$$\lim_{|J|\to 0} \frac{1}{|J|}\,(\sigma_\alpha(J) - \sigma_\alpha(J')) = 0$$

uniformly for $\alpha \in \mathbb{T}$.

Proof. It is well known that the Herglotz integral of a positive measure is in $\mathscr{B}$ if and only if the measure is Zygmund, and it is in $\mathscr{B}_0$ if and only if the measure is in the small Zygmund class (see [**18**, p. 156]). So it is sufficient to observe that the functions $(\alpha + I)(\alpha - I)^{-1}$ are in $\mathscr{B}_0$ and

$$\sup_{\alpha}\ \sup_{1>|z|\geqslant 1-r} (1-|z|^2)\left|\left(\frac{\alpha+I}{\alpha-I}\right)'(z)\right| \to 0 \quad \text{as } r \to 1.$$

Observe that Proposition 3.1 and Theorem 4 also show that σ_α are (uniformly in $\alpha \in \mathbb{T}$) symmetric measures.

The following theorem, whose proof is omitted, is established in a similar manner to Theorem 2.6 and Corollary 2.7. Recall that given an inner function I, $\mathscr{A}(I)$ denotes the σ-algebra generated by the preimages under I of the Lebesgue measurable sets in $\mathbb{T}$ and the sets of measure 0.

THEOREM 3.6. *Let I be an inner function satisfying* (3.3) *and let $f \in L^1(\mathbb{T})$ be measurable with respect to the σ-algebra $\mathscr{A}(I)$. Then*

(a) *the function*

$$G(z) = \int_{\mathbb{T}} \frac{f(\xi)\,d\xi}{1-\bar{\xi}z}$$

belongs to $\mathscr{B}_0$, *and*

(b) *the function*

$$F(x) = \int_0^x f(e^{it})\, dt$$

belongs to $\lambda^*(\mathbb{R})$.

If one chooses f as the characteristic function of $I^{-1}(J)$, one obtains Corollary 3 of § 1.

4. *Proofs of Theorems* 4 *and* 5

To prove Theorem 4 we restate condition (a) as

$$\left| \int_{\mathbb{T}} P(z, \xi) \frac{d\sigma(\xi)}{\tau(z, \xi)} \right| = o(1) \int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi) \quad \text{as } |z| \to 1^- \tag{4.1}$$

where

$$\tau(z, \xi) = \frac{\xi - z}{1 - \bar{z}\xi} \quad (\xi \in \mathbb{T}).$$

It is readily shown that this is equivalent to (a).

Given a point $z = re^{i\theta} \in \mathbb{D}$, denote by $J(z)$ the arc of $\mathbb{T}$ with centre $e^{i\theta}$ and (normalized) length $1 - r$. Also, given an arc $J \subset \mathbb{T}$ and $M > 0$ let MJ be the arc of the same centre and with $|MJ| = M|J|$.

Part I: (b) $\Rightarrow$ (a). Assume that (b) holds. We first prove the following.

LEMMA 4.1. *Given* $\varepsilon > 0$ *there exist* $N > 0$ *and* $\delta > 0$ *such that if* $1 - \delta < |z| < 1$, *then*

$$\int_{\mathbb{T} \setminus NJ(z)} P(z, \xi)\, d\sigma(\xi) < \varepsilon \int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi).$$

The lemma states, roughly speaking, that contributions to the Poisson integral from far away do not matter.

Proof. Given $\varepsilon > 0$, choose δ so that if J is an arc of $\mathbb{T}$ with $|J| < \delta$ then

$$|\sigma(J) - \sigma(J')| < \varepsilon\sigma(J)$$

and hence

$$|\sigma(J \cup J') - 2\sigma(J)| < \varepsilon\sigma(J).$$

Hence, if $2^k|J| < \delta$, we have

$$\sigma(2^k J) < (2 + \varepsilon)^k \sigma(J).$$

We break the integral on the left into dyadic pieces. Let M denote the integer part of $\log_2(\delta/(1 - |z|))$, so that $2^M(1 - |z|) \sim \delta$. Then, using crude estimates we obtain

$$\int_{\mathbb{T} \setminus NJ(z)} P(z, \xi)\, d\sigma(\xi) \leqslant C \left(\sum_{k = \log_2 N}^{M} \frac{\sigma(2^k J(z))}{2^{2k}(1 - |z|)} + \sum_{k > M} \frac{\sigma(2^k J(z))}{2^{2k}(1 - |z|)} \right),$$

where C is an absolute constant.

The first sum is bounded by

$$\frac{\sigma(J(z))}{|J(z)|}\sum_{k=\log_2 N}^{\infty}\left(\frac{2+\varepsilon}{4}\right)^k<\varepsilon\int_{\mathbb{T}}P(z,\xi)\,d\sigma(\xi),$$

if N is sufficiently large.

Observe now that, for any $\varepsilon>0$,

$$\frac{\sigma(J)}{|J|^2}\geqslant\left(\frac{4}{2+\varepsilon}\right)\frac{\mu(2J)}{|2J|^2}$$

if $|J|$ is sufficiently small. Iterating this inequality, we obtain

$$\frac{\sigma(J)}{|J|^2}>C\left(\frac{4}{2+\varepsilon}\right)^n\to\infty\quad\text{as } n\to\infty.$$

Thus

$$\lim_{|J|\to 0}\frac{\sigma(J)}{|J|^2}=\infty. \tag{4.2}$$

The second sum above can be estimated by

$$\frac{2\sigma(\mathbb{T})}{2^{2M}4(1-|z|)}\sim\frac{\sigma(\mathbb{T})}{\delta^2}(1-|z|)$$

and from (4.2) if $1-|z|$ is sufficiently small, this does not exceed

$$\varepsilon\,\frac{\sigma(J(z))}{1-|z|}<\varepsilon\int_{\mathbb{T}}P(z,\xi)\,d\sigma(\xi),$$

as required.

Now let $l>0$ be a small number to be fixed later and divide $NJ(z)$ into N/l arcs each of length $l(1-|z|)$. Call these arcs J_k and let the centre of each arc be $\xi_k=e^{i\theta_k}$. Then

$$\begin{aligned}\left|\int_{J_k}P(z,\xi)\,\frac{d\sigma(\xi)}{\tau(z,\xi)}-P(z,\xi_k)\,\frac{\sigma(J_k)}{\tau(z,\xi_k)}\right|&\leqslant(1-|z|^2)\int_{J_k}\left|\frac{\xi}{(\xi-z)^2}-\frac{\xi_k}{(\xi_k-z)^2}\right|d\sigma(\xi)\\&\leqslant(1-|z|^2)\int_{J_k}\frac{|\xi-\xi_k|\,|\xi\xi_k-z^2|}{|\xi-z|^2\,|\xi_k-z|^2}\,d\sigma(\xi)\\&\leqslant 4l\int_{J_k}P(z,\xi)\,d\sigma(\xi),\end{aligned}$$

since $|\xi-\xi_k|<l(1-|z|)$ and $|\xi\xi_k-z^2|\sim|\xi_k-z|$. Now sum over k to obtain

$$\left|\int_{NJ(z)}P(z,\xi)\,\frac{d\sigma(\xi)}{\tau(z,\xi)}-\sum_{k=1}^{N/l}P(z,\xi_k)\,\frac{\sigma(J_k)}{\tau(z,\xi_k)}\right|<4l\int_{\mathbb{T}}P(z,\xi)\,d\sigma(\xi).$$

The estimate (4.1) follows on taking l such that $4l<\varepsilon$ provided that we can show that

$$\left|\sum_{k=1}^{N/l}P(z,\xi_k)\,\frac{\sigma(J_k)}{\tau(z,\xi_k)}\right|\leqslant\frac{1}{N}\int_{\mathbb{T}}P(z,\xi)\,d\sigma(\xi) \tag{4.3}$$

for any $z\in\mathbb{D}$ such that $|z|$ is close enough to 1.

The number of arcs J_k is large but independent of z. Hence if $|z|$ is close enough to 1, we have

$$|\sigma(J_k) - \sigma(J_i)| < \frac{\varepsilon}{2\pi}\,\sigma(J_k), \quad \text{for } 1 \leqslant k, j \leqslant N/l.$$

We write

$$\sum_{k=1}^{N/l} P(z, \xi_k)\,\frac{\sigma(J_k)}{\tau(z, \xi_k)} = \sum_{k=1}^{N/l} P(z, \xi_k)\,\frac{\sigma(J_k) - \sigma(J_1)}{\tau(z, \xi_k)} + \sigma(J_1)\sum_{k=1}^{N/l}\frac{P(z, \xi_k)}{\tau(z, \xi_k)}$$
$$= \mathbb{T}_1 + \mathbb{T}_2,$$

say.

Now

$$|\mathbb{T}_1| < \varepsilon\,\frac{\sigma(J_1)}{|J_1|} < C\varepsilon \int_{\mathbb{T}} P(z, \xi)\,d\sigma(\xi),$$

where C is an absolute constant, while

$$\mathbb{T}_2 = \frac{\sigma(J_1)}{|J_1|}\sum_{k=1}^{N/l}\frac{1 - |z|^2}{(\xi_k - z)^2}\,\xi_k|J_k|$$

since $|J_k| = |J_1|$ for $1 \leqslant k \leqslant N/l$. The sum above is a Riemann sum of the integral

$$\int_{NJ(z)} \frac{1 - |z|^2}{(\xi - z)^2}\,d\xi,$$

which an easy calculation shows to be bounded by $1/N$. The estimate (4.3) follows on taking N large enough since

$$\frac{\sigma(J_1)}{|J_1|} < 2\,\frac{\sigma(J(z))}{|J(z)|} < C\int_{\mathbb{T}} P(z, \xi)\,d\sigma(\xi),$$

where C is an absolute constant.

Part II: (a) $\Rightarrow$ (b). The proof follows closely the arguments of [**3**]. Consider the pseudohyperbolic disc centred at z of radius $c < 1$, that is,

$$\{w\colon \rho(w, z) < c < 1\} \quad \text{where } \rho(w, z) = \left|\frac{w - z}{1 - \bar{z}w}\right|.$$

Integrate (a) from z to w to obtain, for all $c < 1$,

$$\sup_{\rho(w,z) \leqslant c} \frac{|\operatorname{Re} H(w) - \operatorname{Re} H(z)|}{\operatorname{Re} H(z)} \to 0 \quad \text{as } |z| \to 1.$$

Thus there exists a function $a(r)$ such that

$$\text{(4.4)} \quad \begin{aligned} &\text{(a)} \quad a(r) \to 1 \quad \text{as } r \to 1,\\ &\text{(b)} \quad \sup\left\{\frac{|\operatorname{Re} H(w) - \operatorname{Re} H(z)|}{\operatorname{Re} H(z)} : \rho(w, z) < a(|z|)\right\} \to 0 \quad \text{as } |z| \to 1. \end{aligned}$$

LEMMA 4.2. *Suppose that* (a) *holds. Then, given* $N > 1$ *there exists* $\delta = \delta(N) \in (0, 1)$ *such that if* $1 - \delta < |z| < 1$, *then*

$$\int_{\mathbb{T}\setminus NJ(z)} P(z, \xi)\, d\sigma(\xi) < \frac{C}{N} \int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi),$$

where C *is an absolute constant.*

Proof. Let $\delta = \delta(N)$ be a small number, to be fixed later, with $\delta < 1/N$. Given $z \in \mathbb{D}$, with $1 - |z| < \delta$, consider the point

$$z_N = (1 - N(1 - |z|))(z/|z|).$$

So, $J(z_N) \equiv NJ(z)$ and for $\xi \notin NJ(z)$ we have

$$|\xi - z_N| < C_0 |\xi - z|,$$

where C_0 is an absolute constant. Hence

$$P(z_N, \xi) > C_0^{-2} N P(z, \xi)$$

for $\xi \notin NJ(z)$.

Now, if $\delta > 0$ is sufficiently small and $1 - \delta < |z| < 1$, we have

$$\operatorname{Re} H(z) \geqslant \tfrac{1}{2} \operatorname{Re} H(z_N)$$

and hence

$$\int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi) = \operatorname{Re} H(z) \geqslant \tfrac{1}{2} \operatorname{Re} H(z_N) \geqslant \tfrac{1}{2} C_0^{-2} N \int_{\mathbb{T}\setminus NJ(z)} P(z, \xi)\, d\sigma(\xi).$$

LEMMA 4.3. *With the above notation,*

$$\left| \frac{\sigma(J(z))}{|J(z)|} - \operatorname{Re} H(z) \right| = o(1) \operatorname{Re} H(z) \quad \text{as } |z| \to 1^-.$$

Proof. For a given $z \in \mathbb{D}$, consider the arc

$$L = \{ re^{i\theta} \colon |\theta - \arg z| < \pi(1 - \delta)(1 - |z|) \}$$

where $r = r(z)$, $\delta = \delta(z)$ will be chosen later to satisfy

$$r \to 1, \quad \delta \to 0, \quad \frac{1 - r}{(1 - |z|)\delta} \to 0, \quad \text{as } |z| \to 1^-.$$

Given $\varepsilon > 0$, Lemma 4.2 shows that, for any $w \in L$,

$$\left| \operatorname{Re} H(w) - \int_{J(z)} P(w, \xi)\, d\sigma(\xi) \right| < \varepsilon \operatorname{Re} H(z)$$

provided that $(1 - r)/\delta(1 - |z|)$ is sufficiently small. Thus

$$\sup_{w \in L} \frac{1}{\operatorname{Re} H(z)} \left| \operatorname{Re} H(z) - \int_{J(z)} P(w, \xi)\, d\sigma(\xi) \right| \to 0 \quad \text{as } |z| \to 1^-.$$

Integrating along the arc L we obtain

$$\left| |L| \operatorname{Re} H(z) - \frac{1}{2\pi} \int_{J(z)} \int_L P(w, \xi)\, d\sigma(\xi)\, |dw| \right| = o(1) |L| \operatorname{Re} H(z) \quad \text{as } |z| \to 1^-.$$

Now $|J(z)| - |L| = \delta(1-|z|) \to 0$ and

$$\frac{1}{2\pi}\int_L P(w,\xi)\,|dw| \to 1 \quad \text{as } |z| \to 1^-$$

if $|\theta - \arg z| < \pi(1-c)(1-|z|)$. This shows that for any small number $c > 0$, we have

$$\liminf_{|z|\to 1} \frac{\sigma(J(z))}{|J(z)|\operatorname{Re} H(z)} \geqslant 1 - c$$

and

$$\limsup_{|z|\to 1} \frac{\sigma((1-c)J(z))}{|J(z)|\operatorname{Re} H(z)} \leqslant 1 - c.$$

Consider the point w such that $J(w) = (1-c)J(z)$, that is,

$$w = (1-(1-c)(1-|z|))(z/|z|).$$

The second inequality gives

$$\limsup_{|w|\to 1} \frac{\sigma(J(w))}{(1-c)^{-1}|J(w)|\operatorname{Re} H(w)} \leqslant 1 - c.$$

Thus,

$$1 - c \leqslant \liminf_{|z|\to 1} \frac{\sigma(J(z))}{|J(z)|\operatorname{Re} H(z)} \leqslant \limsup_{|z|\to 1} \frac{\sigma(J(z))}{|J(z)|\operatorname{Re} H(z)} \leqslant 1,$$

for any small number $c > 0$. This proves the lemma.

The proof that (a) $\Rightarrow$ (b) now follows immediately. For contiguous arcs J, J' with centres z and z' (and, as always, the same length),

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(J')}{|J'|}\right| \leqslant \left|\frac{\sigma(J)}{|J|} - \operatorname{Re} H(z)\right| + \left|\frac{\sigma(J')}{|J'|} - \operatorname{Re} H(z')\right| + |\operatorname{Re} H(z) - \operatorname{Re} H(z')|.$$

Lemma 4.3 shows that the first two terms are bounded by $\varepsilon(\operatorname{Re} H(z) + \operatorname{Re} H(z'))$.

Also z and z' are within a bounded hyperbolic distance of each other and hence by (4.4) the last term is also less than $\varepsilon(\operatorname{Re} H(z))$. Summing up, we have

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(J')}{|J'|}\right| < 4\varepsilon \operatorname{Re} H(z) < 5\varepsilon\,\frac{\sigma(J)}{|J|},$$

as required.

A little consideration shows that the proof of Theorem 4 may be applied to prove the following more general result.

THEOREM 4.4. *Let $\{f_z\colon z \in \mathbb{D}\}$ be a family of positive continuous functions on $\mathbb{T}$. Assume that there exist constants $C, M > 0$ such that for all $z \in \mathbb{D}$ and all $\xi_1, \xi_2 \in \mathbb{T}$ we have*

$$M^{-1} \leqslant f_z(\xi_1) \leqslant M,$$

$$|f_z(\xi_1) - f_z(\xi_2)| \leqslant \frac{C}{1-|z|}\,|\xi_1 - \xi_2|.$$

Assume, further, that σ is a symmetric measure on $\mathbb{T}$. Then

(4.5)

$$\lim_{|z|\to 1}\left\{\left(\frac{1}{\sigma(J(z))}\int_{\mathbb{T}} f_z(\xi)P(z,\xi)\,d\sigma(\xi)\right)\Big/\left(\frac{1}{|J(z)|}\int_{\mathbb{T}} f_z(\xi)P(z,\xi)\,\frac{|d\xi|}{2\pi}\right)\right\}=1.$$

Proof. (This is merely sketched.) As in Lemma 4.1 one may replace the integrals in (4.5) by integrals on $NJ(z)$ for large N. The Riemann sum argument used to prove that (b) $\Rightarrow$ (a) can now be applied.

COROLLARY 4.5. *Let σ be a symmetric measure on $\mathbb{T}$ and suppose that f is a continuous function on $\mathbb{T}$. Then*

$$\lim_{|z|\to 1^-}\frac{1-|z|}{\sigma(J(z))}\int_{\mathbb{T}}(f\circ\tau_z)(\xi)P(z,\xi)\,d\sigma(\xi)=\int_{\mathbb{T}} f(\xi)\,\frac{|d\xi|}{2\pi}$$

where, as before,

$$\tau_z(\xi)=\frac{\xi-z}{1-\bar{z}\xi}.$$

Proof. Theorem 4.4 can be applied directly if the continuous function satisfies a Lipschitz condition,

$$|f(\xi_1)-f(\xi_2)|\leqslant C|\xi_1-\xi_2|$$

on $\mathbb{T}$. Moreover for $f\equiv 1$ one obtains

$$\lim_{|z|\to 1}\frac{1-|z|}{\sigma(J(z))}\int_{\mathbb{T}} P(z,\xi)\,d\sigma(\xi)=1. \tag{4.6}$$

Consequently,

$$\sup_{z\in\mathbb{D}}\frac{1-|z|}{\sigma(J(z))}\int_{\mathbb{T}} P(z,\xi)\,d\sigma(\xi)<\infty.$$

Applying the Banach–Steinhaus theorem, we obtain the desired equality for any continuous function f.

COROLLARY 4.6. *Let σ be a symmetric measure on $\mathbb{T}$ and f be a continuous function on $\mathbb{T}$. Then*

$$\lim_{|z|\to 1}\frac{\int_{\mathbb{T}}(f\circ\tau_z)(\xi)P(z,\xi)\,d\sigma(\xi)}{\int_{\mathbb{T}} P(z,\xi)\,d\sigma(\xi)}=\int_{\mathbb{T}} f(\xi)\,\frac{|d\xi|}{2\pi}.$$

Proof. It suffices to apply (4.6) and Corollary 4.5.

Observe that by taking $f(z)=\bar{z}$, this corollary proves (b) $\Rightarrow$ (a) in Theorem 4.

Proof of Theorem 5. This is similar to that of Theorem 4 and so is only sketched.

Part I: (b) $\Rightarrow$ (a). Using the characterization of the inner functions in $\mathscr{B}_0$ given by Bishop in [**3**] one can easily see that $I\in\mathscr{B}_0$. Hence in proving (a) one may assume that $|I(z)|\geqslant\frac{1}{2}$. A computation with logarithmic derivatives shows that

$$(1-|z|^2)|I'(z)|=|I(z)|\left|\int_{\overline{\mathbb{D}}} P(z,\xi)\,\frac{d\mu(\xi)}{\tau(z,\xi)}\right|, \tag{4.7}$$

while

$$1 - |I(z)|^2 \sim \log |I(z)|^{-2} \sim \int_{\overline{\mathbb{D}}} P(z, \xi)\, d\mu(\xi)$$

and it is these last two integrals which one has to compare.

For fixed $\eta > 0$, condition (2.b) of Theorem 5 yields an $N > 0$ such that

$$\int_{\overline{\mathbb{D}} \setminus NQ(z)} P(z, \xi)\, d\mu(\xi) < \eta \int_{\overline{\mathbb{D}}} P(z, \xi)\, d\mu(\xi),$$

if $|z|$ is sufficiently close to 1. For such a z consider the $[N/\eta]$ disjoint Carleson squares, Q_k say, with $k = 1, 2, \ldots, [N/\eta]$, of size $\eta(1 - |z|)$ contained in $NQ(z)$. Since $I \in \mathscr{B}_0$ and $|I(z)| \geqslant \frac{1}{2}$, the zeros of I are (hyperbolically) distant from z and we can assume that the zeros of I in $NQ(z)$ are contained in $\bigcup_k Q_k$. Thus

$$\mu(NQ(z)) = \mu\left(\bigcup_k Q_k\right).$$

As in the previous proof, the principal idea is to discretize the integral in (4.7) and compare it with an integral with respect to Lebesgue measure. If we write $A \sim B$ to mean

$$|A - B| \leqslant \eta \int_{\overline{\mathbb{D}}} P(z, \xi)\, d\mu(\xi),$$

then given points $\xi_k \in Q_k \cap \mathbb{T}$, one can show, as before, that

$$\sum_k \int_{Q_k} P(z, \xi) \frac{d\mu(\xi)}{\tau(z, \xi)} \sim \sum_k P(z, \xi_k) \frac{\mu(Q_k)}{\tau(z, \xi_k)}$$

$$\sim \frac{\mu(Q(z))}{|Q(z)|} \sum_k \frac{1 - |z|^2}{(\xi_k - z)(1 - z\overline{\xi_k})} |Q_k|$$

using (1.b) of Theorem 5 in the second estimate. Finally, one only has to observe that the last sum is a Riemann sum for the integral

$$\int_{NQ(z) \cap \mathbb{T}} \frac{1 - |z|^2}{(\xi - z)^2}\, d\xi$$

and that this is bounded by $1/N$.

Part II: (a) $\Rightarrow$ (b). As in the proof of Theorem 4, one can show that, given $\eta > 0$, there exist $N > 0$ and $\delta > 0$ such that

$$\int_{\overline{\mathbb{D}} \setminus NQ(z)} P(z, \xi)\, d\mu(\xi) < \eta \int_{\overline{\mathbb{D}}} P(z, \xi)\, d\mu(\xi) \tag{4.8}$$

if $0 < 1 - |z| < \delta$. To prove (1.b) of Theorem 5, it is sufficient to show that, for any $\varepsilon > 0$,

$$\sup_{z:\, |I(z)| > \varepsilon} \frac{|(\mu(Q(z))/|Q(z)|) - \log |I(z)|^{-1}|}{\log |I(z)|^{-1}} \to 0 \quad \text{as } |z| \to 1^-. \tag{4.9}$$

The estimate (4.9) can be proved with the same integration technique used in the corresponding implication in Theorem 4. Finally, to prove (2.b) of Theorem 5 we use (4.8) and (4.9) to show that

$$\int_{\overline{\mathbb{D}} \setminus NQ(z)} P(z, \xi)\, d\mu(\xi) < 2\eta \frac{\mu(Q(z))}{|Q(z)|}$$

if $\mu(Q(z)) > \varepsilon |Q(z)|$. One now estimates the left-hand side dyadically to obtain (2.b). The details are omitted.

5. *The decay in Schwarz's lemma and symmetric and Kahane measures*

The existence of the function $H(z)$ of Theorem 4 as well as the existence of the inner function of Theorem 3 both depend ultimately on the existence of singular symmetric measures. In connection with the Beurling–Ahlfors extension theorem for quasi-conformal mappings, L. Carleson has shown [**5**] that such measures do exist. Indeed if $w(t)$ is a continuous increasing function on $[0, 1]$ with $w(0) = 0$, such that $t^{-1/2}w(t)$ is decreasing and such that

$$\int_0 \frac{w^2(t)}{t}\, dt = \infty, \tag{5.1}$$

then there exists a singular measure σ on $\mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \left| \frac{\sigma(x, x+h)}{\sigma(x-h, x)} - 1 \right| \leqslant w(h) \quad \text{for } h > 0. \tag{5.2}$$

Thus choosing, for instance, $w(t) = (\log(1/t))^{-\alpha}$, with $\alpha \leqslant \frac{1}{2}$, one obtains a singular symmetric measure. The integral condition (5.1) is also necessary for the existence of a singular measure satisfying (5.2), as was also established in [**5**]. Actually, if σ is a measure satisfying (5.2) and

$$\int_0 \frac{w^2(t)}{t}\, dt < \infty,$$

then σ is absolutely continuous and its derivative is in L^2_{loc}.

A similar situation occurs for inner functions.

THEOREM 5.1. *Let w be a positive continuous function on $(0, 1]$. Assume that*

$$\int_0 \frac{w^2(t)}{t}\, dt < \infty.$$

Then, there is no non-constant inner function I such that

$$(1 - |z|^2) \frac{|I'(z)|}{1 - |I(z)|^2} \leqslant w(1 - |z|),$$

for all $z \in \mathbb{D}$.

Proof. Assume that such an inner function I exists. Consider a positive singular measure σ such that

$$H(z) = \frac{1 + I(z)}{1 - I(z)} = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta) \quad \text{for } z \in \mathbb{D}.$$

Then, for all $z \in \mathbb{D}$ we have

$$\frac{H'(z)}{H(z)} = \frac{2I'(z)}{1 - I(z)^2}.$$

So,

$$(1 - |z|^2)\,\frac{|H'(z)|^2}{|H(z)|^2} \leqslant \frac{w^2(1 - |z|)}{1 - |z|^2} \quad \text{for } z \in \mathbb{D}.$$

Therefore $\log H$ is an analytic function whose boundary values are of vanishing mean oscillation (see [**9**, Chapter VI]). In particular, H belongs to the Hardy space H^p, for any $p < \infty$. Since σ is a singular measure, $\operatorname{Re} H(e^{i\theta}) = 0$ for almost every $e^{i\theta} \in \mathbb{T}$, and this is a contradiction (see [**9**, p. 95]).

Observe that the previous argument also shows, assuming the integral condition on w, that the only inner functions I satisfying

$$(1 - |z|^2)|I'(z)| \leqslant w(1 - |z|^2) \quad \text{for } z \in \mathbb{D},$$

are the finite Blaschke products.

The converse of Theorem 5.1 is the following.

THEOREM 5.2. *Let w be a positive increasing function on $(0, 1]$, with $w(0^+) = 0$. Assume that there exist constants k and δ such that*

$$\widetilde{w}(t) \leqslant k w(t) \quad \text{if } |t| < \delta,$$

where $\widetilde{w}(t)$ is given by (1.4), *and that*

$$\int_0 \frac{w^2(t)\,dt}{t} = \infty.$$

Then, there exists an inner function I such that

$$(1 - |z|^2)\,\frac{|I'(z)|}{1 - |I(z)|^2} \leqslant w(1 - |z|) \quad \text{for } z \in \mathbb{D}.$$

We can then use the composition process. Let ϕ be a positive continuous function with $\phi(0^+) = 0$ as in Theorem 2, and let B_0 be the interpolating Blaschke product of Theorem 2.

THEOREM 5.3. *With w, B_0, ϕ and I as above, set $B = B_0 \circ I$. Then*

$$\frac{(1 - |z|^2)|B'(z)|}{\phi(1 - |B(z)|^2)} = o(w(1 - |z|^2)) \quad \text{as } |z| \to 1^-.$$

This permits us to establish the analogues of Corollaries 1 and 2 with $\mathscr{B}_0$ replaced by

$$\mathscr{B}_0(w) = \left\{ f\colon f \text{ analytic in } \mathbb{D},\ \lim_{|z| \to 1} \frac{(1 - |z|^2)|f'(z)|}{w(1 - |z|^2)} = 0 \right\},$$

assuming always that w satisfies the conditions in Theorem 5.2.

As before, the case $\phi(t) = t^2$ in Theorem 5.3 is of special interest. If the inner function B is such that

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|B'(z)|}{(1 - |B(z)|^2)^2 w(1 - |z|^2)} = 0,$$

then the corresponding family of positive singular measures σ_α, with $\alpha \in \mathbb{T}$, satisfy, uniformly in α, the following two conditions simultaneously:

$$
\begin{aligned}
&|\sigma_\alpha(J) - \sigma_\alpha(J')| \leqslant w(|J|)\sigma_\alpha(J),\\
&|\sigma_\alpha(J) - \sigma_\alpha(J')| \leqslant w(|J|)|J|.
\end{aligned} \tag{5.3}
$$

The point is, however, that starting from a given symmetric measure σ, a whole family $\{\sigma_\alpha\colon \alpha \in \mathbb{T}\}$ of singular Kahane symmetric measures, with the additional property that σ_α and σ_β are mutually singular if $\alpha \neq \beta$, can be obtained.

The condition (5.3) follows from the following refined version of (a) $\Rightarrow$ (b) of Theorem 4.

THEOREM 5.4. *Let H be analytic in $\mathbb{D}$ with $\operatorname{Re} H(z) > 0$ for $z \in \mathbb{D}$. Let σ be the corresponding measure on $\mathbb{T}$ for which*

$$\operatorname{Re} H(z) = \int_{\mathbb{T}} P(z, \xi)\, d\sigma(\xi).$$

Assume that

$$\frac{(1 - |z|^2)|H'(z)|}{\operatorname{Re} H(z)} \leqslant \alpha(1 - |z|)$$

for all $z \in \mathbb{D}$, where α is a positive increasing function on $(0, \pi]$, with $\alpha(0^+) = 0$. Then

$$|\sigma(J) - \sigma(J')| < C\alpha(\pi|J|)\sigma(J),$$

for any sufficiently small arc J of the unit circle.

Proof. We will use the following result due to N. G. Makarov. Given an arc J of the unit circle, denote by z_J the point $\tau(0)$, equidistant from the ends of J, where τ is the automorphism of the unit disc mapping the arc $\mathbb{T} \cap \{\operatorname{Re} z > 0\}$ onto J. Also, denote the domain $\tau(\{z \in \mathbb{D}\colon \operatorname{Re} z > 0\})$ by $\Delta(J)$.

LEMMA [**15**, p. 6]. *Let b be an analytic function in $\overline{\mathbb{D}}$, and J an arc of $\mathbb{T}$, and assume that*

$$(1 - |z|^2)|b'(z)| \leqslant \alpha \quad \text{for } z \in \Delta(J),$$

for some $\alpha < 2$. Then

$$\left| \frac{1}{|J|} \int_J [\exp(b(\xi) - b(z_J)) - 1] \frac{|d\xi|}{2\pi} \right| \leqslant C(\alpha).$$

Considering $H_r(z) = H(rz)$, with $r < 1$, we may assume that H is analytic in a neighbourhood of the unit disc. Given an arc J of the unit circle, replacing H by $H - i \operatorname{Im} H(z_J)$, we also may assume that $H(z_J) > 0$. Observe that the function $b = \log H$ satisfies

$$(1 - |z|^2)|b'(z)| \leqslant \alpha(1 - |z|).$$

Since $1 - |z_J| \leqslant \pi|J|$, we obtain

$$\left| \frac{1}{|J|} \int_J \operatorname{Re} H(\xi) \frac{|d\xi|}{2\pi} - \operatorname{Re} H(z_J) \right| \leqslant C\alpha(\pi|J|) \operatorname{Re} H(z_J).$$

Hence,

$$\left| \frac{\sigma(J)}{|J|} - \operatorname{Re} H(z_J) \right| \leqslant C\alpha(\pi|J|) \operatorname{Re} H(z_J).$$

Since

$$|\operatorname{Re} H(z_J) - \operatorname{Re} H(z_J')| \leqslant C_2 \alpha(\pi|J|) \operatorname{Re} H(z_J),$$

we deduce that

$$|\sigma(J) - \sigma(J')| \leqslant C_3 \alpha(\pi|J|)\sigma(J).$$

Theorem 5.2 follows from the following refined version of (b) ⇒ (a) of Theorem 4.

THEOREM 5.5. *Let σ be a positive measure of the unit circle. Assume that*

$$|\sigma(J) - \sigma(J')| \leqslant \alpha(|J|)\sigma(J),$$

for any arc J of the unit circle, where α is a positive increasing function on $(0, 1]$, $\alpha(0^+) = 0$. Then, the function

$$H(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\sigma(\xi)$$

satisfies

$$\frac{(1 - |z|^2)|H'(z)|}{\operatorname{Re} H(z)} \leqslant C\widetilde{\alpha}(1 - |z|),$$

for all $z \in \mathbb{D}$, where

$$\widetilde{\alpha}(t) = t \int_t^1 \frac{\alpha(s)}{s^2} \, ds + t\alpha(1).$$

REMARK. Observe that $\widetilde{\alpha}(t) \geqslant \alpha(t)$, for $0 < t < 1$, and $\widetilde{\alpha} \leqslant C\alpha$ if $\alpha(t)/t^{1-\varepsilon}$ is decreasing for some positive ε.

Proof of Theorem 5.2. By the Carleson Theorem, when $w(t)/t^{1/2}$ decreases, or applying Theorem 6.3 observing that $\widetilde{w}(t)/t$ decreases, we see that there exists a positive singular measure σ on $\mathbb{T}$ such that

$$|\sigma(J) - \sigma(J')| \leqslant Cw(|J|)\sigma(J),$$

for any arc J of the unit circle.

Thus, Theorem 5.5 gives

$$(1 - |z|^2) \frac{|H'(z)|}{\operatorname{Re} H(z)} \leqslant C_1 \widetilde{w}(1 - |z|) \leqslant C_2 w(1 - |z|),$$

for all $z \in \mathbb{D}$. So, one can choose $I = (H - 1)(H + 1)^{-1}$ or $I = \exp(-H)$.

Proof of Theorem 5.5. Let J and Δ be arcs of the unit circle, with $J \subset \Delta$. L. Carleson observed in [**5**, Lemma 4] that if $\alpha(\Delta) < \frac{1}{2}$, one has

$$\left| \frac{\sigma(J)}{\sigma(\Delta)} - \frac{|J|}{|\Delta|} \right| \leqslant C\alpha(\tfrac{1}{2}|\Delta|),$$

where C is an absolute constant. Actually, if α increases, then the argument of L. Carleson shows that $C = 1$. We need more information on the measure σ.

LEMMA 5.6. *Assume that the measure* σ *and the function* α *satisfy the conditions of Theorem* 5.5. *Let* J *and* Δ *be arcs of the unit circle, with* $J \subset \Delta$, $|\Delta| \geq 2|J|$ *and* $\alpha(\Delta) < \frac{1}{8}$. *Then,*

$$\frac{\sigma(J)}{|J|}\exp\left(-\int_{|J|}^{|\Delta|}\frac{4\alpha(t)}{t}\,dt\right) \leq \frac{\sigma(\Delta)}{|\Delta|} \leq \frac{\sigma(J)}{|J|}\exp\left(\int_{|J|}^{|\Delta|}\frac{4\alpha(t)}{t}\,dt\right).$$

Proof. Choose a natural number n such that $2^n|J| < |\Delta| \leq 2^{n+1}|J|$, and arcs $J \subset K_0 \subset K_1 \subset \ldots \subset K_n = \Delta$, with $|K_{i+1}| = 2|K_i|$, for $i = 0, \ldots, n-1$, and $|K_0| \leq 2|J|$. Then for $i = 0, \ldots, n-1$ we have

$$\frac{\sigma(K_i)}{|K_i|}(1+\tfrac{1}{2}\alpha(|K_i|))^{-1} \leq \frac{\sigma(K_{i+1})}{|K_{i+1}|} \leq \frac{\sigma(K_i)}{|K_i|}(1+\tfrac{1}{2}\alpha(|K_i|))$$

and

$$\frac{\sigma(J)}{|J|}(1+2\alpha(|J|))^{-1} \leq \frac{\sigma(K_0)}{|K_0|} \leq \frac{\sigma(J)}{|J|}(1+\tfrac{17}{8}\alpha(|J|)).$$

Since,

$$1+\tfrac{1}{2}\alpha(|K_i|) \leq \exp\left(\int_{|K_i|}^{2|K_i|}\frac{\alpha(t)}{t}\,\frac{dt}{2\log 2}\right)$$

and

$$1+\tfrac{17}{8}\alpha(|J|) \leq \exp\left(\int_{|J|}^{2|J|}\frac{17\alpha(t)}{8(\log 2)t}\,dt\right),$$

the lemma follows.

The following result follows from Lemma 5.6.

LEMMA 5.7. *Under the assumptions of Lemma* 5.6, *one has*

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(\Delta)}{|\Delta|}\right| \leq \min\left\{\frac{\sigma(J)}{|J|}, \frac{\sigma(\Delta)}{|\Delta|}\right\}\left[\exp\left(\int_{|J|}^{|\Delta|}\frac{4\alpha(t)}{t}\,dt\right) - 1\right].$$

As in Theorem 4, to prove Theorem 5.5 it is sufficient to show the following estimate:

$$\int_{\mathbb{T}}\frac{\bar{\xi}(1-|z|^2)}{(1-\bar{\xi}z)^2}\,d\sigma(\xi) \leq C\tilde{\alpha}(|J|)\,\frac{\sigma(J)}{|J|},$$

where $J = J(z)$, for all $z \in \mathbb{D}$. Consequently, it is sufficient to prove that

$$\int_{\mathbb{T}}\frac{\bar{\xi}\,d\sigma(\xi)}{(1-\bar{\xi}z)^2} \leq C\tilde{\alpha}(|J|)\,\frac{\sigma(J)}{|J|^2}, \tag{5.4}$$

for all $z \in \mathbb{D}$. Consider the (signed) measure $\mu = \sigma - (2\pi)^{-1}|J|^{-1}\sigma(J)\,|d\xi|$. It is clear that

$$\int_{\mathbb{T}}\frac{\bar{\xi}\,d\sigma(\xi)}{(1-\bar{\xi}z)^2} = \int_{\mathbb{T}}\frac{\bar{\xi}\,d\mu(\xi)}{(1-\bar{\xi}z)^2}.$$

An integration by parts shows that the last integral is bounded by a multiple of

$$|\mu|(\mathbb{T}) + |J|^{-2}\int_0^{1/|J|}\min\{1, s^{-3}\}(|\mu((sJ)_+)| + |\mu((sJ)_-)|)\,ds.$$

Here if $z = re^{it}$, $(sJ)_+$, $(sJ)_-$ denote, respectively, the arcs,

$$(sJ)_+ = \{e^{i(t+\varphi)}\colon 0 \leqslant \varphi \leqslant \pi s(1-|z|)\}, \quad (sJ)_- = \{e^{i(t-\varphi)}\colon 0 \leqslant \varphi \leqslant \pi s(1-|z|)\}.$$

Hence (5.4) will follow if we prove the following two estimates:

$$|\mu|(\mathbb{T}) \leqslant C\widetilde{\alpha}(|J|)\,\frac{\sigma(J)}{|J|^2}, \tag{5.5}$$

$$\int_0^{1/|J|}\min\{1, s^{-3}\}|\mu((sJ)_+)|\,ds \leqslant C\widetilde{\alpha}(|J|)\sigma(J). \tag{5.6}$$

Since $|\mu|(\mathbb{T}) \leqslant \sigma(\mathbb{T}) + \sigma(J)/|J|$, (5.5) follows from the fact that

$$\inf_J\left\{\frac{\alpha(|J|)\sigma(J)}{|J|^2}\right\} > 0.$$

Actually, by Lemma 5.6, one has

$$\frac{\sigma(J)}{|J|} \geqslant C_1\exp\left(-\int_{|J|}^1\frac{4\alpha(t)}{t}\,dt\right) \geqslant C_2\,\frac{|J|}{\widetilde{\alpha}(|J|)}$$

because

$$\liminf_{t\to 0}\frac{\int_t^1\alpha(s)\,ds/s^2}{\exp\left(\int_t^1 4\alpha(s)\,ds/s\right)} > 0,$$

as a simple calculation shows.

Now let us prove (5.6). One can assume that $|J|$ is small. Observe that $\mu((sJ)_+) = \sigma((sJ)_+) - \frac{1}{2}s\sigma(J)$. Thus, for $0 < s < 1$, Lemma 5.7 gives

$$\begin{aligned}|\mu((sJ)_+)| &\leqslant |\sigma((sJ)_+) - s\sigma(J_+)| + s|\sigma(J_+) - \tfrac{1}{2}\sigma(J)|\\ &\leqslant Cs\sigma(J)\left[\exp\left(\int_{s|J|/2}^{|J|}\frac{4\alpha(u)}{u}\,du\right) - 1\right]\\ &\leqslant Cs\sigma(J)((2/s)^{4\alpha(|J|)} - 1).\end{aligned}$$

Consequently,

$$\int_0^1|\mu((sJ)_+)|\,ds \leqslant 3C\alpha(|J|)\sigma(J).$$

Also, using Lemma 5.7, for $1 < s < 2$ one has

$$\begin{aligned}|\mu((sJ)_+)| &\leqslant |\sigma((sJ)_+) - s\sigma(J_+)| + s|\sigma(J_+) - \tfrac{1}{2}\sigma(J)|\\ &\leqslant 4C\alpha(|J|)\sigma(J)\end{aligned}$$

and

$$\int_1^2|\mu((sJ)_+)|\,ds \leqslant 4C\alpha(|J|)\sigma(J).$$

Now, for $s > 2$, Lemma 5.7 gives

$$\begin{aligned}|\mu((sJ)_+)| &\leqslant |\sigma((sJ)_+) - s\sigma(J_+)| + s\,|\sigma(J_+) - \tfrac{1}{2}\sigma(J)| \\ &\leqslant s\sigma(J_+)\left[\exp\left(\int_{|J|/2}^{s|J|/2} \frac{4\alpha(t)}{t}\,dt\right) - 1\right] + s\alpha(|J|)\sigma(J) \\ &\leqslant s\sigma(J_+)\left[\exp\left(\int_{|J|}^{s|J|} \frac{4\alpha(t)}{t}\,dt\right) - 1\right] + s\alpha(|J|)\sigma(J).\end{aligned}$$

Set $s_0 = \widetilde{\alpha}(|J|)^{-1}$. Since

$$\int_{|J|}^{s_0|J|} \frac{\alpha(t)}{t}\,dt \leqslant \int_{|J|}^{s_0|J|} \frac{\widetilde{\alpha}(t)}{t}\,dt \leqslant s_0|J|\,\frac{\widetilde{\alpha}(|J|)}{|J|} = 1, \tag{5.7}$$

we deduce that for $2 < s \leqslant s_0$,

$$|\mu((sJ)_+)| \leqslant Cs\sigma(J)\int_{|J|}^{s|J|} \frac{\alpha(t)}{t}\,dt. \tag{5.8}$$

Consequently,

$$\begin{aligned}\int_2^{s_0} |\mu((sJ)_+)|\,s^{-3}\,ds &\leqslant C\sigma(J)\int_2^{\infty} s^{-2}\int_{|J|/2}^{s|J|} \frac{\alpha(t)}{t}\,dt\,ds \\ &\leqslant C\sigma(J)|J|\int_{|J|}^{1} \frac{\alpha(t)}{t^2}\,dt \leqslant C\widetilde{\alpha}(|J|)\sigma(J).\end{aligned}$$

Observe that Lemma 5.6 and estimates (5.7) and (5.8) imply that $\sigma((s_0J)_+) \leqslant Cs_0\,\sigma(J)$. Take $\delta > 0$ such that $\alpha(\delta) \leqslant \frac{1}{8}$. For $s_0 < s < \delta/|J|$, Lemma 5.6 gives

$$\sigma((sJ)_+) \leqslant \frac{2s}{s_0}\,\sigma((2s_0J)_+)\exp\left(\int_{s_0|J|}^{s|J|} \frac{4\alpha(t)}{t}\,dt\right) \leqslant Cs\sigma(J)(2s/s_0)^{1/2}.$$

Consequently,

$$\int_{s_0}^{\delta/|J|} |\mu((sJ)_+)|\,s^{-3}\,ds \leqslant C\,\frac{\sigma(J)}{s_0} = C\widetilde{\alpha}(|J|)\sigma(J).$$

Finally, applying (5.5), one has

$$\int_{\delta/|J|}^{1/|J|} |\mu((sJ)_+)|\,s^{-3}\,ds \leqslant \frac{1}{\delta^2}\left(\sigma(\mathbb{T}) + \frac{\sigma(J)}{|J|}\right)|J|^2 \leqslant \frac{C}{\delta^2}\,\widetilde{\alpha}(|J|)\,\sigma(J).$$

To prove Corollary 4 stated in the introduction we will use the following version of Theorem 2.6.

THEOREM 5.8. *Let I be an inner function satisfying*

$$\frac{(1 - |z|^2)|I'(z)|}{1 - |I(z)|^2} \leqslant \alpha(1 - |z|),$$

for all $z \in \mathbb{D}$, where α is an increasing function on $(0, \pi]$, with $\alpha(0^+) = 0$, such that $\widetilde{\alpha} \leqslant C\alpha$, where $\widetilde{\alpha}$ is defined in Theorem 5.5. Let $h \in L^1(\mathbb{T})$ be a non-negative *function, measurable with respect to the σ-algebra $\mathscr{A}(I)$. Then*

$$\left| \int_J h\,|d\xi| - \int_{J'} h\,|d\xi| \right| \leqslant C\alpha(\pi|J|) \int_J h\,|d\xi|,$$

for any arc J of the unit circle.

Proof. Take $g \in L^1(\mathbb{T})$ such that $h = g \circ I$ and consider

$$G(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, g(\xi)\,|d\xi| \quad \text{for } z \in \mathbb{D}.$$

Observe that

$$\operatorname{Re} G(I(z)) = \int_{\mathbb{T}} P(z, \xi)\, h(\xi)\,|d\xi|.$$

Since $(1 - |z|^2)|G'(z)| \leqslant 2 \operatorname{Re} G(z)$, for all $z \in \mathbb{D}$, one deduces that

$$\frac{(1 - |z|^2)|(G \circ I)'(z)|}{\operatorname{Re} G(I(z))} \leqslant 2\alpha(1 - |z|),$$

for all $z \in \mathbb{D}$. Now, one can apply Theorem 5.4.

Proof of Corollary 4. Assume (b) holds. Consider the function

$$H(z) = \int_E \frac{\xi + z}{\xi - z}\,|d\xi| \quad \text{for } z \in \mathbb{D}.$$

Then $(1 - |z|)|H'(z)| \leqslant C\alpha(1 - |z|)$ for all $z \in \mathbb{D}$ and hence

$$(1 - |z|)|H'(z)|^2 \leqslant C\,\frac{\alpha^2(1 - |z|)}{1 - |z|} \quad \text{for } z \in \mathbb{D}.$$

Now, if (c) does not hold, one would deduce that H has vanishing mean oscillation, which is a contradiction.

Assume (c) holds. Apply Theorem 5.2 to get an inner function I such that

$$\frac{(1 - |z|^2)|I'(z)|}{1 - |I(z)|^2} \leqslant \alpha(1 - |z|) \quad \text{for } z \in \mathbb{D}.$$

Then, for any measurable set J of the unit circle, with $0 < |J| < 1$, let $E = I^{-1}(J)$ be its preimage. Now (a) follows from Theorem 5.8.

Given $f \in H^\infty$, with $\|f\|_\infty \leqslant 1$, consider the family of positive measures $\{\sigma_\alpha \colon \alpha \in \mathbb{T}\}$ given by

$$\operatorname{Re}\left(\frac{\alpha + f(z)}{\alpha - f(z)}\right) = \int_{\mathbb{T}} P(z, \xi)\, d\sigma_\alpha(\xi).$$

Let w be an increasing function on $(0, 1]$, with $w(0^+) = 0$. Assume that for some $\alpha_0 \in \mathbb{T}$, the measure σ_{α_0} satisfies

$$|\sigma_{\alpha_0}(J) - \sigma_{\alpha_0}(J')| \leqslant w(|J|)\,\sigma_{\alpha_0}(J)$$

for any arc J. Then, there exists a constant C such that

$$|\sigma_\alpha(J) - \sigma_\alpha(J')| \leqslant C\widetilde{w}(|J|)\sigma_\alpha(J),$$

for any arc J and for any $\alpha \in \mathbb{T}$. In particular, if $\widetilde{w} \leqslant Cw$, the above condition does not depend on $\alpha \in \mathbb{T}$.

6. *Riesz products*

Another way of constructing a singular symmetric measure is by means of Riesz products. These are defined on $\mathbb{T}$ as the w^*-limit of the measures

$$\prod_{j=1}^{N} (1 + \mathrm{Re}(a_j \xi^{n_j})) \frac{|d\xi|}{2\pi}$$

as $N \to \infty$. Here a_j are complex numbers, $|a_j| \leqslant 1$ for $j = 1, 2, \ldots,$ and the integers n_j satisfy $n_{j+1}/n_j \geqslant 3$. It is well known that the corresponding measure is singular if $\sum_{j=1}^{\infty} |a_j|^2 = \infty$. We refer to [**11**] for information on Riesz products.

THEOREM 6.1. *With the above notation assume* $|a_j| < 1$ *for all* j *and* $\lim_{j \to \infty} a_j = 0$. *Then the measure*

$$\sigma = \lim_{N \to \infty} \prod_{j=1}^{N} (1 + \mathrm{Re}(a_j \xi^{n_j})) \frac{|d\xi|}{2\pi}$$

is symmetric.

Proof. Set

$$F_k(\xi) = \prod_{j=1}^{k} (1 + \mathrm{Re}(a_j \xi^{n_j})), \quad F_1 \equiv 1$$

and

$$f_k(\xi) = \tfrac{1}{2} a_k \xi^{n_k} F_{k-1}(\xi).$$

It is clear that f_k is an analytic polynomial whose non-vanishing Fourier coefficients lie in the interval $[2^{-1} n_k, 2^{-1} 3 n_k]$. Also $F_k - F_{k-1} = f_k + \overline{f}_k$.

If f is a continuous function in the unit circle, set

$$\|f\|_{l^1} = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|,$$

where

$$\widehat{f}(n) = \int_{\mathbb{T}} f(\xi) \overline{\xi}^n \frac{|d\xi|}{2\pi}$$

are the Fourier coefficients.

We have

$$\|f_k\|_{l^1} \leqslant \tfrac{1}{2} |a_k| \prod_{j=1}^{k-1} (1 + |a_j|) \leqslant 2^{k-2} |a_k|. \tag{6.1}$$

LEMMA 6.2. *Let* J *be a closed arc of the unit circle and* $k \in \mathbb{N}$. *Then the following estimates hold:*

$$\frac{\max_J |F_k|}{\min_J |F_k|} \leqslant \exp\left(2\pi |J| \sum_{j=1}^{k} \frac{|a_j| n_j}{1 - |a_j|} \right),$$

$$\left| \int_J F_k^{-1} \, d\sigma - |J| \right| \leqslant \frac{6}{\pi n_{k+1}} \sup_{j \geqslant k+1} |a_j|.$$

Proof. Considering logarithmic derivatives one gets

$$\left|\frac{d}{dt}\log F_k(e^{it})\right| \leqslant \sum_{j=1}^{k} \frac{|a_j|n_j}{1-|a_j|}.$$

Now, an integration proves the first estimate.

Replacing σ by the Riesz product $F_k^{-1}\sigma$, one shows that it is sufficient to prove the second inequality when $k=0$. Let χ_J be the characteristic function of J. Applying the inequality

$$|\widehat{\chi}_J(k)| \leqslant \frac{1}{\pi|k|} \quad \text{with } k \neq 0,$$

and (6.1), one deduces that

$$\begin{aligned} |\sigma(J) - |J|| &\leqslant \sum_{k\neq 0} |\widehat{\sigma}(k)|\,|\widehat{\chi}_J(k)| \\ &\leqslant \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\|f_j\|_{l^1}}{n_j} \leqslant \frac{1}{\pi}\sum_{j=1}^{\infty} \frac{2^j|a_j|}{n_j} \leqslant \frac{6}{\pi n_1}\sup_{j\geqslant 1}|a_j|. \end{aligned}$$

A similar argument can be found in [**16**].

Now, let J be an arc of the unit circle and let ξ be the common end of J and J'. Take k such that $n_{k+1}^{-1} \leqslant |J| < n_k^{-1}$. Applying Lemma 6.2, one has

$$\frac{\sigma(J)}{|J|} = \frac{1}{|J|}\int_J F_k F_k^{-1}\,d\sigma \simeq F_k(\xi).$$

Here $A_k \simeq B_k$ means that $A_k/B_k \to 1$ as $k\to\infty$. Similarly,

$$\sigma(J')/|J'| \simeq F_k(\xi).$$

Hence σ is symmetric.

Assume that (a_j) satisfy the hypothesis of Theorem 6.1 and $\sum|a_j|^2 = \infty$. Let σ be the corresponding singular symmetric measure. Observe that the measures

$$\sigma_t = \prod_{j=1}^{\infty}[1+\operatorname{Re}(e^{it}a_j\xi^{n_j})]\,\frac{|d\xi|}{2\pi}, \quad \text{where } t\in[0,2\pi),$$

are also singular and symmetric. Actually the proof of Theorem 6.1 shows that

$$\lim_{|J|\to 0}\frac{\sigma_t(J)}{\sigma_t(J')} = 1,$$

uniformly in $t\in[0,2\pi)$. Moreover, if $t\neq s$, the measures σ_t and σ_s are mutually singular.

Given a singular symmetric measure σ, we can use our composition process to obtain families of Kahane symmetric measures. If, on the other hand, one attempts to construct a Kahane measure by means of a Riesz product with $n_{j+1}/n_j \geqslant 3$ for all j, then P. Duren showed that $\sum|a_j|^2 < \infty$ so the measure is absolutely continuous [**6**].

Minor modifications of the proof of Theorem 6.1, show that, essentially, the measures constructed by L. Carleson can also be obtained as Riesz products.

THEOREM 6.3. *Let w be a positive increasing function on $[0,1]$ such that $w(t)/t$ is decreasing and*

$$\int_0 \frac{w^2(t)}{t}\, dt = \infty.$$

Then there exists a sequence of non-negative numbers $\{r_k\}$, with $\sum_{k=0}^{\infty} r_k^2 = \infty$, such that for any sequence a_k of complex numbers, $|a_k| \leqslant r_k$ where $k = 0, 1, 2, \ldots$, the measure σ associated with the Riesz product

$$\prod_{j=1}^{\infty} (1 + \mathrm{Re}(a_j \xi^{3^j}))\, \frac{|d\xi|}{2\pi}$$

satisfies

$$\left| \frac{\sigma(J')}{\sigma(J)} - 1 \right| \leqslant w(|J|),$$

for any arc J of the unit circle. Moreover if $|a_k| = r_k$ for $k = 0, 1, 2, \ldots,$ the measure σ is singular.

Proof. We may assume $\lim_{t\to 0} w(t) = 0$. Consider $\varepsilon_k = 20^{-1} w(3^{-k-1})$ with $k \geqslant 0$. The integral condition on w gives

$$\sum_{k=0}^{\infty} \varepsilon_k^2 = \infty.$$

Choose $r_k = \varepsilon_k - 3^{-1}\varepsilon_{k-1}$ with $k \geqslant 1$. Observe that $r_k \geqslant 0$ because $w(t)/t$ decreases. Also, $\sum_{k=1}^{\infty} r_k^2 = \infty$. Let J be an arc of the unit circle, $3^{-k-1} \leqslant |J| < 3^{-k}$. We now use the notation of the proof of Theorem 6.1. There exists a point $\xi_k \in J$ such that

$$\frac{\sigma(J)}{|J|} = \frac{1}{|J|} \int_J F_k F_k^{-1}\, d\sigma = F_k(\xi_k)\, \frac{1}{|J|} \int_J F_k^{-1}\, d\sigma.$$

Now, Lemma 6.2 gives

$$\left| \frac{\sigma(J)}{|J|} - F_k(\xi_k) \right| \leqslant F_k(\xi_j)\, \frac{6}{\pi} \sup_{j \geqslant k+1} |a_j| \leqslant 2\varepsilon_{k+1} F_k(\xi_k).$$

Similarly, there exists $\xi_k' \in J'$ such that

$$\left| \frac{\sigma(J')}{|J'|} - F_k(\xi_k') \right| \leqslant 2\varepsilon_{k+1} F_k(\xi_k').$$

Writing $t = 4\pi |J| \sum_{j=1}^{k} |a_j| 3^j (1 - |a_j|)^{-1}$, we find that the first estimate of Lemma 6.2 gives

$$\begin{aligned} |F_k(\xi_k) - F_k(\xi_k')| &\leqslant F_k(\xi_k)(e^t - 1) \\ &\leqslant 15 F_k(\xi_k) \sum_{j=1}^{k} r_j 3^{j-k} \leqslant 15 \varepsilon_k F_k(\xi_k). \end{aligned}$$

Thus, if k is sufficiently large, one gets

$$|\sigma(J) - \sigma(J')| \leqslant 19 \varepsilon_k F_k(\xi_k) |J| \leqslant 20 \varepsilon_k \sigma(J) \leqslant w(|J|)\, \sigma(J).$$

Replacing r_k by $r'_k = r_{k-N}$, for $k > N$, where N is sufficiently large, and $r'_k = 0$ if $k < N$, we see that the last inequality holds for any arc J of the unit circle.

References

1. A. B. Aleksandrov, 'Multiplicity of boundary values of inner functions', *Izv. Akad. Nauk Arm. SSR* 22 (1987) 490–503 (Russian).
2. A. F. Beardon and Ch. Pommerenke, 'The Poincaré metric of plane domains', *J. London Math. Soc.* (2) 18 (1978) 475–483.
3. C. J. Bishop, 'Bounded functions in the little Bloch space', *Pacific J. Math.* 142 (1990) 209–225.
4. A. Canton, 'Singular measures and the little Bloch space', *Publ. Mat.* 42 (1998) 211–222.
5. L. Carleson, 'On mappings, conformal at the boundary', *J. Anal. Math.* 19 (1967) 1–13.
6. P. L. Duren, 'Smoothness of functions generated by Riesz products', *Proc. Amer. Math. Soc.* 16 (1965) 1263–1268.
7. J. L. Fernández and A. Nicolau, 'Boundary behaviour of inner functions and holomorphic functions', *Math. Ann.* 310 (1998) 423–445.
8. F. P. Gardiner and D. P. Sullivan, 'Symmetric structures on a closed curve', *Amer. J. Math.* 114 (1992) 683–736.
9. J. B. Garnett, *Bounded analytic functions* (Academic Press, New York, 1981).
10. J. Garnett and M. Papadimitrakis, 'Almost isometric maps of the hyperbolic plane', *J. London Math. Soc.* (2) 43 (1991) 269–282.
11. V. Havin and B. Jöricke, *The uncertainty principle in harmonic analysis* (Springer, Berlin, 1994).
12. G. Hungerford, 'Boundary of smooth sets and singular sets of Blaschke products in the little Bloch class', thesis, California Institute of Technology, 1988.
13. J. P. Kahane, 'Trois notes sur les ensembles parfait linéares', *Enseign. Math.* 15 (1969) 185–192.
14. K. Madigan and A. Matheson, 'Compact composition operators on the Bloch space', *Trans. Amer. Math. Soc.* 347 (1995) 2679–2687.
15. N. G. Makarov, 'Probability methods in the theory of conformal mappings', *Leningrad Math. J.* 1 (1990) 1–56.
16. A. Peyriere, 'Études de quelques propriétés des produits de Riesz', *Ann. Inst. Fourier* 25 (1975) 127–169.
17. G. Piranian, 'Two monotonic, singular, uniformly almost smooth functions', *Duke Math. J.* 33 (1966) 255–262.
18. C. Pommerenke, *Boundary behaviour of conformal mappings* (Springer, Berlin, 1994).
19. W. Smith, 'Inner functions in the hyperbolic little Bloch class', *Michigan Math. J.* 45 (1998) 103–114.

St. Petersburg Branch
Steklov Mathematical Institute
Russian Academy of Sciences
Fontanka 27
191011 *St Petersburg*
Russia
E-mail: alex@alex.pdmi.ras.ru

Department of Mathematics
University College
Gower Street
*London WC*1*E* 6*BT*
E-mail: ucahoff@ucl.ac.uk

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 *Bellaterra*
Barcelona
Spain
E-mail: artur@manwe.mat.uab.es