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INTERPOLATING BLASCHKE PRODUCTS GENERATE H^∞

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The algebra of bounded analytic functions on the open unit disc is generated by the set of Blaschke products having simple zeros which form an interpolating sequence.

Let H^∞ be the algebra of bounded analytic functions in the unit disc $\mathbb{D}$ and set

$$||f|| = \sup_{z\in\mathbb{D}} |f(z)|,$$

for $f \in H^\infty$. A *Blaschke product* is an H^∞ function of the form

$$B(z) = \prod_{\nu=1}^{\infty} \frac{-\overline{z_\nu}}{|z_\nu|} \frac{z - z_\nu}{1 - \overline{z_\nu} z}$$

with $\sum(1 - |z_\nu|) < \infty$. In [**5**] D.E. Marshall proved that H^∞ is the closed linear span of the Blaschke products: given $f \in H^\infty$ and $\varepsilon > 0$, there are constants $c_1, \dots, c_n$ and Blaschke products $B_1, \dots, B_n$ such that

$$||f + c_1 B_1 + \cdots + c_n B_n||_\infty < \varepsilon. \tag{1}$$

In fact, Marshall proved that the unit ball of H^∞ is the uniformly closed convex hull of the set of Blaschke products (including $B \equiv 1$).

A Blaschke product $B(z)$ is called an *interpolating Blaschke product* if

$$\inf_\nu (1 - |z_\nu|^2)\, |B'(z_\nu)| = \delta_B > 0, \tag{2}$$

because of the Carleson theorem that (2) holds if and only if every interpolation problem

$$f(z_\nu) = w_\nu, \qquad \nu = 1, 2, \dots ,$$

for $\{w_\nu\} \in l^\infty$, has a solution $f \in H^\infty$. Although the interpolating Blaschke products comprise a small subset of the set of all Blaschke products, they play a central role in the theory of H^∞. See the last three chapters of [**3**]. The theorem in this paper helps explain why interpolating Blaschke products are so important in that theory.

Theorem. *H^∞ is the closed linear span of the interpolating Blaschke products.*

In other words, (1) is true with the additional proviso that each of $B_1, \dots ,$ B_n is an interpolating Blaschke product.

The theorem solves a problem posed in [**3**] and [**4**]. It is not known if the set of interpolating Blaschke products is norm dense in the set of all Blaschke products. It is also not known if the unit ball of H^∞ is the closed convex hull of the set of all interpolating Blaschke products.

Recently, Marshall and A. Stray [**6**] proved the theorem in the special case that f extends continuously to almost every point of $\partial\mathbb{D}$, and our proof closely follows their reasoning. In particular, the idea of comparing (11) and (12) and the argument deriving the theorem from Lemma 3 below are both due to them. We thank Violant Marti for making the drawings.

The *hyperbolic distance* between $z \in \mathbb{D}$ and $w \in \mathbb{D}$ is

$$\rho(z,w) = \log\left(\frac{1+\left|\dfrac{z-w}{1-\overline{w}z}\right|}{1-\left|\dfrac{z-w}{1-\overline{w}z}\right|}\right),$$

and the *hyperbolic derivative* of an analytic function f is

$$(1-|z|^2)\,|f'(z)|.$$

The hyperbolic derivative is invariant under conformal changes in $z \in \mathbb{D}$.

The Blaschke product with zeros $\{z_\nu\}$ is an interpolating Blaschke product if and only if the following conditions both hold:

$$\inf_{\nu\neq\mu} \rho(z_\mu, z_\nu) > 0 \tag{3}$$

and

$$\sum_{z_\nu\in Q} (1-|z_\nu|) < C\ell(Q) \tag{4}$$

for all $Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q),\, 1-\ell(Q) < r < 1\}$. See [**1**] or Chapter VII of [**3**].

Lemma 1. *Let B be a Blaschke product and let $\{z_\nu\}$ be its zeros, counted with their multiplicities. Then the following are equivalent:*

(a) *$B = B_1 \dots B_N$, with each B_j an interpolating Blaschke product.*

(b) *Condition* (4) *holds.*

(c) *There exist positive constants ρ_0, δ_0 such that for each z_ν there is w_ν with*

$$\rho(z_\nu, w_\nu) \leq \rho_0 \tag{5}$$

and

$$(1-|w_\nu|^2)\,|B'(w_\nu)| \geq \delta_0. \tag{6}$$

In [**6**] it is shown that if B satisfies one of these conditions, then B is the uniform limit of a sequence of interpolating Blaschke products.

Proof of Lemma 1. The equivalence between (a) and (b) is in [**7**]. Assume (c) holds, let

$$Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q),\, 1-\ell(Q) < r < 1\},$$

and set

$$T(Q) = \{re^{i\theta} \in Q : 1-\ell(Q) < r < 1 - 2^{-1}\ell(Q)\}.$$

To prove (4), we may assume there exists $z_\nu \in T(Q)$. Let w_ν satisfy (5) and (6). Then there exists a_ν such that $\rho(a_\nu, z_\nu) < \rho_0$ and $|B(a_\nu)| \geq m = m(\rho_0, \delta_0) > 0$. Then the inequalities

$$\begin{aligned}\log m^{-2} \geq \log|B(a_\nu)|^{-2} &\geq \sum_{z_\mu\in Q} \frac{(1-|z_\mu|^2)\,(1-|a_\nu|^2)}{|1-\overline{a_\nu}z_\mu|^2} \\ &\geq \frac{A(\rho_0,\delta_0)}{\ell(Q)} \sum_{z_\mu\in Q}(1-|z_\mu|)\end{aligned}$$

show that (4) holds.

If (a) holds, there exists $C > 0$ such that

$$|B(z)| \geq C \prod_{j=1}^{N} \inf_{\{B_j(z_\nu)=0\}} \left|\frac{z-z_\nu}{1-\overline{z_\nu}z}\right|.$$

Fix $\delta_0 > 0$. Given z_ν, there exists ζ_ν such that $\rho(z_\nu, \zeta_\nu) \leq \delta_0$ and $|B(\zeta_\nu)| \geq m = m(\delta_0) > 0$, and then the geodesic arc from z_ν to ζ_ν contains a point w_ν at which (6) holds. □

We write $\mathcal{F}$ for the set of finite products of interpolating Blaschke products. By the remark following Lemma 1, it is enough to prove (1) with each $B_j \in \mathcal{F}$, and by Marshall's theorem it is also enough to prove (1) when $f = B$ is a Blaschke product.

Fix a Blaschke product B and let $0 < \alpha < \beta < 1$, $M = 2^K > 1$, and $\delta < 1$ be constants which will be determined later. We may assume $|B(0)| > \beta$. Consider "squares" of the form

$$Q_{n,j} = \{re^{i\theta} : 2\pi j 2^{-n} \leq \theta < 2\pi(j+1)2^{-n};\ 1-2^{-n} \leq r < 1\}$$

and their top halves

$$T(Q_{n,j}) = Q_{n,j} \cap \{re^{i\theta} : 1 - 2^{-n} \le r < 1 - 2^{-n-1}\}.$$

Let $G_1 = \left\{Q_1^{(1)}, Q_2^{(1)}, \dots\right\}$ be the set of maximal $Q_{n,j}$ for which

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

The squares in G_1 have disjoint interiors. Write $S_{p,j}^{(1)}$, $1 \le p \le M = 2^K$, for 2^K different $Q_{n+K,j} \subset Q_k^{(1)} = Q_{n,j}$. If M is fixed and $1-\beta$ is small, then by Harnack's inequality

$$\sup_{T\left(S_{p,j}^{(1)}\right)} |B(z)| < \beta. \tag{7}$$

Now let $H_1 = \left\{V_1^{(1)}, V_2^{(1)}, \dots\right\}$ be the set of maximal $Q_{n,j}$ such that

$$V_k^{(1)} \subset Q_p^{(1)}$$

for some $Q_p^{(1)}$ and

$$\inf_{T\left(V_k^{(1)}\right)} |B(z)| > \beta.$$

Since $|B|$ has nontangential limit 1 almost everywhere,

$$\sum_{V_k^{(1)} \subset Q_p^{(1)}} \ell\left(V_k^{(1)}\right) = \ell\left(Q_p^{(1)}\right).$$

If $(1-\beta)/(1-\alpha)$ is small, then

$$l\left(V_k^{(1)}\right) < \frac{1}{M} l\left(Q_p^{(1)}\right)$$

when $V_k^{(1)} \subset Q_p^{(1)}$, again by Harnack's inequality. Hence $V_k^{(1)} \subset S_{p,j}^{(1)}$, for some p, j, because of (7).

Next let $G_2 = \left\{Q_1^{(2)}, Q_2^{(2)}, \dots\right\}$ be the set of maximal $Q_{n,j}$ such that

$$Q_{n,j} \subset V_k^{(1)} \in H_1$$

and

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

If $(1-\beta)/(1-\alpha)$ is small, then

$$\sum_{Q_j^{(2)} \subset V_k^{(1)}} \ell\left(Q_j^{(2)}\right) < \varepsilon \ell\left(V_k^{(1)}\right) \tag{8}$$

(see [**3**, p. 334]).

We form the $S_{p,k}^{(2)}$ as before and continue, obtaining $Q_j^{(m)}$, $S_{p,j}^{(m)}$ and $V_k^{(m+1)}$ with

$$Q_j^{(m)} \supset S_{p,j}^{(m)} \supset V_k^{(m+1)}.$$

See Figure 1. Then $B(z)$ has zeros only in

$$\bigcup_{m,j} \left(Q_j^{(m)} \Big\backslash \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)} \right).$$

In fact, if $1-\alpha$ is small enough, all zeros from

$$Q_j^{(m)} \Big\backslash \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)}$$

fall into

$$\bigcup_{p=1}^{M} R_{p,j}^{(m)} = \bigcup_{p=1}^{M} \left(S_{p,j}^{(m)} \Big\backslash \bigcup_{V_k^{(m+1)} \subset S_{p,j}^{(m)}} V_k^{(m+1)} \right),$$

and we require $1-\alpha$ to be that small.

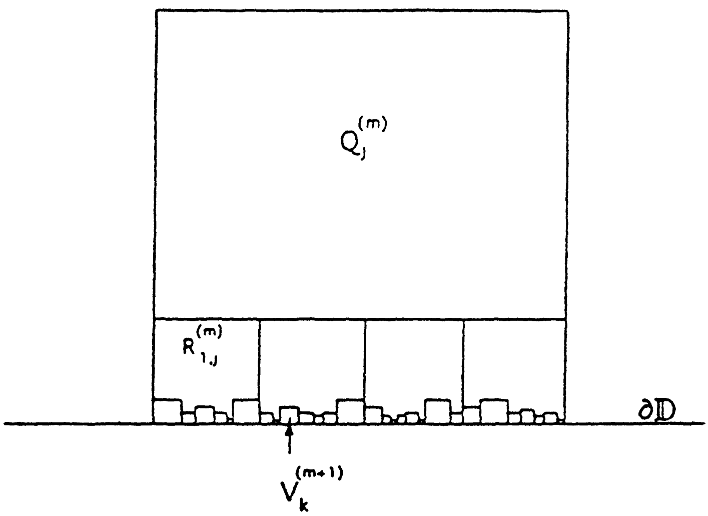

Figure 1.

Now factor

$$B = B_1 B_2 \cdots B_M,$$

where for fixed p, B_p has zeros only in $\bigcup_{m,j} R_{p,j}^{(m)}$. Fix p, set

$$\Gamma_{p,j}^{(m)} = \partial R_{p,j}^{(m)} \Big\backslash \partial S_{p,j}^{(m)} ,$$

and mark points $z_\nu^* = z_\nu^*(m,p,j)$ on $\Gamma_{p,j}^{(m)}$ with

$$\rho(z_\nu^*, z_{\nu+1}^*) = \delta. \tag{9}$$

Let B_p^* be the Blaschke product with zeros $\bigcup_{m,j} z_\nu^*(m,p,j)$. Then by (3), (4), (8) and (9), B_p^* is an interpolating Blaschke product.

Lemma 2. *$|B_p^*| \le \delta^{1/4}$ on $\bigcup_{m,j} R_{p,j}^{(m)}$.*

Proof. Clearly $|B_p^*| < \delta$ on $\bigcup_{m,j} \Gamma_{p,j}^{(m)}$. Fix one $R_{p,j}^{(m)}$. Then for any $\varepsilon > 0$, the harmonic measure

$$\omega\left(z, \Gamma_{p,j}^{(m)}, \mathbb{D} \Big\backslash \bigcup \overline{\left\{V_k^{(m+1)} \subset S_{p,j}^{(m)}\right\}}\right) > \frac{1}{4} - \varepsilon$$

for all $z \in R_{p,j}^{(m)}$, provided $(1-\beta)/(1-\alpha)$ is small. Since $\log|B_p^*(z)|$ is harmonic, that shows $|B_p^*| \le \delta^{1/4}$ on $R_{p,j}^{(m)}$. □

Lemma 3. *There exist $A = A(\alpha,\beta,\delta,M)$ and $\eta = \eta(\alpha,\beta,\delta,M) > 0$ so that if*

$$\inf_{\xi \in \bigcup_{m,j} R_{p,j}^{(m)}} \rho(z,\xi) > A \tag{10}$$

and if

$$|B_p B_p^*(z)| = \delta^{1/8},$$

then

$$(1-|z|^2)\left|(B_p B_p^*)'(z)\right| \ge \eta.$$

Proof. We have

$$\frac{1}{4}\log\frac{1}{\delta} = \log|B_p B_p^*(z)|^{-2} \sim \sum_\nu \frac{(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu} z|^2}, \tag{11}$$

where $\{z_\nu\}$ is the zero set of $B_p B_p^*$. On the other hand,

$$(1-|z|^2)\frac{(B_p B_p^*)'(z)}{B_p B_p^*(z)} = \overline{z}\sum_\nu \frac{(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu} z|^2}\left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu}\right). \tag{12}$$

By (10) there is A' so that if $|z - z_\nu| < A'(1 - |z|)$, then $z_\nu \in R^{(m)}_{p,j}$ where $\ell\left(S^{(m)}_{p,j}\right) < 1 - |z|$. See Figure 2.

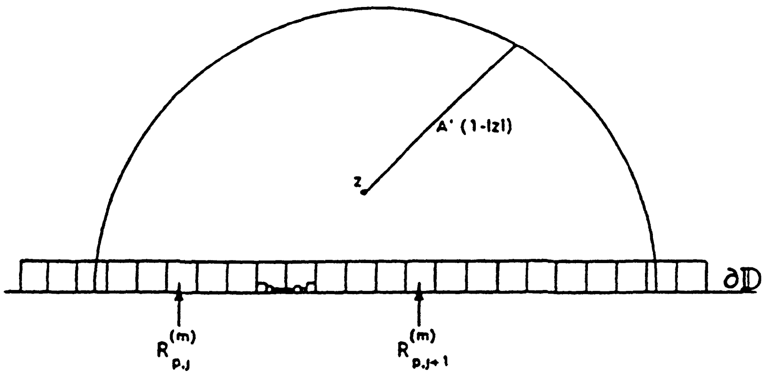

Figure 2.

If $(1 - \alpha)$ is small compared to $1/M$, then $\inf_{T\left(S^{(m)}_{p,j}\right)} |B(z)| \geq C(\alpha) > 0$ and

$$\sum_{\left\{z_n \in R^{(m)}_{p,j};\, B(z_n)=0\right\}} (1 - |z|^2) \leq C_1(\alpha)\ell\left(S^{(m)}_{p,j}\right),$$

where $C_1(\alpha)$ tends to 0 if α tends to 1. Therefore

$$\begin{aligned}\sum_{|z_\nu - z| < A'(1-|z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \overline{z_\nu} z|^2} &\leq \frac{1}{1 - |z|^2} \sum_{|z_\nu - z| < A'(1-|z|)} (1 - |z_\nu|^2) \\ &\leq \frac{1}{\delta M}(1 + \varepsilon + \varepsilon^2 + \cdots) \\ &\quad + \frac{C_1(\alpha)}{M}(1 + \varepsilon + \varepsilon^2 + \cdots).\end{aligned}$$

Take M so large (and consequently $1 - \alpha$ so small) that

$$\sum_{|z_\nu - z| < A'(1-|z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \overline{z_\nu} z|^2} < \frac{1}{16} \log \frac{1}{\delta}.$$

If $|z - z_\nu| > A'(1 - |z|)$ then

$$\left| \arg \left(\frac{\frac{1}{\bar{z}} - z_\nu}{z - z_\nu} \right) \right| < c(A')$$

where $c(A') \to 0$ as $A' \to \infty$. Hence

$$\left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu}z|^2} \left(\frac{\frac{1}{z}-z_\nu}{z-z_\nu} \right) \right|$$
$$\geq \cos^{-1}(c(A')) \sum_{|z-z_\nu| \geq A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu}z|^2} \left(\frac{\frac{1}{z}-z_\nu}{z-z_\nu} \right) \right|.$$

Consequently,

$$(1-|z|^2) \left| (B_p B_p^*)'(z) \right|$$
$$\geq \delta^{1/8} \left(\left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu}z|^2} \left(\frac{\frac{1}{z}-z_\nu}{z-z_\nu} \right) \right| \right.$$
$$\left. - \sum_{|z-z_\nu| < A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\overline{z_\nu}z|^2} \left(\frac{\frac{1}{z}-z_\nu}{z-z_\nu} \right) \right| \right)$$
$$\geq \delta^{1/8} \left(\cos^{-1}(c(A')) \frac{11}{16} \log(1/\delta) - \frac{1}{16} \log(1/\delta) \right),$$

and if A' is large, that proves the lemma. □

With Lemma 3, the remainder of the proof is just like in the Marshall-Stray paper [**6**]. There is γ, $|\gamma| = \delta^{1/8}$, so that

$$\frac{B_p B_p^* - \gamma}{1 - \bar{\gamma} B_p B_p^*} = C_p$$

is a Blaschke product, by a theorem of Frostman [**2**]. Suppose $C_p(z) = 0$. Then

$$|B_p B_p^*(z)| = \delta^{1/8}$$

and

$$(1-|z|^2) \left| C_p'(z) \right| = \frac{(1-|z|^2)}{1-|\gamma|^2} \left| (B_p B_p^*)'(z) \right|.$$

Thus by Lemma 3

$$(1-|z|^2) \left| C_p'(z) \right| \geq \frac{\eta}{1-\delta^{1/4}}$$

if (10) holds. But if (10) fails, then there is $\xi \in \bigcup_{m,j} R_{p,j}^{(m)}$ with $\rho(z, \xi) < A$. By Lemma 2, $|B_p B_p^*(\xi)| \leq \delta^{1/4}$. Somewhere along the hyperbolic geodesic from z to ξ there is a point w with

$$(1-|w|^2) \left| (B_p B_p^*)'(w) \right| > \eta' > 0$$

and $\rho(z,w) < A$. So by Lemma 1, C_p is a finite product of interpolating Blaschke products and $B_pB_p^* \in \mathcal{F}$.

For σ very small, replace B_p^* by

$$\widetilde{B}_p^* = \frac{B_p^* - \sigma}{1 - \overline{\sigma}B_p^*},$$

which is again an interpolating Blaschke product by [**3**, p. 404]. Repeating the above argument with $\widetilde{B}_p^*$, we see that

$$\widetilde{C}_p = \frac{B_p\widetilde{B}_p^* - \widetilde{\gamma}}{1 - \overline{\widetilde{\gamma}}B_p\widetilde{B}_p^*}$$

is also a finite product of interpolating Blaschke products for some $\widetilde{\gamma}$. Thus also $B_p\widetilde{B}_p^* \in \mathcal{F}$. But then since

$$B_p\widetilde{B}_p^* = -\sigma B_p + \left(1 - |\sigma|^2\right) B_pB_p^* + \cdots ,$$

we conclude that $B_p \in \mathcal{F}$. □

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