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INTERPOLATING SEQUENCES FOR ANALYTIC SELFMAPPINGS OF THE DISC

By Pere Menal Ferrer, Nacho Monreal Galán, and Artur Nicolau

Abstract. Schwarz's Lemma leads to a natural interpolation problem for analytic functions from the disc into itself. The corresponding interpolating sequences are geometrically described in terms of a certain hyperbolic density.

1. Introduction. Let H^∞ be the algebra of bounded analytic functions in the unit disc $\mathbb{D}$ of the complex plane. Let

$$\mathcal{B} = \left\{ f \in \mathsf{H}^\infty \colon \ \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \leq 1 \right\}$$

be its closed unit ball. Given two sets of points $\{z_1, \ldots, z_N\}$ and $\{w_1, \ldots, w_N\}$ in the unit disc, the Nevanlinna-Pick interpolation problem consists in finding $f \in \mathcal{B}$ with $f(z_n) = w_n$, $n = 1, \ldots, N$. Nevanlinna and Pick independently proved that the interpolation problem has a solution if and only if the matrix

$$\left(\frac{1 - w_i \overline{w}_j}{1 - z_i \overline{z}_j} \right)_{i,j=1,\ldots,N}$$

is positive semidefinite ([13], [14]). This is a very nice result which is the root of a very active research area (see for instance the book by J. Agler and J. McCarthy [1] and the references therein), with connections with other topics. However, in some concrete situations, as the one we will present, it is not easy to verify the matrix condition and one needs to use more direct methods.

Let $\beta(z, w)$ be the hyperbolic distance between two points $z, w \in \mathbb{D}$. A sequence of points $\{z_n\}$ in the unit disc is called an interpolating sequence if for any bounded sequence of values $\{w_n\}$ there exists a function $f \in \mathsf{H}^\infty$ with $f(z_n) = w_n$, $n = 1, 2, \ldots$. A celebrated result of L. Carleson [5] asserts that $\{z_n\}$ is an interpolating sequence if and only if $\{z_n\}$ is a separated sequence and there exists a

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constant $M > 0$ such that

$$\sum_{z_n \in Q} (1 - |z_n|) \leq M\ell(Q)$$

for any Carleson box Q, that is, a box Q of the form

$$Q = \left\{ re^{i\theta} \colon\ 0 < 1 - r < \ell(Q),\ |\theta - \theta_0| < \ell(Q) \right\}. \tag{1}$$

A sequence of points $\{z_n\}$ in the unit disc is called a separated sequence if $\inf_{n \neq m} \beta(z_n, z_m) > 0$. A standard application of the open mapping theorem tells that whenever $\{z_n\}$ is an interpolating sequence there exists a constant $C = C(\{z_n\}) > 1$ with the following property: for any bounded sequence $\{w_n\}$ there exists $f \in \mathsf{H}^\infty$, such that $f(z_n) = w_n$ for $n = 1, 2, \ldots$ with $\|f\|_\infty \leq C \sup_n |w_n|$.

The main purpose of this paper is to consider a situation which is intermediate between these two classical results. On one hand, as in Carleson's Theorem, we want to interpolate a concrete and natural target space of values $\{w_n\}$. On the other, as in the Nevanlinna-Pick interpolation problem, we want to do it by functions in the unit ball $\mathcal{B}$ of H^∞.

Our discussion starts with Schwarz's Lemma (see for example [8, p. 1]). If $f \in \mathcal{B}$ the classical Schwarz's Lemma tells that

$$\beta(f(z), f(w)) \leq \beta(z, w), \quad \text{for any } z, w \in \mathbb{D}.$$

So, for any sequence of points $\{z_n\}$ in the unit disc, the corresponding values $w_n = f(z_n)$, $n = 1, 2, \ldots$, satisfy $\beta(w_n, w_m) \leq \beta(z_n, z_m)$, for $n, m = 1, 2 \ldots$. However, given a sequence of points $\{z_n\} \subset \mathbb{D}$ we can not expect to interpolate any sequence of values $\{w_n\}$ satisfying the above compatibility condition unless $\{z_n\}$ reduces to two points. Actually having equality in Schwarz's Lemma for two different points forces the function $f \in \mathcal{B}$ to be an automorphism and hence we can not expect to interpolate any further value. In other words, the trace space arising from Schwarz's Lemma is too large and we are lead to the following notion.

A sequence of distinct points $\{z_n\}$ in the unit disc will be called an *interpolating sequence for* $\mathcal{B}$ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ such that for any sequence of values $\{w_n\} \subset \mathbb{D}$ satisfying the compatibility condition

$$\beta(w_m, w_n) \leq \varepsilon \beta(z_m, z_n), \quad n, m = 1, 2, \ldots, \tag{2}$$

there exists a function $f \in \mathcal{B}$ such that $f(z_n) = w_n,\ n = 1, 2, \ldots$.

Observe that this notion is conformally invariant, that is, if $\{z_n\}$ is an interpolating sequence for $\mathcal{B}$ then so is $\{\tau(z_n)\}$ for any automorphism τ of the unit disc. Moreover the constant in the definition verifies $\varepsilon(\{\tau(z_n)\}) = \varepsilon(\{z_n\})$.

It is obvious that if a separated sequence is an interpolating sequence for $\mathcal{B}$ it is also an interpolating sequence for H^∞. As we will see, the converse is far from being true.

Let Δ denote a hyperbolic disc in $\mathbb{D}$, that is, $\Delta = \{w\colon\ \beta(w,z) < \rho\}$ for some $z \in \mathbb{D}$ and $\rho > 0$. Let also $A_h(\Delta)$ denote the hyperbolic area of Δ. The main result of the paper is the following geometric description of interpolating sequences for $\mathcal{B}$.

THEOREM 1. *A sequence $\{z_n\}$ of distinct points in the unit disc is an interpolating sequence for $\mathcal{B}$ if and only if the following two conditions hold:*

(a) *$\{z_n\}$ is the union of two separated sequences.*

(b) *There exist constants $M > 0$ and $0 < \alpha < 1$ such that for any hyperbolic disc Δ with $A_h(\Delta) \geq M$ we have*

$$\#\{z_k\colon\ z_k \in \Delta\} \leq A_h(\Delta)^\alpha.$$

The density condition (b) is the essential one. Let us discuss its geometrical meaning. If the sequence $\{z_n\}$ was merely separated we would have that there exists a constant $M > 0$ such that for any hyperbolic disc Δ, the estimate

$$\#\{z_k\colon\ z_k \in \Delta\} \leq M(A_h(\Delta) + 1)$$

holds. Hence interpolating sequences for $\mathcal{B}$ are exponentially more sparse than separated sequences. The density condition *b*) can also be stated in the following way: there exist constants $M > 0$ and $0 < \alpha < 1$ such that for any Carleson box of the form (1) we have

$$\#\left\{z_k \in Q\colon\ 2^{-n-1}\ell(Q) < 1 - |z_k| \leq 2^{-n}\ell(Q)\right\} \leq M2^{\alpha n} \tag{3}$$

for any $n = 1, 2, \ldots$.

Let us briefly describe the connection of this result with the characterization of interpolating sequences in the Bloch space. An analytic function f in $\mathbb{D}$ is in the Bloch space if

$$\sup\,(1 - |z|^2)|f'(z)| < \infty,$$

where the supremum is taken over all $z \in \mathbb{D}$. A sequence of points $\{z_n\}$ in the unit disc is an interpolating sequence for the Bloch space if, whenever a corresponding sequence of complex numbers $\{w_n\}$ satisfies

$$|w_n - w_n| \leq C\beta(z_n, z_m), \quad n, m = 1, 2 \ldots$$

for some constant $C > 0$, there exists a function f in the Bloch space solving the interpolation problem $f(z_n) = w_n$, for $n = 1, 2, \ldots$. It was proved in [3] that the conditions in Theorem 1 also characterize interpolating sequences for the Bloch space. The nice book by K. Seip [15] contains a discussion of this result, its relations to interpolation in other spaces, as well as several conditions equivalent to the density condition (b). Although the proof of Theorem 1 contains many of the ideas of [3], it is worth mentioning that we can not deduce our result from the one for Bloch functions. Roughly speaking, euclidean distance and the interpolation constant C in the case of the Bloch space are replaced in this work by hyperbolic distance and the interpolation constant ε in (2). So the problem we treat is analogue to the one in [3]. However the fact that hyperbolic distance now appears in both sides of (2), as well as that we are in a non linear setting, makes the problem more difficult.

Let us now explain the main ideas in the proof. The necessity is proven by taking values $\{w_n\}$ for which $\beta(w_n, w_m)/\beta(z_n, z_m)$ is maximal and applying standard techniques involving the non-tangential maximal function. The proof of the sufficiency is considerably harder. Given a sequence of points $\{z_n\}$ satisfying both conditions (a) and (b) and a sequence of values $\{w_n\} \subset \mathbb{D}$ with $\beta(w_n, w_m) \leq \varepsilon\beta(z_n, z_m)$ we have to find a function $f \in \mathcal{B}$ with $f(z_n) = w_n$. The main step of the proof is the construction of a non-analytic function φ in the unit disc with $\varphi(z_n) = w_n$ such that

$$\int_Q \frac{|\nabla\varphi(z)|}{1 - |\varphi(z)|^2} dA(z) \leq C\varepsilon\ell(Q), \tag{4}$$

for any Carleson box Q of the form (1). Once this is done, standard techniques involving BMO and $\bar{\partial}$-equations provide a solution of the interpolation problem in the unit ball of H^∞. The construction of the function φ is made in two different steps. Using a certain collection of dyadic Carleson boxes, we first construct a non-analytic interpolating function φ_0. It is more convenient to work in the upper half plane $\mathbb{R}^2_+$ than in the unit disc $\mathbb{D}$. Let us assume that $\{z_n\}$ is contained in the unit square $[0, 1)^2$. Let $I^0 = [0, 1)$ be the unit interval and for $n = 1, 2, \ldots$ consider the 2^n dyadic intervals $I^n_j = [(j-1)/2^n, j/2^n)$ with $j = 1, \ldots, 2^n$. Given an interval I of the line, let $Q(I) = \{x+iy \in \mathbb{R}^2_+ \colon\ x \in I, 0 < y \leq |I|\}$ be its associated Carleson box and $T(I) = \{x + iy \in Q(I) \colon\ y > |I|/2\}$ its top part. We will also denote by $z(I)$ the center of $T(I)$, that is, if $I = [a, b)$ then $z(I) = (a + b)/2 + i3(b - a)/4$.

Let $\mathcal{A}$ be the collection of dyadic intervals I such that $T(I) \cap \{z_n\} \neq \emptyset$. For the sake of simplicity, let us assume that $\{z_n\}$ is a separated sequence and in fact, that each $T(I)$ contains at most one single point of the sequence $\{z_n\}$. The construction of the function φ_0 is based on an useful combinatorial result which is proved in [3]. It consists on considering a bigger family of dyadic intervals $\mathcal{G} \supseteq \mathcal{A}$ and the function φ_0 will be constructed by, roughly speaking, moving in the vertical direction at most ε hyperbolic units in each $T(I)$ for $I \in \mathcal{G}$. The family $\mathcal{G}$ verifies two properties that point to two opposite directions. On the one hand the family $\mathcal{G}$ should be large to guarantee that φ_0 reaches the corresponding values w_n in the points $z_n \in T(I)$, $I \in \mathcal{A}$, as soon as these values satisfy the compatibility condition (2). On the other hand, since $\partial_y \varphi_0(z)$ will vanish in $\mathbb{R}^2_+ \setminus \cup T(I)$, where the union is taken over all $I \in \mathcal{G}$, the family $\mathcal{G}$ should be small enough to guarantee

$$\int_Q \frac{|\partial_y \varphi_0(z)|}{1 - |\varphi_0(z)|^2} dA(z) \leq C\varepsilon \ell(Q)$$

for any Carleson box $Q \subset \mathbb{R}^2_+$. However, since there is no control on the jumps of the function φ_0 on the vertical sides of $Q(I)$, $I \in \mathcal{G}$, we can not expect that $\partial_x \varphi_0$ satisfies an analogue estimate. To overcome this difficulty, in the euclidean setting we would produce a smooth interpolating function by averaging the functions φ_t corresponding to different sequences $\{z_n + t\colon\ n = 1, 2, \ldots\}$, $t \in [0, 1)$, that is, by taking

$$\varphi(z) = \int_0^1 \varphi_t(z + t) dt, \quad z \in \mathbb{R}^2_+,$$

where φ_t are constructed as explained above and verify $\varphi_t(z_n + t) = w_n$ (see [15, p. 76]). However, in our hyperbolic setting this does not make sense and the averaging procedure is much more subtle.

Given a point $z \in \mathbb{R}^2_+$, we would like to define $\varphi(z)$ as the *center of mass* of the set of points $\{\varphi_t(z + t)\colon\ t \in [0, 1)\}$, suitably weighted, in such a way that for any pair of points $z, w \in \mathbb{R}^2_+$ the following inequality holds

$$\beta(\varphi(z), \varphi(w)) \leq \int_0^1 \beta(\varphi_t(z + t), \varphi_t(w + t)) dt. \tag{5}$$

This inequality and standard arguments will lead to estimate (4). There are several possible notions of *center of mass* in hyperbolic space, and which one is preferable depends on our specific purpose. Let us discuss briefly two of them. One possibility consists in defining the center of mass of a finite number of point masses inductively. The main difficulty of this approach is to determine the representative mass of the center of the masses (the naive attempt of taking the sum of the masses lead to a definition that depends on the partition used in the inductive step). See [7] for a full exposition of this. With this definition, the center of mass of three points with equal mass coincides with the barycenter of the triangle that they define; this fact can be used to prove that this definition can not yield an estimate of the form of (5). Another approach is to define the center of mass of a finite measure μ as the unique minimum of the function

$$H(x) = \int_{\mathbb{R}^2_+} \beta(x, y)^2 d\mu(y).$$

As we will see in Section 3, with this definition, if we take μ as the pushforward measure of the Lebesgue measure on $[0, 1)$ by the map $t \mapsto \varphi_t(z + t)$, which simply corresponds to assign a weight to the points $\varphi_t(z+t)$, then inequality (5) is satisfied. The proof of this fact is a straightforward generalization of a similar result stated in [11].

The paper is organized as follows. The necessity in Theorem 1 is proved in Section 2. Section 3 is devoted to the construction of the suitable center of mass of a certain measure in the hyperbolic space. Since these notions have been considered in the literature in general spaces of negative curvature, we will present the results in the general context of Hadamard spaces. This construction holds in any space of negative curvature. Section 4 contains the proof of the sufficiency in Theorem 1, and is the most technical part of the paper.

The letters $C, C_1, C_2, \ldots$ will denote absolute constants while $C(\delta)$ will denote a constant depending on δ.

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2. Necessity. We use the following normalization of the hyperbolic distance

$$\beta(z,w) = \log_2 \frac{1 + \left|\frac{z-w}{1-\overline{w}z}\right|}{1 - \left|\frac{z-w}{1-\overline{w}z}\right|},$$

where $z, w \in \mathbb{D}$, because it fits conveniently with the dyadic decomposition of the disc, which will be a basic tool. Let

$$I_j^n = [e^{i\pi(j-1)/2^n}, e^{i\pi j/2^n}), \quad \text{with } n \in \mathbb{N} \text{ and } j = 1, \ldots, 2^n,$$

be the standard collection of dyadic arcs on the unit circle $\mathbb{T}$ so that $|I_j^n| = 2^{-n}$, where $|\cdot|$ is the normalized linear measure on $\mathbb{T}$. Given a dyadic arc $I \subset \mathbb{T}$ the corresponding $Q(I)$ is called a dyadic Carleson square, and $\{Q(I_j^n)\colon j = 1, \ldots, 2^n, n \in \mathbb{N}\}$ is the dyadic decomposition of $\mathbb{D}$. Given a dyadic arc I_j^n we will say that $z(I_j^n) = (1 - |I_j^n|)\exp(i\pi(j+1)/2^{n+1})$ is the center of the Carleson square $Q(I_j^n)$. It is easy to deduce that if I, J are dyadic arcs, $I \subseteq J$ and $|I| = 2^{-k}|J|$, for some $k \in \mathbb{N}$, then $|\beta(z(I), z(J)) - k| \leq C$, where C is a universal constant independent of I, J and k.

First of all let us show the equivalence between condition (b) in Theorem 1 and (3). Assume (b) is verified, by conformal invariance we may take $Q = \mathbb{D}$ in (3). Now

$$A_h(\Delta) = C \int_\Delta \frac{1}{(1-|z|^2)^2} dA(z),$$

where $C > 0$ is a universal constant and $dA(z)$ is the euclidean area measure (see [2]). Then it is easy to show that for any $n = 1, 2, \ldots$ we have that $A_h(D(0, 1 -$

$2^{-n})) = C_1 2^n$, where C_1 is a universal constant. So condition (b) implies that there exists $M_1 > 0$ such that

$$\#\left\{z_k \in \mathbb{D}\colon\ 2^{-n-1} < 1 - |z_k| \le 2^{-n}\right\} \le M_1 2^{\alpha n}.$$

Conversely, assume (3) holds and let Δ be an euclidean disc centered at the origin and with euclidean radius $r < 1$. Then $A_h(\Delta) = C_2(1-r)^{-1}$, where C_2 is an absolute constant. Pick $n \in \mathbb{N}$ such that $2^{-n-1} < 1 - r \le 2^{-n}$, then applying (3) to the sets $\{z_k \in \mathbb{D}\colon\ 2^{-j-1} < 1 - |z_k| \le 2^{-j}\}$ for $j = 0, 1, \ldots, n$ and summing in j we deduce that $\#\{z_k\colon\ z_k \in \Delta\} \le M_2 2^{\alpha n} \le M_3 A_h(\Delta)^{\alpha}$.

With the normalization of the hyperbolic distance given above, we may prove that estimate (3), and equivalently condition (b) in Theorem 1, can be expressed in the following way: There exist constants $M > 0$ and $0 < \alpha < 1$ such that for any $z \in \mathbb{D}$

$$\#\{z_k\colon\ \beta(z, z_k) \le n\} \le M 2^{\alpha n}, \quad n = 1, 2, \ldots \tag{6}$$

2.1. Union of two separated sequences. We start with the easiest part of the necessity which is the separation condition (a) in Theorem 1. This condition appears because our target space is defined in terms of first differences, while Cauchy's formula tells that $(1 - |z|)^n |f^{(n)}(z)| \le C(n)\|f\|_\infty$ for any $z \in \mathbb{D}$, $n = 1, 2, \ldots$. So if three points of the sequence $\{z_n\}$ were in a small hyperbolic disc, the corresponding values should satisfy a more restrictive smoothness condition which could be expressed in terms of second differences. More concretely, we will show that there exists $\delta > 0$ such that any hyperbolic disc of radius δ has at most two points of the sequence. Let z_1, z_2, z_3 be three points of the sequence $\{z_n\}$ with

$$\max\{\beta(z_1, z_2), \beta(z_2, z_3)\} \le \beta(z_1, z_3) = \rho.$$

We will show that ρ is bounded below. By conformal invariance we may assume $z_1 = 0$. Now take the values $w_1 = w_3 = 0$ and $w_2 = \varepsilon\beta(z_1, z_2)$, where $\varepsilon = \varepsilon(\{z_n\})$ is the interpolation constant, that is, the quantifier appearing in (2). It is clear that these values satisfy the compatibility condition (2), so then there exists a function $f \in \mathcal{B}$ with $f(0) = f(z_3) = 0$ and $f(z_2) = \varepsilon\beta(0, z_2)$. Hence there is a point ζ in the radius from 0 to z_2 with $(1 - |\zeta|)|f'(\zeta)| > C_1\varepsilon$. On the other hand since

$$|f(z)| \le |z| \left|\frac{z - z_3}{1 - \bar{z}_3 z}\right|, \quad z \in \mathbb{D},$$

we have that

$$|f'(z)| \le C_2\rho \quad \text{if } |z| < \rho.$$

So we deduce that there exists a constant $C_3 > 0$ such that $\rho \geq C_3\varepsilon$. Hence we have proved that $\{z_n\}$ can be written as $\{z_n\} = \{z_n^{(1)}\} \cup \{z_n^{(2)}\}$, where both sequences are separated.

2.2. Density condition.

2.2.1. Some lemmas. The proof of the density condition (b) is based on the following two auxiliary results. The first one is a convenient version of a well known estimate of the non-tangential maximal function. The second one is an elementary combinatorial statement. Fix $M > 1$. Given a set $E \subseteq \mathbb{D}$, let $\Pi(E) = \Pi_M(E)$ denote the set of points $\xi \in \mathbb{T}$ such that the Stolz angle $\Gamma_M(\xi) = \{z \in \mathbb{D}\colon\ |z - \xi| < M(1 - |z|)\}$ intersects E.

LEMMA 2. *There is a constant $C(M) > 0$ such that for any $f \in \mathcal{B}$ with $f(0) = 0$ and any $\eta > 0$ we have*

$$|\Pi\left(\{z \in \mathbb{D}\colon\ |f(z) - 1| < \eta\}\right)| \leq C\eta.$$

Proof. Consider the function $g = (1 - f)^{-1}$ which maps the disc into the right half plane. Let $\mathscr{M}g$ be the non-tangential maximal function of g, that is

$$\mathscr{M}g(\xi) = \sup\{|g(z)|\colon\ z \in \Gamma(\xi), \xi \in \partial\mathbb{D}\},$$

where $\Gamma(\xi)$ is the Stolz angle with vertex at ξ. Since g has positive real part, then $\mathscr{M}g$ satisfies the weak type estimate

$$|\{\xi \in \mathbb{T}\colon\ |\mathscr{M}g(\xi)| > \lambda\}| \leq \frac{C}{\lambda}$$

for any $\lambda > 0$. Here C is a universal constant independent of g. Since

$$\Pi\left(\{z \in \mathbb{D}\colon\ |f(z) - 1| < \eta\}\right) = \left\{\xi \in \mathbb{T}\colon\ \mathscr{M}g(\xi) > \frac{1}{\eta}\right\}$$

the proof is completed. □

LEMMA 3. *Let $M > 0$ and $0 < \alpha < 1$ be fixed constants. Let $\mathcal{A}$ be a collection of dyadic arcs of the unit circle $\mathbb{T}$. Assume that for any dyadic arc I and any positive integer n at least one of its two halves, which we denote by $\widetilde{I}$, satisfies*

$$\#\{J \in \mathcal{A}\colon\ J \subset \widetilde{I},\ |J| = 2^{-n}|I|\} \leq M2^{n\alpha}. \tag{7}$$

Then for any dyadic arc I and any positive integer n we have

$$\#\{J \in \mathcal{A}\colon\ J \subset I,\ |J| = 2^{-n}|I|\} \leq \frac{2M}{1 - 2^{-\alpha}}2^{n\alpha}.$$

Proof. Fix a dyadic arc I. By hypothesis at least one of its two halves, denoted by I_1, verifies estimate (7). If the other half also verifies the estimate we denote it by I_2 and we stop the decomposition. If not, by hypothesis at least one of its two halves, denoted now by I_2, must verify that

$$\#\{J \in \mathcal{A}\colon\ J \subset I_2,\ |J| = 2^{-n}|I|\} \leq M2^{(n-1)\alpha}.$$

Repeating this process at most n times we cover I by $m \leq n+1$ pairwise disjoint dyadic intervals $\{I_j\colon\ j = 1, \dots, m\}$ with $|I_j| = 2^{-j}|I|$ if $j < m$ and $|I_m| = 2^{1-m}|I|$ if $m \leq n$, or $|I_m| = 2^{-n}|I|$ if $m = n+1$. Also these intervals satisfy that

$$\#\left\{J \in \mathcal{A}\colon\ J \subset I_j,\ |J| = 2^{-n}|I|\right\} \leq 2M2^{(n-j)\alpha}, \quad j = 1, \dots, m.$$

Hence,

$$\#\left\{J \in \mathcal{A}\colon\ J \subset I,\ |J| = 2^{-n}|I|\right\} \ \leq\ 2M \sum_{j=1}^{m} 2^{(n-j)\alpha}. \qquad \square$$

2.2.2. Necessity of the density condition. Let $\{z_n\}$ be an interpolating sequence for $\mathcal{B}$ and let $\varepsilon = \varepsilon(\{z_n\}) > 0$ be the interpolation constant appearing in (2). We first prove that there exist constants $M = M(\varepsilon) > 0$ and $\alpha = \alpha(\varepsilon)$ with $0 < \alpha < 1$ such that for any positive integer number n and any dyadic arc $I \subset \mathbb{T}$ at least one of its two halves, say $\tilde{I}$, satisfies

$$\#\left\{z_k \in Q(\tilde{I})\colon\ n-1 < \beta(z(I), z_k) \leq n\right\} \leq M2^{n\alpha}. \tag{8}$$

By conformal invariance we may assume that $I = \mathbb{T}$ and $z(I) = 0$. Let $\gamma > 0$ be a small number to be fixed later with $\gamma < \varepsilon$. Now let $\mathcal{F}_m = \mathcal{F}_m(n)$, $m = 1, 2$, be the two collections of points of $\{z_n\}$ given by

$$\begin{aligned}
\mathcal{F}_1 &= \left\{z_k\colon\ n-1 < \beta(0, z_k) \leq n,\ |\arg z_k| < \frac{\pi}{2} - 2^{-n\gamma}\right\},\\
\mathcal{F}_2 &= \left\{z_k\colon\ n-1 < \beta(0, z_k) \leq n,\ |\arg z_k - \pi| < \frac{\pi}{2} - 2^{-n\gamma}\right\}.
\end{aligned}$$

See Figure 1. Notice that if $z \in \mathcal{F}_1$ and $\tilde{z} \in \mathcal{F}_2$ we have $\beta(z, \tilde{z}) \geq Cn\gamma$, where C is an absolute constant. Now consider the values $\{w_k\}$ given by

$$\begin{aligned}
w_k &= 1 - 2^{-C\varepsilon\gamma n} \quad \text{if } z_k \in \mathcal{F}_1,\\
w_k &= -1 + 2^{-C\varepsilon\gamma n} \quad \text{if } z_k \in \mathcal{F}_2.
\end{aligned}$$

It is easy to show that the compatibility condition (2) holds, that is, $\beta(w_j, w_k) \leq \varepsilon\beta(z_j, z_k)$ if $z_j, z_k \in \mathcal{F}_1 \cup \mathcal{F}_2$. Hence by hypothesis there exists a function $f \in \mathcal{B}$

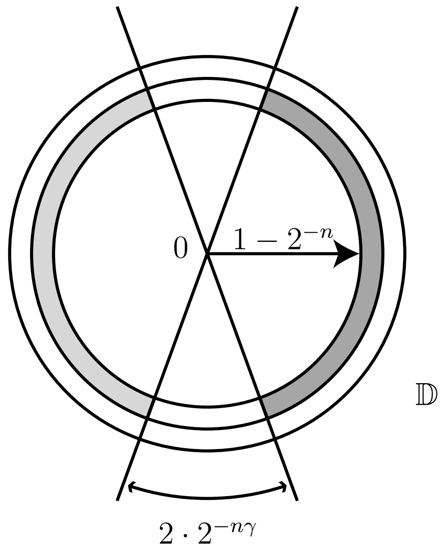

Figure 1. The family $\mathcal{F}_1$ are the points of the sequence in the dark grey zone, while the family $\mathcal{F}_2$ are the ones in the light grey zone.

with $f(z_k) = w_k$ for any $z_k \in \mathcal{F}_1 \cup \mathcal{F}_2$. Notice that there exists at least one index $m = 1, 2$ for which $|\operatorname{Re} f(z_k) - \operatorname{Re} f(0)| > 1 - 2^{-C\varepsilon\gamma n}$ for any $z_k \in \mathcal{F}_m$. So let us assume $m = 1$. Write $w = 1 - 2^{-C\varepsilon\gamma n}$ and let τ be the automorphism of the disc which maps $f(0)$ to the origin and which satisfies $\arg \tau(w) = 0$. Observe that $C_1\varepsilon\gamma n \leq \beta(f(0), w) = \beta(0, \tau(w))$. Now take $h = \tau \circ f$ and notice that

$$\beta(0, h(z_k)) \geq C_1\varepsilon\gamma n, \quad z_k \in \mathcal{F}_1.$$

Since $\arg h(z_k) = 0$, we have

$$|1 - h(z_k)| \leq C_2 2^{-C_1\varepsilon\gamma n}, \quad z_k \in \mathcal{F}_1.$$

Now we can apply Lemma 2 to the function h and the parameter $\eta = C_2 2^{-C_1\varepsilon\gamma n}$ to deduce that

$$|\Pi(\mathcal{F}_1)| \leq C_3 2^{-C_1\varepsilon\gamma n}.$$

Notice that the projection Π of a single point of $\mathcal{F}_1$ is an arc of length comparable to 2^{-n}. Since the sequence $\{z_n\}$ is the union of two separated sequences, the corresponding intervals $\{\Pi(z_n)\}_n$ are quasidisjoint, that is, the function

$$\sum_n \chi_{\Pi(z_n)}(\xi)$$

is bounded on the unit circle. Here χ_E represents the characteristic function of a set E. So we deduce

$$2^{-n}\#\mathcal{F}_1 \leq C_4 2^{-C_1\varepsilon\gamma n}.$$

Moreover there are at most $C_5 2^{n(1-\gamma)}$ points such that $n-1 < \beta(0,z_k) \leq n$ and $\pi/2 - 2^{-n\gamma} \leq |\arg z_k| \leq \pi/2$. So now the estimate (8) follows because the term on the left hand side is bounded by $C_5 2^{n(1-\gamma)} + C_4 2^{n(1-C_1\varepsilon\gamma)}$. We only need to choose $\gamma < 1$ so that $C_1\varepsilon\gamma < 1$ and pick $\alpha = \max\{1-\gamma, 1-C_1\varepsilon\gamma\}$ and $M = 2\max\{C_4, C_5\}$.

We now prove the necessity of the density condition (b). So, given a point $z \in \mathbb{D}$ we wish to estimate the number of points z_k in the sequence $\{z_n\}$ such that $\beta(z_k, z) \leq n$. Equivalently, according to (3), given a Carleson box Q we need to estimate $\#\mathcal{F}_n$, where $\mathcal{F}_n = \{z_k \in Q\colon\ 2^{-n-1}\ell(Q) \leq 1-|z_k| \leq 2^{-n}\ell(Q)\}$. Since any arc of the circle is contained in the union of at most four dyadic arcs of comparable total length, we can assume that $\overline{Q} \cap \mathbb{T}$ is a dyadic arc. Denoting by $\mathcal{A}_n$ the set of the dyadic arcs I of length 2^{-n} such that $Q(I) \cap \mathcal{F}_n \neq \emptyset$, we have $\#\mathcal{F}_n \leq C\#\mathcal{A}_n$, where C is a constant only depending on $\{z_n\}$. An application of Lemma 3 will provide an estimate of $\#\mathcal{A}_n$. For this, we only have to observe that estimate (8) gives the hypothesis (7) in Lemma 3. So estimate (3) is proved.

3. Center of mass. Let μ be a probability measure on $\mathbb{R}^n$ with bounded support. We may define the center of mass of μ as the unique point in $\mathbb{R}^n$ that minimizes the function

$$H_\mu(x) = \int_{\mathbb{R}^n} \|x-y\|^2 d\mu(y).$$

The fact that guarantees the existence and uniqueness of that minimum is the strong convexity of the euclidean distance (see [4, p. 332]). This means that the same construction can be performed in a more general setting, namely, in a metric space (with certain regularity) such that the distance is strongly convex. For instance, we can define it in hyperbolic space.

Although the results of this section will be applied only to the case of hyperbolic plane, we are going to work in the more general framework of Hadamard spaces. There are at least two reasons for doing that. The first one is that the restriction to hyperbolic plane seems not to simplify the arguments. The second reason is that we are going to follow a small part of the work done in [11], where the results are placed in this broader context. Readers not interested in this generality may skip the definitions given below concerning Hadamard spaces, and just replace *Hadamard space* by *hyperbolic space* in the statements.

The aim of this section is to define the center of mass of a probability measure in a Hadamard space and to prove the contractive property stated in Proposition 6 below. Let us start with some definitions.

A *geodesic space* is a metric space in which any two points can be connected by a minimizing geodesic (see [9, p. 31]). Next, we want to give the definition of nonpositive curvature in the sense of Alexandrov. Let (X, d) be a geodesic space and Γ a triangle in X formed by minimizing geodesics with vertices A_1, A_2, A_3. Consider a euclidean triangle Γ' with vertices A_1', A_2', A_3' such that $d(A_i', A_j') = d(A_i, A_j)$ for $i, j = 1, 2, 3$. Let P be a point in the segment $\overline{A_2A_3}$, and P' the point in the segment $\overline{A_2'A_3'}$ such that $d(A_2, P) = d(A_2', P')$. The geodesic space (X, d) is said to have *nonpositive curvature* if for every $x \in X$ there exists $r_x > 0$ such that every triangle Γ contained in $B(x, r_x)$ satisfies, with the above notation, the following inequality

$$d(A_1, P) \leq d(A_1', P').$$

If we parameterize the segment $\overline{A_2A_3}$ by a geodesic $\sigma\colon [0, 1] \to X$ such that $\sigma(0) = A_2$ and $\sigma(1) = A_3$, the above inequality turns out to be

$$d(A_1, \sigma(t))^2 \leq (1 - t)d(A_1, A_2)^2 + td(A_1, A_3)^2 - t(t - 1)d(A_2, A_3)^2, \tag{9}$$

for all $t \in [0, 1]$, as an easy computation in euclidean geometry shows. Note that inequality (9) says that the function $f(x) = d(A, x)^2$, when restricted to a minimizing geodesic, is strongly convex (see [4, p. 332]).

A complete simply connected space of nonpositive curvature is called a *Hadamard space*.

Remark 4. Using hyperbolic trigonometry we can verify that hyperbolic space has indeed nonpositive curvature, and consequently it is a Hadamard space. More generally, it is a theorem by E. Cartan [6] that a Riemannian manifold of sectional curvature bounded above by 0 is a space of nonpositive curvature, in the sense of Alexandrov.

An important fact about Hadamard spaces is that the nonpositive curvature condition holds for every triangle. In other words, for every point x the constant r_x that appears in the definition of nonpositive curvature can be taken arbitrarily large (see [4, p. 329]). The following results have been taken almost verbatim from [11], (Lemmas 2.1, 4.1 and 4.2).

LEMMA 5. *Let μ be a probability measure of bounded support on a Hadamard space (X, d). Then the function defined by*

$$H_\mu(x) = \int_X d(x, y)^2 d\mu(y),$$

has a unique minimum. Moreover, if we denote this point by c_μ, for all $z \in X$ we have

$$H_\mu(z) \geq H_\mu(c_\mu) + d(c_\mu, z)^2. \tag{10}$$

Proof. First of all note that from inequality (9) and the fact that $\mu(X) = 1$, we have

$$H_\mu(\sigma(t)) \leq (1-t)H_\mu(z) + tH_\mu(z') - t(1-t)d(z,z')^2,$$

where $z, z' \in X$ and $\sigma\colon [0,1] \to X$ is a geodesic joining them starting at z. From the above inequality, the uniqueness of the minimum of H_μ follows immediately. For the existence, let m be the infimum of H_μ and $\{z_i\}$ a minimizing sequence. For i, j sufficiently large, $H_\mu(z_i), H_\mu(z_j) < m + (\epsilon/2)^2$. Now, taking $t = 1/2$ in the above inequality, we get

$$m \leq m + (\epsilon/2)^2 - \frac{1}{4}d(z_i, z_j)^2,$$

which shows that $\{z_i\}$ is a Cauchy sequence. Since X is complete this sequence converges to a point. This proves that the minimum of H_μ is attained at some point which will be denoted by c_μ. Now, let z be a point in X. Taking a geodesic from z to c_μ, from the above inequality we obtain

$$H_\mu(c_\mu) \leq (1-t)H_\mu(c_\mu) + tH_\mu(z) - t(1-t)d(z, c_\mu)^2,$$

for $t \in [0,1]$. It yields

$$H_\mu(c_\mu) \leq H_\mu(z) - (1-t)d(z, c_\mu)^2,$$

and inequality (10) follows. (Note that the same argument can be applied to an arbitrary strongly convex function bounded from below, see [4, p. 333].) □

The point c_μ is called the *center of mass* of μ.

In order to prove the main result of this section, we need the following instance of Reshetnyak's quadrilateral inequality, see Lemma 2.1 of [11].

LEMMA A. *Let (Y, d) be a Hadamard space and $y, y', z, z' \in Y$. Then*

$$d(y,z')^2 + d(y',z)^2 \leq d(y,z)^2 + d(y',z')^2 + 2d(y,y')d(z,z').$$

Let (X, μ) be a measure space of finite measure, and (Y, d) be a Hadamard space. Given a bounded map $f\colon X \longrightarrow Y$ we denote by c_f the center of mass of the push-forward measure $f_*\mu$ defined as $(f_*\mu)(E) = \mu(f^{-1}(E))$, for any measurable set $E \subset Y$.

PROPOSITION 6. *Let (X, μ) be a measure space with $\mu(X) = 1$ and let (Y, d) be a Hadamard space with distance d. If $f, g \in \mathsf{L}^\infty(X, Y)$ are two essentially bounded* *maps from X to Y, then*

$$d(c_f, c_g) \leq \int_X d(f(x), g(x))d\mu(x).$$

Proof. Consider the quadrilateral defined by the points $f(x), g(x), c_f, c_g$ and apply Lemma A to obtain

$$\begin{aligned} d(f(x), c_g)^2 + d(g(x), c_f)^2 \ \leq\ & d(f(x), c_f)^2 + d(g(x), c_g)^2 \\ & + 2d(f(x), g(x))d(c_f, c_g). \end{aligned}$$

Integrating on X and making a change of variables, we get

$$\begin{aligned} H_{f_*\mu}(c_g) + H_{g_*\mu}(c_f) \ \leq\ & H_{f_*\mu}(c_f) + H_{g_*\mu}(c_g) + \\ & 2d(c_f, c_g) \int_X d(f(x), g(x))d\mu(x). \end{aligned}$$

On the other hand, inequality (10) yields

$$(H_{f_*\mu}(c_g) - H_{f_*\mu}(c_f)) + (H_{g_*\mu}(c_f) - H_{g_*\mu}(c_g)) \geq 2d(c_g, c_f)^2.$$

Combining these two last inequalities, we obtain

$$d(c_g, c_f) \ \leq\ \int_X d(f(x), g(x))d\mu(x). \qquad \square$$

4. Sufficiency. The proof of the sufficiency is presented in the upper half plane $\mathbb{R}^2_+$. We will also denote by $\beta(z, w)$ the hyperbolic distance between two points $z, w \in \mathbb{R}^2_+$, that is,

$$\beta(z, w) = \frac{1}{2} \log_2 \frac{1 + \left|\frac{z-w}{z-\overline{w}}\right|}{1 - \left|\frac{z-w}{z-\overline{w}}\right|}.$$

Recall also that a measure μ defined in the upper half plane is a Carleson measure if there exists an absolute constant $C > 0$ such that for any Carleson box $Q \subset \mathbb{R}^+_2$ we have that

$$\mu(Q) \leq C\ell(Q).$$

The Carleson norm of the measure $\|\mu\|_C$ is the infimum of the constants $C > 0$ for which this inequality holds.

Let $\{z_n\}$ be a sequence of points in $\mathbb{R}^2_+$ satisfying conditions (a) and (b) in Theorem 1. We want to find a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ so that for any sequence of values $\{w_n\} \subset \mathbb{D}$ for which the compatibility condition

$$\beta(w_n, w_m) \leq \varepsilon\beta(z_n, z_m)$$

holds, there exists a function $f \in \mathcal{B}$ with $f(z_n) = w_n$ for $n = 1, 2, \ldots$. We first assume that $\{z_n\}$ is a separated sequence with constant of separation δ. This is not the general case, but contains the main ideas of the proof. By a normal families argument we can assume that $\{z_n\}$ has only a finite number of points. The main part of our argument is to construct a non-analytic mapping $\varphi\colon \mathbb{R}^2_+ \longrightarrow \mathbb{D}$ that verifies the following three conditions:

(A) For $n = 1, 2 \ldots$ we have $\varphi \equiv w_n$ in $\Delta(z_n, C\delta)$, where $C < 1$ is a fixed positive constant that will be chosen later.

(B) For any $z, w \in \mathbb{R}^2_+$ we have $\beta(\varphi(z), \varphi(w)) \leq C\varepsilon\beta(z, w)$.

(C) The measure $|\nabla\varphi(z)|dA(z)/(1 - |\varphi(z)|^2)$ is a Carleson measure with Carleson norm smaller than $C\varepsilon$.

The next three subsections are devoted to the construction of the function φ when $\{z_n\}$ is a separated sequence, while in the last two sections $\bar{\partial}$-techniques are applied to obtain analytic solutions of our interpolation problem.

4.1. Reduction to well separated sequences. The main purpose of this subsection is to reduce the construction of the smooth interpolating function φ to the case when the sequence $\Lambda = \{z_n\}$ is separated, with large constant of separation. Moreover we will show that we can assume that the sequence Λ consists of the centers of a certain collection of dyadic Carleson squares.

First of all we may assume that the whole sequence $\{z_n\}$ is contained in $Q([0, 1))$. We also can add the point $z_0 = 1/2 + i3/2$ with value $w_0 = 0$ in the sequence $\{w_n\}$.

So consider a separated sequence $\{z_n\}$. Given a positive number $N > 0$, we are going to construct a sequence $\Lambda_0 = \{z(n), n = 0, 1, \ldots\}$ such that

$$\Lambda \subseteq \bigcup_{z(n)\in\Lambda_0} \overline{\Delta\,(z(n), N)}$$

and

$$\inf\{\beta(z(n), z(m))\colon\ z(n), z(m) \in \Lambda_0,\ n \neq m\} \geq N.$$

The construction of Λ_0 is as follows. Assume that the sequence Λ is ordered such that $\operatorname{Im} z_n \geq \operatorname{Im} z_{n+1}$, $n = 0, 1, 2 \ldots$. Take $z(0) = z_0 \in \Lambda_0$, and $w(0) = 0$. We consider the family $\mathcal{F}_1$ of dyadic intervals so that $T(I)$ contains a point of $\Lambda \setminus \Delta(z(0), N)$. Take I_1 so that $|I_1| = \max\{|I|\colon I \in \mathcal{F}_1\}$ and let $z(1)$ be the center of $T(I_1)$. Take $w(1) = w_{j(1)}$ such that $z_{j(1)}$ is the closest point of $\{z_n\}$ to $z(1)$. Now let $\mathcal{F}_2$ be the family of dyadic intervals I so that $T(I)$ contains a point of $\Lambda \setminus (\Delta(z(0), N) \cup \Delta(z(1), N))$. We continue this construction by induction. Take

$\Lambda_0 = \{z(0), z(1), \ldots\}$. Then Λ_0 satisfies the conditions above. For each $z(n) \in \Lambda_0$ choose $w(n) = w_{j(n)}$ such that $z_{j(n)}$ is the closest point of Λ to $z(n)$.

Now assume that we have constructed a smooth function $\varphi_0\colon \mathbb{R}^2_+ \longrightarrow \mathbb{D}$ such that interpolates the corresponding values $\{w(n)\}$ for points in Λ_0 and verifies properties (A), (B), (C) above. Next we will construct a smooth function which interpolates the prescribed values at the sequence Λ and satisfies analogous estimates.

Since Λ is separated, there exists $\delta > 0$ such that the hyperbolic discs $\{\Delta(z_n, 2\delta)\colon\ z_n \in \Lambda\}$ are pairwise disjoint. We define the function φ as $\varphi \equiv \varphi_0$ on $\mathbb{R}^2_+ \setminus \cup\Delta(z_n, 2\delta)$, where the union is taken over all points $z_n \in \Lambda$. For each point $z_n \in \Lambda$ we define $\varphi \equiv w_n$ on $\Delta(z_n, \delta)$ and (A) follows. So φ is now defined in $\mathbb{R}^2_+ \setminus \cup\{z\colon\ \delta \le \beta(z_n, z) < 2\delta\}$. A basic Lemma of McShanne [12] and Valentine [18] (see [9, p. 43]) says that if X is a metric space and $E \subset X$, any Lipschitz function on E can be extended to X with the same Lipschitz constant. The compatibility condition (2) and property (B) for φ_0 give that we can extend φ to $\Delta(z_n, 2\delta) \setminus \Delta(z_n, \delta)$ so that

$$\beta(\varphi(z), \varphi(\tilde{z})) \le C(\delta)\varepsilon\beta(z, \tilde{z})$$

for any $z, \tilde{z} \in \mathbb{R}^2_+$. So (B) holds for φ. Dividing by $\beta(z, \tilde{z})$ and taking $\tilde{z} \to z$ we deduce that

$$(\operatorname{Im} z)|\nabla\varphi(z)| \le C_1(\delta)\varepsilon(1 - |\varphi(z)|^2)$$

for any $z \in \mathbb{R}^2_+$. So we have that

$$\frac{|\nabla\varphi(z)|}{(1 - |\varphi(z)|^2)} \le \frac{|\nabla\varphi_0(z)|}{(1 - |\varphi_0(z)|^2)} + C_2(\delta)\varepsilon \sum_n \frac{\chi_{\Delta(z_n,2\delta)}(z)}{\operatorname{Im} z}.$$

Since φ_0 verifies property (C), we only have to show that the second term gives rise to a Carleson measure. So let Q be a Carleson box. If $\Delta(z_n, 2\delta) \cap Q \neq \emptyset$ notice that $z_n \in C(\delta)Q$. Hence

$$\sum_n \int_Q \frac{\chi_{\Delta(z_n,2\delta)}(z)}{\operatorname{Im} z} dA(z) \le C_3(\delta) \sum_{z_n \in C(\delta)Q} \operatorname{Im} z_n.$$

Since the density condition (b) in Theorem 1 implies Carleson's condition, then this term is bounded by $MC_3(\delta)C(\alpha)\ell(Q)$, so φ verifies (C).

Hence, without loss of generality, we may assume that the sequence $\Lambda = \{z_n\}$ is a well separated sequence formed by the centers of some dyadic Carleson squares.

4.2. A non-smooth interpolating function. As explained before, given a sequence of values satisfying the compatibility condition (2), we wish to construct a smooth interpolating function φ satisfying conditions (A), (B) and (C). The main purpose of this section is to construct a piecewise continuous interpolating function φ in the upper half plane which satisfies condition (A) and, roughly speaking, conditions (B) and (C) if we restrict attention to the vertical direction. These two conditions will fail in the horizontal direction, but we will still have certain control which will be used later. This is better expressed in terms of collections of dyadic squares. For this purpose, we will use one of the main auxiliary results from [3], a covering Lemma that comes in handy in the present construction as well. As explained in the previous section, we may assume that the points $\{z_n\}$ are centers of $T(I)$, for a certain collection $\mathcal{A}$ of dyadic intervals. The density condition (3) then translates to the following one: for any dyadic interval I we have that

$$\#\{I_k \in \mathcal{A}\colon\ I_k \subset I, |I_k| = 2^{-n}|I|\} \leq M_1 2^{\alpha n}, \quad n = 0, 1, 2 \ldots \tag{11}$$

The covering Lemma is the following:

LEMMA B. *Let $\mathcal{A}$ be a collection of dyadic intervals. The following conditions are equivalent:*

(a) *There exist constants $M_1 > 0$ and $0 < \alpha < 1$ such that for any dyadic interval I and any $n = 0, 1, 2, \ldots$ we have*

$$\#\{I_k \in \mathcal{A}\colon\ I_k \subset I, |I_k| = 2^{-n}|I|\} \leq M_1 2^{\alpha n}$$

(b) *There exist a family $\mathcal{G}$ of dyadic arcs with $\mathcal{A} \subset \mathcal{G}$ and a constant $C > 0$ such that the following two conditions hold*

(b1) *For any dyadic arc J we have*

$$\sum_{I \subseteq J, I \in \mathcal{G}} |I| \leq C|J|.$$

(b2) *For any pair of intervals $I_0 \in \mathcal{A}$ and $I_1 \in \mathcal{G}$ with $I_0 \subseteq I_1$ we have*

$$\#\{I \in \mathcal{G}\colon\ I_0 \subseteq I \subseteq I_1\} \geq C^{-1} \log \frac{|I_1|}{|I_0|}.$$

We will call $\mathcal{G}$ the *intermediate family* associated to $\mathcal{A}$. The properties (b1) and (b2) point to two opposite directions. On the one hand the family $\mathcal{G}$ should be large to guarantee (b2) but on the other hand, $\mathcal{G}$ should be small to guarantee (b1). As it turns out, our density condition (b) in Theorem 1 is what we need to arrive to a compromise. The intermediate family will help us to construct the non-smooth interpolating function, as we may see in the following lemma.

LEMMA 7. *Let $\mathcal{A}$ be a family of dyadic intervals satisfying condition (11), and let $\{z_n\}$ be the sequence of centers $\{z(I)\colon I \in \mathcal{A}\}$. Let $\mathcal{G}$ be the intermediate family given by Lemma B. Let $\{w_n\}$ be a sequence of values in the unit disc satisfying the compatibility condition*

$$\beta(w_n, w_m) \le \varepsilon \beta(z_n, z_m),\ n, m = 0, 1, 2, \ldots$$

Then there exists a piecewise continuous function $\varphi\colon \mathbb{R}^2_+ \longrightarrow \mathbb{D}$ whose partial derivatives are complex measures, which satisfies $\varphi \equiv w_n$ on $\Delta(z_n, 1/10)$, for $n = 0, 1, 2, \ldots$ and

(a) *The support of the measure $|\nabla\varphi|$ is contained in $\cup T(I) \cup \partial_v Q(I)$, where the union is taken over all intervals $I \in \mathcal{G}$. Here $\partial_v Q(I)$ means the vertical part of the boundary of $Q(I)$.*

(b) *The function φ has a vertical derivative at any $z \in \mathbb{R}^2_+$ and*

$$\frac{(\operatorname{Im} z)|\partial_y \varphi(z)|}{(1 - |\varphi(z)|^2)} \le C\varepsilon.$$

(c) *The measure $|\partial_y \varphi(z)|(1 - |\varphi(z)|^2)^{-1} dA(z)$ is a Carleson measure with Carleson norm at most $C\varepsilon$.*

(d) *There exists a universal constant $C_2 > 0$ such that*

$$\frac{|\partial_x \varphi(z)|}{1 - |\varphi(z)|^2} \le C_2 \varepsilon \sum ds(\partial_v Q(I))(z)$$

as positive measures, where the sum is taken over all $I \in \mathcal{G}$. Here $ds(\partial_v Q(I))$ means the linear measure in $\partial_v Q(I)$.

Proof. Given the interpolation problem for the sequence $\{z_n\}$ we first extend it to a suitable interpolation problem on a bigger sequence defined in terms of the intermediate family. We proceed as in [3]. To this end, observe that the intermediate family $\mathcal{G}$ can be viewed as a tree. The root node corresponds to the unit interval (this is the reason why the point $z_0 = 1/2 + i3/2$ was added to the sequence). Every interval $I \in \mathcal{G}$ corresponds to a node $a(I)$ in the tree. Two nodes $a(I)$ and $a(\tilde{I})$ with $I, \tilde{I} \in \mathcal{G}$ and $\tilde{I} \subset I$ are joined by an edge in the tree if $\tilde{I}$ is maximal among all $J \in \mathcal{G}$ with $J \subset I$. As usual, the distance between two nodes in the tree is defined to be the minimal number of edges joining them. Observe that the condition (b2) in Lemma B tells that the distance between two edges $a(I), a(\tilde{I})$ in the tree is bigger than a fixed multiple of the hyperbolic distance between the associated points $z(I), z(\tilde{I})$. We consider the set $\mathcal{V}$ of vertices of the tree corresponding to points of the original sequence $\{z_n\}$ and we define a function φ_0 on $\mathcal{V}$ by $\varphi_0(a(I)) = w_n$ if z_n is the center of $T(I)$. Then the compatibility condition $\beta(w_n, w_m) \le \varepsilon\beta(z_n, z_m)$, and (b2) in Lemma B gives that the function $\varphi_0\colon V \longrightarrow \mathbb{D}$ is Lipschitz with constant $C_1\varepsilon$ when considering the metric of the tree in $\mathcal{V}$ and the hyperbolic metric in the image domain $\mathbb{D}$. We can apply here again the Lemma by McShanne and Valentine enounced in the previous section to extend the function φ_0 to the whole tree $\{a(I)\colon\ I \in \mathcal{G}\}$. The extended function will be also denoted by φ_0.

Let us now return to the upper half plane. Let $\Lambda = \{z_n\}$ be the original sequence and let $\Lambda^+ = \{z(I)\colon\ I \in \mathcal{G}\}$ be the sequence formed by the center of the Carleson squares corresponding to intervals in the intermediate family $\mathcal{G}$, and rename $\Lambda^+ = \{z_n^+\}$. If $z_n^+ = z(I)$, $I \in \mathcal{G}$, we denote by $w_n^+ = \varphi_0(z_n^+) = \varphi_0(a(I))$. Notice that if $z_n^+, z_m^+ \in \Lambda^+$ correspond to two consecutive nodes in the tree, then $\beta(w_n^+, w_m^+) \le C_2\varepsilon$. We define a function $\varphi_1\colon\ \mathbb{R}^2_+ \longrightarrow \mathbb{D}$ as $\varphi_1(z_0) = 0$ and $\varphi_1(z) = w_n^+$ if $z \in Q(z_n^+)\setminus \cup\, Q(z_m^+)$, where the union is taken over all $Q(z_m^+)$, $z_m^+ \in \Lambda^+$, contained in $Q(z_n^+)$. We next proceed as in [15, p. 75] to smooth the function φ_1 in the vertical direction.

Let Q_0 be the unit square and set

$$\psi_0(x+iy) = \chi_{Q_0}(x+iy)\min\Big\{1, 6(1-y)\Big\}.$$

For each point $z_n^+ \in \Lambda^+$ we denote by $Q(n)$ the dyadic square such that $z_n^+ \in T(Q(n))$. If $Q(n) = Q([a_n, b_n))$, we define $\tau_n(z) = (z - a_n)/(b_n - a_n)$ and $\psi_n(z) = \psi_0(\tau_n(z))$. So ψ_n vanishes outside the square $Q(n)$, has constant value 1 in $Q(n) \cap \{z\colon\ \operatorname{Im} z < 5(b_n - a_n)/6\}$ and it is linear in the vertical direction in $Q(n) \cap \{z\colon\ \operatorname{Im} z \ge 5(b_n - a_n)/6\}$.

We set $b_0^+ = w_0 = 0$ and $b_n^+ = w_n^+ - w_m^+ = \varphi_1(z_n^+) - \varphi_1(z_m^+)$, $n \ge 1$, where $I(z_m^+)$ is the smallest dyadic interval in the family $\mathcal{G}$ which contains the interval $I(z_n^+)$. In other words, z_m^+ corresponds to the vertex in the tree above z_n^+. Let us consider

$$\varphi(z) = \sum_n b_n^+ \psi_n(z), \qquad z \in \mathbb{R}^2_+.$$

It is clear that $\partial_x\varphi$ and $\partial_y\varphi$, both as distributions, are complex measures. Observe that φ has a pointwise vertical partial derivative at any point of the upper half plane and

$$\partial_y\varphi(z) = 0, \text{ for } z \notin \bigcup Q(I) \cap \Big\{z\colon \frac{5}{6}\ell(I) < \operatorname{Im} z\Big\},$$

while

$$\partial_x\varphi(z) = 0, \text{ for } z \notin \bigcup \partial_v Q(I).$$

Here both unions are taken over all intervals $I \in \mathcal{G}$. So (a) holds. Let us now check (b). Observe that if $I(m) \in \mathcal{G}$ and $I(n) \subseteq I(m)$ is a maximal interval among the ones in $\mathcal{G}$ contained in $I(m)$, then by construction we have $\beta(\varphi(z_n^+), \varphi(z_m^+)) \leq C_2\varepsilon$. Since φ is linear in the vertical direction, we deduce that

$$\beta(\varphi(z), \varphi(\tilde{z})) \leq C_3\varepsilon\beta(z, \tilde{z}),$$

whenever $z, \tilde{z} \in T(I(n))$ for some $I(n) \in \mathcal{G}$. Dividing the equality by $\beta(z, \tilde{z})$ and taking the limit when $\tilde{z}$ tends to z along the vertical direction we have that

$$\frac{(\operatorname{Im} z)|\partial_y\varphi(z)|}{(1 - |\varphi(z)|^2)} \leq C_4\varepsilon.$$

The property (c) is a direct consequence of (b), property (b1) in Lemma B and the fact that the support of the measure $\partial_y\varphi$ is contained in $\cup T(I)$, where the union is taken over all intervals $I \in \mathcal{G}$. Notice that the behavior of the measure $\partial_x\varphi$ is worse because φ may have jumps across the vertical sides of $Q(I)$, $I \in \mathcal{G}$. However, since at each step the jump is of a fixed hyperbolic length, if we consider z and $\tilde{z}$ close enough, then $\beta(\varphi(z), \varphi(\tilde{z}))$ is at most $C\varepsilon$ times the number of dyadic intervals I in the intermediate family $\mathcal{G}$ such that $\partial_v Q(I)$ separates z and $\tilde{z}$. Hence

$$\frac{|\partial_x\varphi(z)|}{(1 - |\varphi(z)|^2)} \leq C_5\varepsilon \sum ds(\partial_v Q(I))(z)$$

as positive measures, where the sum is taken over all $I \in \mathcal{G}$. □

4.3. Averaging. The next step of the proof will be smoothing the function φ defined in the above paragraph, so that $|\partial_x\varphi(z)|/(1 - |\varphi(z)|^2)$ verifies also properties b) and c) in Lemma 7. As proved in Section 4.1 we may assume that $\Lambda = \{z_n\}$ is a well separated sequence consisting of centers of a certain collection of dyadic squares. So assume that $\beta(z_n, z_m) > 5$, thereby for each $n = 1, 2, \dots$ we can add to the sequence Λ the points $z_n^- := z_n - 4 \operatorname{Im} z_n/3$ and $z_n^+ := z_n + 4 \operatorname{Im} z_n/3$. These points are respectively the centers of the two dyadic squares of the same generation adjacent to $Q(n)$, denoted by $Q(n)^-$ and $Q(n)^+$. The extended sequence $\Lambda\cup\{z_n^-\}\cup\{z_n^+\}$, which will be also denoted by Λ, will still be a separated sequence. For each $n = 1, 2 \dots$, attached to the extra points z_n^- and z_n^+ consider the corresponding value w_n in the sequence $\{w_n\}$. Hence the function φ constructed in the previous section verifies that $\varphi(z_n) = \varphi(z_n^-) = \varphi(z_n^+) = w_n$, for n=1,2... This is a trick used by C. Sundberg in [17].

For $0 \leq t < 1$ let now Λ_t be the sequence obtained by translating the sequence Λ by t euclidean units, that is, $\Lambda_t = \{z_n + t\colon\ n = 0, 1, 2, \dots\}$. Let $\mathcal{G}_t$ be the intermediate family of dyadic intervals given in Lemma B associated to the sequence Λ_t. Finally let φ_t be the function given by Lemma 7. Notice that we could also have looked at it from an equivalent point of view: fix the sequence Λ and let $\mathcal{D}_t = \{I_t\colon\ I_t - t \in \mathcal{D}\}$ be the translation of the standard dyadic intervals

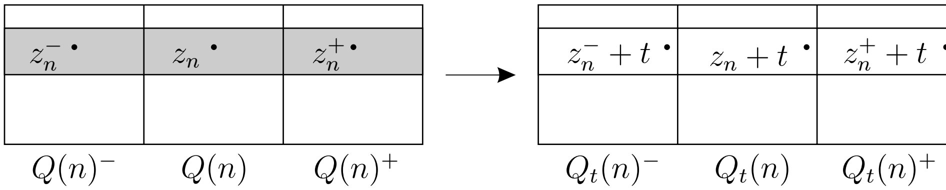

Figure 2. The function $\varphi_t(z+t)$ has constant value w_n in the grey strip for every $t \in [0,1)$.

by t euclidean units. We could have considered the analogues of Lemma 7 for the sequence Λ and the translated dyadic family $\mathcal{D}_t$.

Observe that for any $z \in \mathbb{R}^2_+$ the set of possible values $\{\varphi_t(z+t)\colon\ 0 \leq t < 1\}$ is bounded. Then we may apply results in Section 3 to define $\varphi(z)$. Take as measure space the unit interval $[0,1)$ with the Lebesgue measure, and as Hadamard space the hyperbolic disc $\mathbb{D}$. For each $z \in \mathbb{R}^2_+$ consider the mapping $\varphi(z,\cdot)\colon\ [0,1) \longrightarrow \mathbb{D}$ defined as $\varphi(z,t) = \varphi_t(z+t)$, $t \in [0,1)$. Then we define $\varphi(z)$ as the center of mass of the pushforward measure μ on $\mathbb{D}$ defined as $\mu(E) = |\{t \in [0,1)\colon\ \varphi_t(z+t) \in E\}|$, $E \in \mathbb{R}^2_+$. Now we will prove that the function φ verifies properties (A), (B) and (C).

In Figure 2, the left hand side represents a point z_n of the original sequence and its two corresponding extra points z_n^- and z_n^+. The right hand side corresponds to translate the sequence by t euclidean units. Since we have added the extra points to the original sequence, for every $t \in [0,1)$ the function φ_t verifies that $\varphi_t(z+t) = w_n$ for every $z \in Q(n) \cup Q(n)^- \cup Q(n)^+$ with $\ell(Q)/2 < \operatorname{Im} z < 5\ell(Q)/6$. In other words, for every $t \in [0,1)$ the function $\varphi_t(z+t)$ has constant value w_n in the grey strip of Figure 2. In particular, $\varphi_t(z+t) = w_n$ for every $z \in \Delta(z_n, 1/10)$ and every $t \in [0,1)$. This proves (A).

Now we can check properties (B) and (C). Applying Proposition 6 to the mappings $\varphi(z,\cdot)$ and $\varphi(w,\cdot)$ we obtain that

$$\beta(\varphi(z),\varphi(w)) \leq \int_0^1 \beta(\varphi_t(z+t),\varphi_t(w+t))\,dt. \tag{12}$$

If $|\nabla\varphi(z)|$ was replaced by $|\partial_y\varphi(z)|$, then properties (B) and (C) would follow from estimate (12) and the corresponding properties in Lemma 7. Property (d) in Lemma 7 and the inequality (12) will provide a similar estimate for $|\partial_x\varphi(z)|$.

First let $z, w \in \mathbb{R}^2_+$ with $\operatorname{Im} z = \operatorname{Im} w$, and take $k \in \mathbb{Z}$ so that $2^{-k-1} < \operatorname{Im} z \leq 2^{-k}$. It is enough to prove (B) for $z, w \in \mathbb{R}^2_+$ such that $|z-w| \leq \operatorname{Im} z$. For $j = 1,\ldots,k$ let A_j be the set of $t \in [0,1)$ such that there exists $I \in \mathcal{G}_t$, $|I| = 2^{-j}$ such that at least one of the two vertical sides of $Q(I)$ is between $z+t$ and $w+t$. Since φ_t can jump at most $C\varepsilon$ hyperbolic units in $T(I)$ with $I \in \mathcal{G}_t$, then if $t \in A_j$ we have that

$$\beta(\varphi_t(z+t),\varphi_t(w+t)) \leq C_2\varepsilon(k-j).$$

Since $|A_j| \leq 2^j|z-w| \leq C2^{j-k}\beta(z,w)$, we deduce that

$$\beta(\varphi(z),\varphi(w)) \leq C_3\varepsilon \sum_{j=1}^{k}(k-j)2^{j-k}\beta(z,w) \leq C_4\varepsilon\beta(z,w).$$

So (B) holds, and hence φ is differentiable almost everywhere in $\mathbb{D}$. To check now property (C) for $|\partial_x\varphi(z)|$, let Q be a Carleson square. Divide by $\beta(z,w)$ in both sides of estimate (12). Making w tend to z and then using property (d) in Lemma 7 we obtain that

$$\begin{aligned}\int_Q \frac{|\partial_x\varphi(z)|}{1-|\varphi(z)|^2} &\leq \int_0^1 \left(\int_Q \frac{|\partial_x\varphi_t(z+t)|}{1-|\varphi_t(z+t)|^2}\right) dt \\ &\leq C_2\varepsilon \int_0^1 \sum \ell(\partial_v Q(I)\cap(Q-t))dt,\end{aligned}$$

where the sum is taken all over the dyadic interval $I \in \mathcal{G}_t$. Split the above sum into two terms, the last integral can be written as $(P_1)+(P_2)$, where (P_1) corresponds to those intervals $I \in \mathcal{G}_t$ such that $|I| < \ell(Q)$ and (P_2) to those such that $|I| \geq \ell(Q)$. Applying property (b1) in Lemma B to (P_1), for any $t \in [0,1)$ we have

$$\sum \ell(\partial_v Q(I)\cap(Q-t)) \leq C\ell(Q),$$

where the sum is taken over all $I \in \mathcal{G}_t$ with $|I| < \ell(Q)$. Hence $(P_1) \leq C\ell(Q)$. To estimate (P_2) we proceed as in property (B). Let $k \in \mathbb{Z}$ so that $2^{-k-1} < \ell(Q) \leq 2^{-k}$. For each dyadic interval I of generation $j = 1,\ldots,k$ let $A(I)$ be the set of $t \in [0,1)$ such that $I \in \mathcal{G}_t$ and $\partial_v Q(I)\cap(Q-t) \neq \emptyset$. Take $A_j = \cup A(I)$, where the union is taken over all dyadic intervals of generation j. Notice that $|A_j| \leq 2^{j-k}$. Hence

$$(P_2) \leq \sum \ell(Q)|A(I)|,$$

where the sum is taken over the dyadic intervals I of generation smaller than k. Hence

$$(P_2) \leq \ell(Q)\sum_{j=1}^{k}|A_j| \leq 2\ell(Q).$$

So property (C) holds.

4.4. A suitable $\bar\partial$ problem. The main purpose of this section is to construct an analytic interpolating function. The main tools will be a certain $\bar\partial$-equation and BMO techniques. We start with the following well known result (see [19]) whose proof is presented for the sake of completeness. Recall that $\mathrm{BMO}(\mathbb{R})$ is defined as the set of functions $f \in \mathrm{L}^1(\mathbb{R})$ such that

$$\|f\|_{\mathrm{BMO}} = \sup \frac{1}{|I|} \int_I |f(x) - f_I| dx,$$

where $f_I = \frac{1}{|I|} \int_I f$ and the supremum is taken over all intervals $I \subset \mathbb{R}$. Let us also denote by $P_z(f)$ the Poisson integral of f at the point z.

LEMMA 8. *Let F be a smooth function in $\mathbb{R}^2_+$ such that*

$$f(x) = \lim_{y \to 0} F(x + iy)$$

exists almost every $x \in \mathbb{R}$. Assume $f \in \mathrm{L}^1(\mathbb{R})$ and $|\nabla F(z)| dA(z)$ is a Carleson measure. Then $f \in \mathrm{BMO}(\mathbb{R})$ and

$$\|f\|_{\mathrm{BMO}} \leq C \| |\nabla F(z)| dA(z) \|_C.$$

Moreover, for any $z \in \mathbb{R}^2_+$ there exists a point $\tilde{z} \in T(I(z))$ such that

$$|P_z(f) - F(\tilde{z})| \leq C \| |\nabla F(z)| dA(z) \|_C.$$

Proof. Let $I \subset \mathbb{R}$ be an interval and let z_I be the center of the square $T(I)$, and x_I the center of the interval I. Since $|\nabla F(z)| dA(z)$ is a Carleson measure we have that

$$\int_{T(I)} |\nabla F(z)| dA(z) = \int_{|I|/2}^{|I|} \int_I |\nabla F(x + it)| \, dx \, dt \leq \|\nabla F\|_C |I|.$$

Hence there exists $t_0 \in \big(|I|/2, |I|\big)$ such that $\int_I |\nabla F(x + it_0)| \, dx \leq 2 \|\nabla F\|_C$, so for any $x \in I$ we have that

$$|F(x + it_0) - F(x_I + it_0)| \leq \int_I |\nabla F(s + it_0)| \, ds \leq 2 \|\nabla F\|_C.$$

Moreover for any $x \in I$ we have that

$$|f(x) - F(x + it_0)| \leq \int_0^{t_0} |\nabla F(x + is)| \, ds.$$

The last two inequalities imply that

$$\begin{aligned}\frac{1}{|I|}\int_I |f(x)-F(x_I+it_0)|\,dx &\\ \leq \frac{1}{|I|}\int_I |f(x)-F(x+it_0)|\,dx &+ \frac{1}{|I|}\int_I |F(x+it_0)-F(x_I+it_0)|\,dx\\ \leq \frac{1}{|I|}\int_{Q(I)} |\nabla F(z)|\,dA(z) &+ 2\|\nabla F\|_C \leq 3\|\nabla F\|_C,\end{aligned}$$

which shows that $f\in \mathsf{BMO}$ and that $\|f\|_{\mathsf{BMO}} \leq C\|\nabla F\|_C$. The last estimate in the theorem follows from the well known fact that

$$\left|\frac{1}{|I|}\int_I f - P_{z_I}(f)\right| \leq C\|f\|_{\mathsf{BMO}}. \qquad \square$$

Let $\{z_n\}$ be a separated sequence satisfying the density condition (b) in Theorem 1. Given a sequence of values $\{w_n\}$ satisfying the compatibility condition (2), we next construct a function $f\in \mathsf{H}^\infty$ with $\|f\|_\infty \leq 1$ and $f(z_n)=w_n$ for $n=1,2,\ldots$. Let φ be the smooth interpolating function constructed in subsections 4.2 and 4.3. We will apply Lemma 8 to the function $F(z)=\log(1-|\varphi(z)|^2)$, which by (C) satisfies

$$\||\nabla F(z)|dA(z)\|_C \leq C\varepsilon.$$

Actually, since the function F verifies that $|F(z)-F(\tilde z)|\leq C$ if $\beta(z,\tilde z)\leq 1$, the last estimate in Lemma 8 gives that there exists a universal constant $C_1>0$ such that

$$|P_z(\log(1-|\varphi|^2)) - \log(1-|\varphi(z)|^2)| \leq C_1\varepsilon, \quad z\in\mathbb{R}^2_+. \tag{13}$$

Let $E(1-|\varphi|^2)$ be the outer function given by

$$E(1-|\varphi|^2)(z) = \exp\Big(\int_{\mathbb{R}} \frac{-i}{x-z}\log(1-|\varphi(x)|^2)\,dx\Big), \quad z\in\mathbb{R}^2_+. \tag{14}$$

The estimate (13) reads

$$e^{-C_1\varepsilon} \leq \frac{|E(1-|\varphi|^2)(z)|}{1-|\varphi(z)|^2} \leq e^{C_1\varepsilon}, \quad z\in\mathbb{R}^2_+. \tag{15}$$

Therefore, again by (C) we have that

$$\frac{|\nabla\varphi(z)|}{E(1-|\varphi|^2)(z)}dA(z)$$

is a Carleson measure with Carleson norm bounded by $C_2\varepsilon$. Assuming that $\{z_n\}$ is a separated sequence, the rest of the proof is fairly standard. Let $B(z)$ be the Blaschke product with zeros $\Lambda = \{z_n\}$. Since $\{z_n\}$ is an interpolating sequence for H^∞, there exists $C_3 > 0$ such that $|B(z)| \geq C_3$ for $z \notin \Delta(z_n, C)$. Recall that φ is constant on the hyperbolic discs $\Delta(z_n, C)$, hence the measure

$$\left|\frac{\bar\partial\varphi(z)}{B(z)E(1-|\varphi|^2)(z)}\right| dA(z)$$

is a Carleson measure with Carleson norm smaller than $C_4\varepsilon$. Hence we can find a smooth function b in $\mathbb{R}^2_+$ which extends continuously to $\mathbb{R}$ with $\|b\|_{\mathsf{L}^\infty(\mathbb{R})} \leq C\varepsilon$, such that

$$\bar\partial b(z) = \frac{2\bar\partial\varphi(z)}{B(z)E(1-|\varphi|^2)(z)}, \quad z \in \mathbb{R}^2_+.$$

See [8, p. 311]. Then $f = \varphi - 2^{-1}BE(1-|\varphi|^2)b$ is an analytic function and at almost every $x \in \mathbb{R}$ we have

$$|f(x)| \leq |\varphi(x)| + 2^{-1}(1-|\varphi(x)|^2)C\varepsilon \leq 1$$

if ε is taken sufficiently small, so that $C\varepsilon \leq 1$. So, under the assumption that $\{z_n\}$ is a separated sequence, we have constructed a function $f \in \mathcal{B}$ fulfilling the interpolation.

For later purposes it will be useful to state the following fact.

Remark 9. There exists a function $f \in \mathcal{B}$ satisfying $f(z_n) = w_n$, for $n = 1, 2\ldots$ and

$$|E(1-|f|)(z)| \geq C(1-|f(z)|),\ z \in \mathbb{R}^2_+. \tag{16}$$

Moreover, there exists a constant $\eta > 0$ depending on $\{z_n\}$ such that

$$\beta(f(z), f(z_n)) \leq C\varepsilon\beta(z, z_n), \quad z \in \Delta(z_n, \eta) \tag{17}$$

for any $n = 1, 2, \ldots$.

To show (16) notice that since $\|b\|_\infty \leq C\varepsilon < 1$ there exists a constant $C_1 > 0$ such that $1-|f(x)| \geq C_1(1-|\varphi(x)|)$ for any $x \in \mathbb{R}$. Hence $|E(1-|f|)(z)| \geq C_2|E(1-|\varphi|)(z)|$ for any $z \in \mathbb{R}^2_+$. Applying (15) we deduce that

$$|E(1-|f|)(z)| \geq C_3(1-|\varphi(z)|)$$

for any $z \in \mathbb{R}^2_+$. Since $|\varphi(z) - f(z)| \leq C\varepsilon(1-|\varphi(z)|)$, then (16) holds.

The proof of (17) is more subtle and depends on a beautiful result by P. Jones on bounded solutions of $\bar\partial$-equations.

LEMMA 10. *Let F be a continuous function in the upper half plane such that $|F(z)|dA(z)$ is a Carleson measure. Let $\{z_n\}$ be a sequence of points in the half plane $\mathbb{R}^2_+$. Assume that there exists $\delta > 0$ such that $F \equiv 0$ on $\Delta(z_n, \delta)$. Then there exists a function $b \in \mathcal{C}(\mathbb{R}^2_+)$ with $\bar{\partial} b = F$ and*

$$\|b\|_{L^\infty(\mathbb{R})} + \sup\{|b(z)|\colon z \in \cup\Delta(z_n, \delta/2)\} \le C(\delta)\||F(z)|dA(z)\|_C. \tag{18}$$

Proof. Without loss of generality we may assume that $\||F(z)|dA(z)\|_C = 1$. P. Jones [10] found an explicit solution of the $\bar{\partial}$-equation with uniform estimates. This formula is

$$b(z) = \frac{1}{\pi}\int_{\mathbb{R}^2_+} \frac{(\operatorname{Im}\xi)F(\xi)}{(\xi - z)(z - \bar{\xi})} K(\xi, z)dA(\xi),$$

where

$$K(\xi, z) = \exp\left\{\int_{S(\xi)} \left(\frac{i}{\xi - \bar{w}} - \frac{i}{z - \bar{w}}\right)|F(w)|dA(w)\right\},$$

and $S(\xi) = \{w \in \mathbb{R}^2_+\colon \operatorname{Im} w < \operatorname{Im}\xi\}$. The estimate $\|b\|_{L^\infty(\mathbb{R})} \le C$ is proved in [10]. We will use the same argument that we are using to estimate the second term in the left hand side of (18). Write $E = \mathbb{D} \setminus \cup\Delta(z_n, \delta)$ and notice that $|\xi - z| \ge C_1(\delta)|\bar{\xi} - z|$ if $\xi \in E$ and $z \in \cup\Delta(z_n, 2^{-1}\delta)$. Hence for such points $z \in \cup\Delta(z_n, 2^{-1}\delta)$ we have

$$|b(z)| \le C_2(\delta)\int_E \frac{(\operatorname{Im}\xi)|F(\xi)|}{|\bar{\xi} - z|^2}|K(\xi, z)|dA(\xi).$$

Since any $w \in S(\xi)$ verifies $\operatorname{Im} w > \operatorname{Im}\xi$, we deduce that

$$\int_{S(\xi)} \frac{\operatorname{Im} w}{|\bar{w} - \xi|^2}|F(w)|dA(w) \le \int_{\mathbb{R}^2_+} \frac{\operatorname{Im}\xi}{|\bar{w} - \xi|^2}|F(w)|dA(w) \le C$$

for any $\xi \in \mathbb{R}^2_+$. Then

$$|b(z)| \le C_3 \int_{\mathbb{R}^2_+} \frac{(\operatorname{Im}\xi)|F(\xi)|}{|\bar{\xi} - z|^2} \exp\left(\int_{S(\xi)} \frac{-(\operatorname{Im} w)|F(w)|}{|\bar{w} - z|^2} dA(w)\right) dA(\xi),$$

where $C_3 = C_3(\delta)$. Arguing as in [10], the integral above compares to $\int_0^\infty e^{-x}dx = 1$, and the proof is completed. □

We now continue with the proof of (17). Recall that φ was constant on hyperbolic discs centered at the points $\{z_n\}$ of a fixed radius. Hence the function $F = \bar{\partial}\varphi/(BE(1 - |\varphi|^2))$ satisfies the conditions of Lemma 10, so let b be the solution given by this Lemma. Pick $\eta \leq 2^{-1}\delta$ and take $z \in \Delta(z_n, \eta)$ for some fixed n. Then $|b(z)| \leq C_4\varepsilon$ and we deduce

$$|f(z) - \varphi(z)| \leq C_4\varepsilon|B(z)||E(1 - |\varphi|)(z)| \leq C_5\varepsilon|B(z)|(1 - |\varphi(z)|).$$

Since

$$|B(z)| \leq \left|\frac{z - z_n}{z - \bar{z}_n}\right| \leq C_6\beta(z_n, z),$$

we have that

$$\beta(f(z), \varphi(z)) \leq C_7\varepsilon\beta(z_n, z).$$

Since $\beta(\varphi(z), \varphi(w)) \leq C\varepsilon\beta(z, w)$ for any $z, w, \in \mathbb{R}^2_+$, the estimate (17) follows.

4.5. Union of two separated sequences. This section is devoted to proving the sufficiency of conditions (a) and (b) in Theorem 1. This will end the proof of Theorem 1. So let $\{z_n^{(1)}\}\cup\{z_n^{(2)}\}$ be the union of two separated sequences verifying the density condition (b). Pick a number $\delta > 0$ smaller than the quantifier η appearing in (17) as well as smaller than the separation constant of the sequence $\{z_n^{(1)}\}$. Adding the points in $\{z_n^{(2)}\} \setminus \cup\Delta(z_n^{(1)}, \delta)$ to the sequence $\{z_n^{(1)}\}$, we can assume that the sequence $\{z_n^{(2)}\}$ is contained in $\cup\Delta(z_n^{(1)}, \delta)$. So for each $z_k^{(2)} \in \{z_n^{(2)}\}$ there is a point in $\{z_n^{(1)}\}$, denoted by $z_{n(k)}^{(1)}$, with $\beta(z_{n(k)}^{(1)}, z_k^{(2)}) \leq \delta$. Let B be the Blaschke product with zeros $\{z_n^{(1)}\}$. Since $\{z_n^{(1)}\}$ is a separated sequence which satisfies the density condition (b) in Theorem 1, it is an interpolating sequence for H^∞. Hence there exists a constant $C_1 > 0$ such that

$$B(z_k^{(2)}) \geq C_1\rho(z_{n(k)}^{(1)}, z_k^{(2)}), \quad k = 1, 2, \ldots,$$

where $\rho(a, b) = |a - b|/|a - \bar{b}|$ is the pseudohyperbolic distance in $\mathbb{R}^2_+$. Now let $\{w_n^{(1)}\} \cup \{w_n^{(2)}\}$ be a sequence of values in the unit disc that satisfy the compatibility condition (2). Then there exists $f \in \mathcal{B}$ with $f(z_n^{(1)}) = w_n^{(1)}$, for $n = 1, 2\ldots$, and satisfying the conditions given in Remark 9 for the sequence $\{z_n^{(1)}\}$. However, since $\beta(z_{n(k)}^{(1)}, z_k^{(2)}) \leq \delta$ for any $k = 1, 2, \ldots$, we can assume that conditions (16) and (17) hold for the whole sequence $\{z_n^{(1)}\} \cup \{z_n^{(2)}\}$, once the constant C is replaced by other absolute constants. Notice that estimate (16) gives

$$\left|\frac{w_n^{(2)} - f(z_n^{(1)})}{B(z_n^{(2)})E(1 - |f|)(z_n^{(2)})}\right| \leq C_2\frac{\rho(w_n^{(2)}, w_n^{(1)})}{\rho(z_n^{(2)}, z_n^{(1)})}, \quad n = 1, 2, \ldots.$$

Since $\beta(a,b)$ is comparable to $\rho(a,b)$ whenever $\beta(a,b) \leq 1$, the compatibility condition (2) yields

$$\sup_n \left| \frac{w_n^{(2)} - f(z_n^{(1)})}{B(z_n^{(2)})E(1-|f|)(z_n^{(2)})} \right| \leq C_3 \varepsilon.$$

Also, using (17) instead of the compatibility condition, the argument above tells that

$$\sup_n \left| \frac{f(z_n^{(2)}) - f(z_n^{(1)})}{B(z_n^{(2)})E(1-|f|)(z_n^{(2)})} \right| \leq C_4 \varepsilon.$$

So the values

$$w_n^* = \frac{w_n^{(2)} - f(z_n^{(2)})}{B(z_n^{(2)})E(1-|f|)(z_n^{(2)})}, \quad n = 1, 2, \ldots$$

satisfy $\sup_n |w_n^*| \leq 2C_5\varepsilon$. Since $\{z_n^{(2)}\}$ is a separated sequence satisfying the density condition (b) in Theorem 1, it is an interpolating sequence for H^∞. Hence, fixing $\varepsilon > 0$ sufficiently small there exists $g \in \mathcal{B}$ such that $g(z_n^{(2)}) = w_n^*$, for $n = 1, 2, \ldots$. Then the function

$$h = f + BgE(1-|f|)$$

is in $\mathcal{B}$ and will interpolate the whole sequence of values $\{w_n^{(1)}\} \cup \{w_n^{(2)}\}$ at the whole sequence $\{z_n^{(1)}\} \cup \{z_n^{(2)}\}$. This ends the proof of Theorem 1.

4.6. *Remark.* It is worth mentioning that if $\{z_n\}$ is a sequence that verifies (a) and (b) in Theorem 1 and $\{w_n\}$ is a sequence of values in the unit disc which satisfy the compatibility condition

$$\beta(w_n, w_m) \leq \varepsilon \beta(z_n, z_m), \quad n, m = 1, 2, \ldots$$

our construction provides a non extremal point f in $\mathcal{B}$ with $f(z_n) = w_n$ for $n = 1, 2, \ldots$. Then applying a refinement of Nevanlinna's theorem due to A. Stray [16] we may find a Blaschke product I such that $I(z_n) = w_n$, for $n = 1, 2, \ldots$.

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