Iterates of Blaschke products and Peano curves

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Abstract

Let f be a finite Blaschke product with $f(0)=0$ which is not a rotation and let f^n be its n-th iterate. Given a sequence $\{a_n\}$ of complex numbers consider $F=\sum a_n f^n$. If $\{a_n\}$ tends to 0 but $\sum |a_n| = \infty$, we prove that for any complex number w there exists a point ξ in the unit circle such that $\sum a_n f^n(\xi)$ converges and its sum is w. If $\sum |a_n| < \infty$ and the convergence is slow enough in a certain precise sense, then the image of the unit circle by F has a non empty interior. The proofs are based on inductive constructions which use the beautiful interplay between the dynamics of f as a selfmapping of the unit circle and those as a selfmapping of the unit disc.

1 Introduction

A lacunary power series is a power series of the form

$$F(z)=\sum_{n=1}^{\infty} a_n z^{k_n}, \tag{1}$$

where $\{a_n\}$ is a sequence of complex numbers and $\{k_n\}$ is a sequence of positive integers satisfying $\inf k_{n+1}/k_n > 1$. The behavior of lacunary series has been extensively studied and it has been shown that in many senses, they behave as sums of independent random variables. If the coefficients $\{a_n\}$ satisfy $\sum |a_n| = \infty$ but $a_n \to 0$ as $n\to\infty$, a theorem of Paley, proved by Weiss in [W], says that for any $w \in \mathbb{C}$ there exists a point ξ in the unit circle $\partial\mathbb{D}$ such that $\sum a_n \xi^{k_n}$ converges and its sum is w. Salem and Zygmund proved that boundary values of certain lacunary series are Peano curves ([SZ]). Their result was refined by Kahane, Weiss and Weiss who showed that if $\sum |a_n| < \infty$ but the convergence is slow enough (in a certain precise sense), then $F\colon \partial\mathbb{D} \to \mathbb{C}$ as defined in (1) is a Peano curve, that is, $F(\partial\mathbb{D})$ contains a (non-degenerate) disc. See [KWW]. More recent related results have been proved by Barański ([Ba]), Belov ([Be]), Murai ([Mu1], [Mu2], [Mu3]) and Younsi ([Y]).

Let f be a finite Blaschke product with $f(0)=0$ which is not a rotation and let f^n denote its n-th iterate. The main purpose of this paper is to present analogous results for series of the form

$$F(z)=\sum_{n=1}^{\infty} a_n f^n(z). \tag{2}$$

It is worth mentioning that several recent results show that linear combinations of iterates, as defined in (2), behave as lacunary series. See [NS] and [N]. Our first result is a version of Paley's Theorem in this context.

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Theorem 1.1. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers tending to 0 such that $\sum |a_n| = \infty$. Then for any $w \in \mathbb{C}$ there exists $\xi \in \partial\mathbb{D}$ such that $\sum a_n f^n(\xi)$ converges and $\sum_{n=1}^{\infty} a_n f^n(\xi) = w$.*

Let m denote Lebesgue measure on the unit circle normalized so that $m(\partial\mathbb{D}) = 1$. It has been proved in [N] that if $\sum |a_n|^2 < \infty$, then the series (2) converges at almost every point of the unit circle. Conversely, if $\sum |a_n|^2 = \infty$, then the series (2) diverges at almost every point of the unit circle. Hence if the coefficients $\{a_n\}$ tend to 0 but $\sum |a_n|^2 = \infty$, Theorem 1.1 provides a set $E \subset \partial\mathbb{D}$ with $m(E) = 0$ such that for any $w \in \mathbb{C}$ there exists $\xi \in E$ such that $\sum a_n f^n(\xi)$ converges and its sum is w. Our next result says that under appropriate conditions on the coefficients $\{a_n\}$, series of the form (2) lead to Peano curves.

Theorem 1.2. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers with $\sum |a_n| < \infty$. Assume*

$$\lim_{n\to\infty} \frac{|a_n|}{\sum_{k>n} |a_k|} = 0. \tag{3}$$

Then $F = \sum_{n=1}^{\infty} a_n f^n \colon \partial\mathbb{D} \to \mathbb{C}$ is a Peano curve, that is, $F(\partial\mathbb{D})$ contains a (non-degenerate) disc.

The proofs of these results are based on delicate inductive constructions which use some techniques due to Weiss ([W]) but we also need several new ideas which arise from the beautiful interplay between dynamical properties of a Blaschke product as a selfmapping of $\partial\mathbb{D}$ and those as a selfmapping of the unit disc $\mathbb{D}$. It is worth mentioning that no lacunarity assumption is needed in our results. The following more technical result plays a central role in our arguments.

Theorem 1.3. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Then there exist constants $\varepsilon = \varepsilon(f) > 0$ and $0 < c = c(f) < 1$ such that the following statement holds. Let $M < N$ be positive integers, let $z \in \mathbb{D}$ with $|f^M(z)| < \varepsilon$ and let $\{a_n : M \le n \le N\}$ be a collection of complex numbers. Then there exists a point $\xi \in \partial\mathbb{D}$ with $|\xi - z| \le c^{-1}(1 - |z|)$ such that*

$$\mathrm{Re}\left(\sum_{n=M}^{N} a_n f^n(\xi)\right) \ge c \sum_{n=M}^{N} |a_n|. \tag{4}$$

The paper is organized as follows. Several auxiliary results are collected in Section 2. These are used in Section 3 where we present the proofs of our main results. Finally, in Section 4 we prove a version of the classical Abel's Theorem in our context, sketch a proof of a generalization of Theorem 1.1 and conclude mentioning several open problems.

2 Auxiliary results

Let g be an analytic mapping from $\mathbb{D}$ into itself with $g(0) = 0$ which is not a rotation. The classical Denjoy-Wolff Theorem says that the iterates g^n converge uniformly to 0 on compacts of $\mathbb{D}$. Actually Pommerenke proved the following exponential decay: there exist constants $0 < a = a(f) < 1$ such that $|g^n(z)| \le a^n (1 - |z|)^{-13}$ for any $z \in \mathbb{D}$. See [P].

A finite Blaschke product f is a finite product of automorphisms of $\mathbb{D}$, that is,

$$f(z) = \prod_{n=1}^{N} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in \mathbb{C},$$

where $z_1, \dots, z_n \in \mathbb{D}$ are the zeros of f. Observe that

$$i\xi \frac{f'(\xi)}{f(\xi)} = \sum_{n=1}^{N} P(z_n, \xi), \quad \xi \in \partial\mathbb{D}, \tag{5}$$

where $P(z, \xi) = (1 - |z|^2)|\xi - z|^{-2}$ is the Poisson kernel at the point z. Hence

$$f(e^{i\theta})\overline{f(1)} = e^{i\psi(\theta)}, \quad 0 \le \theta \le 2\pi,$$

where

$$\psi(\theta) = \int_0^\theta \sum_{n=1}^{N} P(z_n, e^{it})\, dt.$$

Note that ψ is a real analytic branch of the argument of $f(e^{i\theta})\overline{f(1)}$ which is increasing and satisfies

$$f(e^{i\theta})\overline{f(e^{i\varphi})} = \exp\left(i \int_\varphi^\theta \sum_{n=1}^{N} P(z_n, e^{it})\, dt \right), \quad 0 \le \varphi \le \theta \le 2\pi. \tag{6}$$

Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Note that (5) gives that $\min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\} > 1$ and the mapping $f \colon \partial\mathbb{D} \to \partial\mathbb{D}$ is expanding in the sense that $m(f(I)) > m(I)$ for any arc $I \subset \partial\mathbb{D}$ with $m(I) < 1$. Our first auxiliary result points in this direction.

Lemma 2.1. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Consider $K = K(f) = \min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\} > 1$. Let N be a positive integer and let $I \subset \partial\mathbb{D}$ be an arc such that $m(f^N(I)) = \delta < 1$. Then*

$$|f^k(\xi) - f^k(\xi')| \le 2\pi\delta K^{k-N} \tag{7}$$

for any $\xi, \xi' \in I$ and any integer $0 < k \le N$.

Proof. Since there exists an increasing continuous branch of $\operatorname{Arg}(f^N)$ and $m(f^N(I)) < 1$, the mapping $f^N \colon I \to f^N(I)$ is one to one. Then for any integer $0 < k \le N$ we have

$$\begin{aligned} \delta = \int_I |(f^N)'|\, dm &= \int_I |(f^{N-k})'(f^k)||(f^k)'|\, dm \\ &\ge K^{N-k} \int_I |(f^k)'|\, dm = K^{N-k} m(f^k(I)). \end{aligned}$$

We deduce that $m(f^k(I)) \le \delta K^{k-N}$. Since $f^k(I) \subset \partial\mathbb{D}$ is an arc, the estimate (7) follows. □

For future reference we now state two easy consequences of Lemma 2.1.

Corollary 2.2. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Then there exists a constant $c = c(f) > 0$ such that if $I \subset \partial\mathbb{D}$ is an arc, $M < N$ are positive integers with $m(f^N(I)) = \delta < 1$ and $\{a_n : M \le n \le N\}$ is a collection of complex numbers, then*

$$\left| \sum_{n=M}^{N} a_n (f^n(\xi) - f^n(\xi')) \right| \le c\delta \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}, \quad \xi, \xi' \in I.$$

Proof. Consider $K = \min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\}$. Since $f(0) = 0$ and f is not a rotation, identity (5) gives $K > 1$. Lemma 2.1 and Cauchy–Schwarz inequality give

$$\sum_{n=M}^{N} |a_n||f^n(\xi) - f^n(\xi')| \le \frac{2\pi\delta}{(K^2 - 1)^{1/2}} \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}.$$

Taking $c = 2\pi(K^2 - 1)^{-1/2}$, the proof is completed. □

Corollary 2.3. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Fix $0 < \delta < 1$. Let $\{a_n\}$ be a sequence of complex numbers tending to 0. For $N = 1, 2, \dots,$ let $I_N \subset \partial\mathbb{D}$ be an arc such that $m(f^N(I_N)) = \delta$. Then*

$$\max\left\{\left|\sum_{n=1}^{N} a_n(f^n(\xi) - f^n(\xi'))\right| : \xi, \xi' \in I_N\right\} \underset{N\to\infty}{\longrightarrow} 0.$$

Proof. Consider $K = \min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\}$. As before, identity (5) gives $K > 1$. Let $L = L(N)$ be a positive integer to be fixed later satisfying $1 < L < N$. Since $m(f^N(I_N)) = \delta$, Lemma 2.1 gives

$$\begin{aligned}\sum_{n=1}^{N} |a_n||f^n(\xi) - f^n(\xi')| &\le 2\pi\delta \sup_n |a_n| \sum_{n=1}^{L} K^{n-N} + 2\pi\delta \sup_{n>L} |a_n| \sum_{n=L+1}^{N} K^{n-N} \\ &\le 2\pi\delta \sup_n |a_n| \frac{K^{L-N+1}}{K-1} + 2\pi\delta \sup_{n>L} |a_n| \frac{K}{K-1}\end{aligned}$$

which tends to 0 as N tends to ∞, if L is chosen such that both L and $N - L$ tend to ∞ as N tends to ∞. □

Given a point $z \in \mathbb{D}\backslash\{0\}$, let $I(z)$ denote the arc of the unit circle centered at $z/|z|$ with $m(I(z)) = 1 - |z|$. Conversely given an arc $I \subsetneq \partial\mathbb{D}$ let $z(I)$ be the point in $\mathbb{D}$ satisfying $I(z(I)) = I$. Let $\rho(z, w)$ denote the pseudohyperbolic distance between $z, w \in \mathbb{D}$ given by

$$\rho(z, w) = \frac{|z - w|}{|1 - \bar{w}z|}.$$

Lemma 2.4. *(a) For any $0 < \gamma < 1$, there exists $\delta = \delta(\gamma) > 0$ such that if f is a finite Blaschke product and $z \in \mathbb{D}$ satisfies $|f(z)| \le \gamma$, then $m(f(I(z))) \ge \delta$.*

(b) Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Then, given $0 < \delta < 1$, there exists $0 < \gamma = \gamma(\delta, f) < 1$ such that if N is a positive integer, $I \subset \partial\mathbb{D}$ is an arc with $m(f^N(I)) = \delta$, then $|f^N(z(I))| \le \gamma$.

Remark 2.5. It is worth mentioning that the converse of the estimate in (a) does not hold, that is, $|f(z)|$ could be arbitrarily close to 1 and $m(f(I(z))) \ge 1/2$. For instance, if $f(z) = (z - r)^n/(1 - rz)^n$, where $0 < r < 1$ and $n > 2$, we have $|f(s)| = |(s - r)/(1 - rs)|^n$, which could be arbitrarily close to 1 if $\rho(s, r)$ is sufficiently close to 1, while $f(I(s)) = \partial\mathbb{D}$ if $s < r$. Part (b) says that the converse of the estimate (a) holds uniformly for iterates of a finite Blaschke product f, if the constant γ is allowed to depend on f.

Proof. (a) Let $\{z_n\}$ be the zeros of f. We can assume that the mapping $f : I(z) \to f(I(z))$ is one to one since otherwise using (6) we would deduce that $f(I(z)) = \partial\mathbb{D}$. We can also assume $\inf\{\rho(z, z_n) : n \ge 1\} \ge 1/2$. The identity (5) gives

$$m(f(I(z))) = \int_{I(z)} \sum_n P(z_n, \xi)\, dm(\xi).$$

Note that there exists a universal constant $c_1 > 0$ such that $|\xi - z_n| \le c_1|1 - z_n\bar{z}|$ for any $\xi \in I(z)$ and any n. We deduce

$$m(f(I(z))) \ge c_1^{-2} \sum_n \frac{1 - |z_n|^2}{|1 - z_n\bar{z}|^2}(1 - |z|). \tag{8}$$

Since $\inf\{\rho(z, z_n) : n \geq 1\} \geq 1/2$, the elementary estimate $-\log x \leq c_2(1-x^2)$, $1/2 \leq x \leq 1$, provides a universal constant $c_3 > 0$ such that

$$\sum_n \frac{(1-|z_n|^2)(1-|z|)}{|1-z_n\bar{z}|^2} \geq c_3 \log |f(z)|^{-1}. \tag{9}$$

This finishes the proof of (a). We now prove (b). We first show that there exists a constant $c_4 = c_4(f, \delta) > 1$ such that

$$|(f^N)'(\xi)| \leq c_4 |(f^N)'(\xi^*)|, \quad \xi, \xi^* \in I, \quad N \geq 1. \tag{10}$$

Using that $|\log x - \log y| \leq |x-y|$ for any $x, y > 1$, we obtain

$$\begin{aligned}\left|\log \frac{|(f^N)'(\xi)|}{|(f^N)'(\xi^*)|}\right| &= \left|\sum_{k=1}^{N-1} \log \frac{|f'(f^k(\xi))|}{|f'(f^k(\xi^*))|}\right| \\ &\leq \sum_{k=1}^{N-1} |f'(f^k(\xi)) - f'(f^k(\xi^*))| \leq c_5 \sum_{k=1}^{N-1} |f^k(\xi) - f^k(\xi^*)|,\end{aligned}$$

where $c_5 = \max\{|f''(\xi)| : \xi \in \partial\mathbb{D}\}$. Since $m(f^N(I)) = \delta < 1$, Lemma 2.1 gives the estimate (10).

Let $\xi(I)$ be the center of I. Applying (10), for any measurable subset $J \subset I$, we have

$$m(f^N(J)) = \int_J |(f^N)'|\, dm \geq c_4^{-1} |(f^N)'(\xi(I))| m(J) \geq c_4^{-2} m(f^N(I)) \frac{m(J)}{m(I)} = c_4^{-2} \delta \frac{m(J)}{m(I)}.$$

We deduce that there exists a constant $c_6 = c_6(\delta, f) > 0$ such that

$$\frac{1}{m(I)} \int_I |f^N - f^N(z(I))|^2\, dm \geq c_6. \tag{11}$$

Since there exists a universal constant $c_7 > 0$ such that $P(z(I), \xi) \geq c_7 m(I)^{-1}$ for any $\xi \in I$, we obtain

$$\begin{aligned}\frac{1}{m(I)} \int_I |f^N - f^N(z(I))|^2\, dm &\leq c_7^{-1} \int_{\partial\mathbb{D}} |f^N(\xi) - f^N(z(I))|^2 P(z(I), \xi)\, dm(\xi) \\ &= c_7^{-1}(1 - |f^N(z(I))|^2).\end{aligned}$$

Using (11) we deduce $1 - |f^N(z(I))|^2 \geq c_6 c_7$ and the proof is completed. □

Next auxiliary result will be used in Section 4 and it is not needed in the proofs of our main results.

Lemma 2.6. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Fix $0 < \delta < (10 \max\{|f'(\xi)| : \xi \in \partial\mathbb{D}\})^{-1}$. Then there exits a constant $C = C(\delta, f) > 0$ such that the following statement holds. Given $z \in \mathbb{D}$ let $N(z)$ be the smallest positive integer such that $m(f^{N(z)}(I(z))) \geq \delta$. Then*

$$\sum_{k=1}^{N(z)} |f^k(\xi) - f^k(z)| \leq C, \quad \xi \in I(z). \tag{12}$$

Proof. Let $K_1 = \max\{|f'(\xi)| : \xi \in \partial\mathbb{D}\}$. Note that $N(z) \to \infty$ as $|z| \to 1$ and $m(f^{N(z)}(I(z))) \leq \delta K_1$. Part (b) of Lemma 2.4 gives a constant $0 < \gamma = \gamma(\delta, f) < 1$ such that $|f^{N(z)}(z)| \leq \gamma$. Note that $f^{N(z)}$ can not have zeros at pseudohyperbolic distance less than $1/2$ of z, because in this case $m(f^{N(z)}(I(z)))$ would be large. Then, estimates (8) and (9) in the proof of part (a) of Lemma 2.4 give that $\delta K_1 \geq m(f^{N(z)}(I(z))) \geq c_1^{-2} c_3 \log |f^{N(z)}(z)|^{-1}$. Hence

$$e^{-\delta K_1 c_1^2 / c_3} \leq |f^{N(z)}(z)| \leq \gamma. \tag{13}$$

Pommerenke estimates of the Denjoy–Wolff Theorem (Lemma 2 of [P]) provide constants $C(\gamma) > 0$ and $0 < a = a(f) < 1$ such that

$$|f^n(z)| \leq C(\gamma) a^{n-N(z)}, \quad n \geq N(z). \tag{14}$$

Let $K = \min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\}$. Since $f(0) = 0$ and f is not a rotation, identity (5) gives $K > 1$. Lemma 2.1 gives

$$|f^k(\xi) - f^k(\xi')| \leq 2\pi\delta K_1 K^{k-N(z)}, \quad \xi, \xi' \in I(z), \quad 0 < k \leq N(z).$$

Hence $|f^k(\xi) - f^k(z)| \leq |f^k(\xi') - f^k(z)| + 2\pi\delta K_1 K^{k-N(z)}$, for any $\xi, \xi' \in I(z)$ and $0 < k \leq N(z)$. Integrating over $\xi' \in I(z)$ and using that there exists a constant $c_4 > 0$ such that $P(z,\xi) \geq c_4/(1-|z|), \xi \in I(z)$, we deduce

$$\begin{aligned}
|f^k(\xi) - f^k(z)| &\leq (1-|z|)^{-1} \int_{I(z)} |f^k(\xi') - f^k(z)|\, dm(\xi') + 2\pi\delta K_1 K^{k-N(z)} \\
&\leq c_4^{-1} \left(\int_{\partial\mathbb{D}} |f^k(\xi') - f^k(z)|^2 P(z,\xi')\, dm(\xi') \right)^{1/2} + 2\pi\delta K_1 K^{k-N(z)} \\
&= c_4^{-1} (1 - |f^k(z)|^2)^{1/2} + 2\pi\delta K_1 K^{k-N(z)},\ \xi \in I(z),\ 0 < k \leq N(z).
\end{aligned}$$

Using Lemma 2.1(a) of [N] and (13) one finds constants $c_5 = c_5(\delta) > 0$ and $0 < b = b(f) < 1$ such that $1 - |f^k(z)|^2 \leq c_5 b^{N(z)-k}$, if $0 < k \leq N(z)$. Hence there exists a constant $C = C(\delta, f) > 0$ such that

$$\sum_{k=1}^{N(z)} |f^k(\xi) - f^k(z)| \leq C, \quad \xi \in I(z). \tag{15}$$

□

Next auxiliary result is Lemma 3.3 of [N].

Lemma 2.7. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exist two constants $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ such that the following statement holds. Let $M < N$ be positive integers, $z \in \mathbb{D}$ satisfying $|f^M(z)| < \varepsilon$ and $\{a_n : M \leq n \leq N\}$ a collection of complex numbers. Then there exists a set $E = E(z, \{a_n\}) \subset \partial\mathbb{D}$ with $E \subset c^{-1} I(z)$ such that*

$$\operatorname{Re}\left(\sum_{n=M}^{N} a_n f^n(\xi) \right) \geq c \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}, \quad \xi \in E.$$

Our last auxiliary result is elementary and it is stated for future reference.

Lemma 2.8. *Let $\{b_n\}$ be a sequence of positive numbers with $\sum b_n < \infty$. Consider $S_n = \sum_{k>n} b_k$. Assume*

$$\lim_{n\to\infty} \frac{b_n}{S_n} = 0.$$

Fix $K > 1$. Then

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} b_n K^{n-N}}{S_N} = 0$$

Proof. Fix $\varepsilon > 0$ such that $K(1+\varepsilon)^{-1} > 1$. By assumption there exists $n_0 = n_0(\varepsilon) > 0$ such that $S_n \geq (1+\varepsilon)^{-1} S_{n-1}$ for any $n \geq n_0$. Hence $S_n \geq (1+\varepsilon)^{-n+n_0} S_{n_0}$, $n \geq n_0$. We deduce that

$$\lim_{n\to\infty} K^n S_n = \infty. \tag{16}$$

Since $b_n \leq \varepsilon S_n$ for $n \geq n_0$ and $S_n \leq (1+\varepsilon)^{N-n} S_N$ for $n_0 \leq n \leq N$, we have

$$\sum_{n=n_0}^{N} b_n K^{n-N} \leq \varepsilon S_N \sum_{n=n_0}^{N} \left(\frac{K}{1+\varepsilon}\right)^{n-N}.$$

Now the identity (16) finishes the proof. □

3 Proofs of the main results

This section is devoted to the proofs of Theorems 1.1, 1.2 and 1.3. It is worth mentioning that under additional assumptions on the function f, one can give a short proof of Theorem 1.3. Actually, let f be a finite Blaschke product and assume that there exists a constant $\varepsilon > 0$ such that

$$f(I) = \partial\mathbb{D} \tag{17}$$

for any arc $I \subset \partial\mathbb{D}$ with $m(I) \geq 1/2 - \varepsilon$. For instance, according to (6), any function f of the form $f(z) = z^2 g(z)$, where g is a non-constant Blaschke product, satisfies (17). Let $\{a_n\}$ be a sequence of complex numbers. For any n such that $a_n \neq 0$, consider the closed arc I_n centered at $\overline{a}_n/|a_n|$ with $m(I_n) = 1/2 - \varepsilon$. Note that there exists a constant $c = c(\varepsilon) > 0$ such that

$$\mathrm{Re}\left(\sum_n a_n \xi_n\right) \geq c \sum_n |a_n|, \tag{18}$$

for any choice of points $\xi_n \in I_n$, $n \geq 1$. Take $J_1 = I_1$. We now construct inductively a sequence of nested closed arcs J_n such that $f^{n-1}(J_n) = I_n$, $n > 1$. Since $f(I_1) = \partial\mathbb{D}$, there exists an arc $J_2 \subset I_1$ with $f(J_2) = I_2$. Assume $J_2, \dots, J_{n-1}$ have been constructed satisfying $f^{n-2}(J_{n-1}) = I_{n-1}$. Then (17) gives $f^{n-1}(J_{n-1}) = f(I_{n-1}) = \partial\mathbb{D}$ and we can find an arc $J_n \subset J_{n-1}$ such that $f^{n-1}(J_n) = I_n$. This finishes the construction of the nested sequence of arcs $\{J_n\}$. Pick $\xi \in \partial\mathbb{D}$ such that $f(\xi) \in \bigcap_n J_n$. Since $f^n(\xi) \in I_n$ for any $n \geq 1$, estimate (18) gives

$$\mathrm{Re}\left(\sum_n a_n f^n(\xi)\right) \geq c \sum_n |a_n|.$$

This finishes the proof of Theorem 1.3 under the additional assumption (17). However there are finite Blaschke products f which do not satisfy (17) and the proof of Theorem 1.3 requires different ideas.

Proof of Theorem 1.3. The proof of Theorem 1.3 is organized in three steps.

1. Splitting. We use an idea of M. Weiss ([W]). Let $T' < T$ be two (large) positive integers to be fixed later such that T/T' is also an integer. Since one can obviously add terms with vanishing coefficients to the left-hand side term of (4), one can assume that $N - M + 1$ is a multiple of T. Split the sum $F = \sum_M^N a_n f^n$ into blocks of length T, that is, $F = \sum_{k\geq 0} G_k$, where

$$G_k = \sum_{n=0}^{T-1} a_{M+kT+n} f^{M+kT+n}, \quad k = 0, 1, 2, \dots$$

Next split G_{2k}, $k \geq 1$, into successive blocks of length T' and pick the subblock such that the sum of the modulus of the coefficients is the least, that is, pick $\mathcal{S}_k$ a subset of T' consecutive integers in $[M+2kT, M+(2k+1)T)$ of the form $\{M+2kT+jT', \dots, M+2kT+(j+1)T'-1\}$ such that

$$\sum_{n\in\mathcal{S}_k} |a_n| \leq \sum_{\ell\in\mathcal{S}} |a_\ell|,$$

for any other subset $\mathcal{S}$ of T' consecutive integers in $[M+2kT, M+(2k+1)T)$ of the same form. Since the number of disjoint subblocks of this type is T/T' we deduce

$$\sum_{n\in\mathcal{S}_k} |a_n| \leq \frac{T'}{T} \sum_{n=0}^{T-1} |a_{M+2kT+n}|, \quad k \geq 1. \tag{19}$$

The corresponding block

$$S_k = \sum_{n\in\mathcal{S}_k} a_n f^n, \quad k \geq 1$$

will be called a short block. The long blocks are defined as the blocks between two consecutive short blocks as well as the first and last block. More concretely if $\mathcal{S}_k = \{n \in \mathbb{Z} : N_k < n < M_{k+1}\}$, $k \geq 1$, then

$$L_1 = \sum_{n=M}^{N_1} a_n f^n, \quad L_k = \sum_{n=M_k}^{N_k} a_n f^n, \quad k > 1,$$

and

$$S_k = \sum_{n=N_k+1}^{M_{k+1}-1} a_n f^n, \quad k \geq 1.$$

Write $M_1 = M$ and $\mathcal{L}_k = \{n \in \mathbb{Z} : M_k \leq n \leq N_k\}$, $k \geq 1$. Note that for any $k \geq 1$, the set $\mathcal{S}_k$ has T' indexes while the number of indices in $\mathcal{L}_k$ is between T and $3T$. Note also that (19) gives

$$\sum_k \sum_{n\in\mathcal{L}_k} |a_n| \geq \left(1 - \frac{T'}{T}\right) \sum_{n=M}^{N} |a_n|. \tag{20}$$

2. The inductive construction. Let $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ be the constants appearing in Lemma 2.7. We will show that there exist constants $c_0 = c_0(f) > 0$ and $0 < \gamma = \gamma(f) < 1$ and a sequence of nested closed arcs $I_k \subset c^{-1}I(z)$, such that

$$|f^{N_k}(z(I_k))| \leq \gamma, \tag{21}$$

$$\operatorname{Re} \sum_{j=1}^{k} L_j(\xi) \geq \frac{c_0}{T^{1/2}} \sum_{j=1}^{k} \sum_{n\in\mathcal{L}_j} |a_n|, \quad \xi \in I_k. \tag{22}$$

We argue by induction. Since $|f^M(z)| < \varepsilon$, Lemma 2.7 provides a point $\xi_1 \in c^{-1}I(z)$ with

$$\operatorname{Re} L_1(\xi_1) \geq c \left(\sum_{n=M}^{N_1} |a_n|^2\right)^{1/2}. \tag{23}$$

Note that $|f^{N_1}(z)| \leq |f^M(z)| < \varepsilon$. By part (a) of Lemma 2.4, there exists a constant $\delta_0 = \delta_0(\varepsilon) > 0$ such that $m(f^{N_1}(I(z))) \geq \delta_0$. Fix $0 < \delta_1 < \delta_0/2$ and pick an arc I_1 with $\xi_1 \in I_1 \subset c^{-1}I(z)$ with $m(f^{N_1}(I_1)) = \delta_1$. If $\delta_1 > 0$ is chosen sufficiently small, Corollary 2.2 and (23) give

$$\operatorname{Re} L_1(\xi) \geq \frac{c}{2} \left(\sum_{n=M}^{N_1} |a_n|^2\right)^{1/2}, \quad \xi \in I_1.$$

Since $\mathcal{L}_1$ has at most $3T$ indexes, Cauchy–Schwarz's inequality gives

$$\sum_{n\in\mathcal{L}_1} |a_n| \le (3T)^{1/2} \left(\sum_{n\in\mathcal{L}_1} |a_n|^2 \right)^{1/2}$$

and (22) holds for $k=1$ if we pick $0 < c_0 < c/2\sqrt{3}$. Since $m(f^{N_1}(I_1)) = \delta_1$, part (b) of Lemma 2.4 provides a constant $\gamma = \gamma(\delta_1, f) < 1$ such that the estimate (21) holds for $k=1$. Assume now that the arcs $I_k \subset I_{k-1} \subset \cdots \subset I_1$ have been constructed so that (21) and (22) hold. Next we will construct I_{k+1}. Fix a constant $0 < c_1 = c_1(c) < 1$ such that there exists a point $z_k^* \in \mathbb{D}$ with $\rho(z_k^*, z(I_k)) \le c_1$ with $c^{-1}I(z_k^*) \subset I_k$. Since $|f^{N_k}(z(I_k))| \le \gamma$, Schwarz's Lemma gives a constant $0 < \gamma_1 = \gamma_1(\gamma, c_1) < 1$ with $|f^{N_k}(z_k^*)| \le \gamma_1$. Recall that $M_{k+1} = N_k + T'$. Since $f^n \to 0$ uniformly on compacts of $\mathbb{D}$, we can choose a positive integer $T' = T'(f)$ sufficiently large but independent of k, so that

$$|f^{M_{k+1}}(z_k^*)| \le \varepsilon.$$

Apply Lemma 2.7 to find a point $\xi_k^* \in c^{-1}I(z_k^*) \subset I_k$ such that

$$\operatorname{Re}(L_{k+1}(\xi_k^*)) \ge c \left(\sum_{n\in\mathcal{L}_{k+1}} |a_n|^2 \right)^{1/2}. \tag{24}$$

Note that since $f(0)=0$, Schwarz's lemma gives $|f^{N_{k+1}}(z_k^*)| \le |f^{M_{k+1}}(z_k^*)| \le \varepsilon$. By part (a) of Lemma 2.4, there exists a constant $\delta_0 = \delta_0(\varepsilon) > 0$ such that $m(f^{N_{k+1}}(I(z_k^*))) \ge \delta_0$. Fix $0 < \delta_1 < \delta_0/2$ and pick an arc I_{k+1} with $\xi_k^* \in I_{k+1} \subset I_k$ such that $m(f^{N_{k+1}}(I_{k+1})) = \delta_1$. If $\delta_1 > 0$ is chosen sufficiently small, Corollary 2.2 and estimate (24) give

$$\operatorname{Re}(L_{k+1}(\xi)) \ge \frac{c}{2} \left(\sum_{n\in\mathcal{L}_{k+1}} |a_n|^2 \right)^{1/2}, \quad \xi \in I_{k+1}.$$

Since $\mathcal{L}_{k+1}$ has at most $3T$ indexes, Cauchy–Schwarz's inequality gives

$$\sum_{n\in\mathcal{L}_{k+1}} |a_n| \le (3T)^{1/2} \left(\sum_{n\in\mathcal{L}_{k+1}} |a_n|^2 \right)^{1/2}$$

and we deduce

$$\operatorname{Re} L_{k+1}(\xi) \ge \frac{c}{2} \frac{1}{(3T)^{1/2}} \sum_{n\in\mathcal{L}_{k+1}} |a_n|, \quad \xi \in I_{k+1}.$$

If we pick $0 < c_0 < c/2\sqrt{3}$, the inductive assumption gives

$$\operatorname{Re} \sum_{j=1}^{k+1} L_j(\xi) \ge \frac{c_0}{T^{1/2}} \sum_{j=1}^{k+1} \sum_{n\in\mathcal{L}_j} |a_n|, \quad \xi \in I_{k+1}$$

which is (22) for the index $k+1$. Since $m(f^{N_{k+1}}(I_{k+1})) = \delta_1$, the estimate (21) for the index $k+1$, follows from part (b) of Lemma 2.4. This finishes the proof of the existence of the nested sequence of arcs I_k satisfying (21) and (22).

3. The final argument. Pick $\xi \in \bigcap_k I_k$. The estimate (22) gives

$$\operatorname{Re}\left(\sum_{j=1}^{k} L_j(\xi) + \sum_{j=1}^{k-1} S_j(\xi) \right) \ge \frac{c_0}{T^{1/2}} \sum_{j=1}^{k} \sum_{n\in\mathcal{L}_j} |a_n| - \sum_{j=1}^{k-1} \sum_{n\in\mathcal{S}_j} |a_n|, \quad k > 1.$$

Since $\sum_{n=M}^{N} a_n f^n$ is a sum of long and short blocks, the estimates (19) and (20) give

$$\operatorname{Re}\left(\sum_{n=M}^{N} a_n f^n(\xi)\right) \geq \frac{c_0}{T^{1/2}}\left(1-\frac{T'}{T}\right)\sum_{n=M}^{N}|a_n| - \frac{T'}{T}\sum_{n=M}^{N}|a_n|.$$

Choosing T such that $T'/T^{1/2} < c_0/4$, we deduce

$$\operatorname{Re}\left(\sum_{n=M}^{N} a_n f^n(\xi)\right) \geq \frac{c_0}{2T^{1/2}}\sum_{n=M}^{N}|a_n|.$$

This finishes the proof. □

The proof of Theorem 1.1 uses the following easy consequence of Theorem 1.3.

Corollary 3.1. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Then there exists a constant $0 < \eta = \eta(f) < 1$ such that the following statement holds. Let $M < N$ be positive integers, let $z \in \mathbb{D}$ with $|f^M(z)| \leq \eta$, let $\{a_n : M \leq n \leq N\}$ be a collection of complex numbers and let $w \in \mathbb{C}$ with*

$$\sum_{n=M}^{N}|a_n| \leq \eta|w|. \tag{25}$$

Then there exists a point $\xi \in \eta^{-1}I(z)$ such that

$$\left|w - \sum_{n=M}^{N} a_n f^n(\xi)\right| \leq |w| - \eta\sum_{n=M}^{N}|a_n|.$$

Proof. We can assume $w = 1$. Let $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ be the constants appearing in Theorem 1.3. Pick $0 < \eta < \min\{\varepsilon, c\}$. Applying Theorem 1.3 one finds a point $\xi \in c^{-1}I(z)$ such that

$$\operatorname{Re}\left(\sum_{n=M}^{N} a_n f^n(\xi)\right) \geq c\sum_{n=M}^{N}|a_n|. \tag{26}$$

Write $A = \sum\limits_{n=M}^{N}|a_n|$. Since $A \leq \eta < 1$, we have

$$\left|1 - \sum_{n=M}^{N} a_n f^n(\xi)\right|^2 \leq (1-cA)^2 + A^2 - (cA)^2 = 1 - A(2c - A).$$

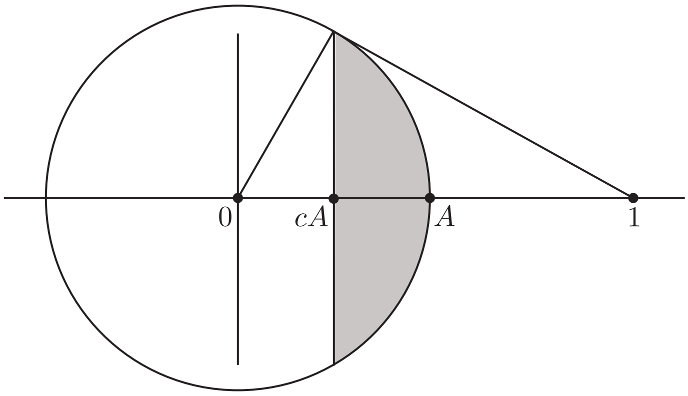
See Figure 1. Using the elementary estimate $\sqrt{1-x} \leq 1 - x/2$, $0 \leq x < 1$, we deduce that $\left|1 - \sum\limits_{n=M}^{N} a_n f^n(\xi)\right| \leq 1 - A(c - A/2) \leq 1 - \eta A$ if $0 < \eta < 2c/3$.

Figure 1: $\sum a_n f^n(\xi)$ lies in the shadowed region. □

We are now ready to prove Theorem 1.1. We will again combine some ideas of Weiss ([W]) with the interplay between dynamical properties of f as a selfmapping of $\mathbb{D}$ with those as a selfmapping of $\partial\mathbb{D}$.

Proof of Theorem 1.1. Fix $w \in \mathbb{C}$. We first divide the sum $\sum a_n f^n$ into blocks

$$P_j = \sum_{n\in\mathcal{P}_j} a_n f^n, \quad P_j^* = \sum_{n\in\mathcal{P}_j^*} a_n f^n,$$



where $\mathcal{P}_j = \{n \in \mathbb{Z} : M_j \le n \le N_j\}$, $M_1 = 1$ and $\mathcal{P}_j^* = \{n \in \mathbb{Z} : N_j < n < M_{j+1}\}$, $j \ge 1$. This splitting is similar to the decomposition used in the proof of Theorem 1.3 but this time the number of indices in $\mathcal{P}_j$ may grow as $j \to \infty$ while, as before, the number of indices in $\mathcal{P}_j^*$ is a large number independent of j to be chosen later. In other words, the blocks will be chosen in such a way that $M_{j+1} - N_j = T = T(f)$ is large but independent of j, while $N_j - M_j$ may tend to infinity as $j \to \infty$. Write

$$F_k = \sum_{j=1}^{k} P_j + \sum_{j=1}^{k-1} P_j^* = \sum_{n=1}^{N_k} a_n f^n.$$

The blocks P_j and P_j^* will be constructed inductively, as will a sequence of positive numbers δ_k tending to 0, and a nested sequence of closed arcs $I_k \subset \partial\mathbb{D}$ in such a way that there exists a constant $0 < \gamma = \gamma(f) < 1$ such that if $d_k = \min\{|F_k(\xi) - w| : \xi \in I_k\}$, the following estimates hold

$$d_{k+1} \le \gamma d_k + \delta_k, \tag{27}$$

$$\sum_{n \in \mathcal{P}_k^* \cup \mathcal{P}_{k+1}} |a_n| \underset{k\to\infty}{\longrightarrow} 0, \tag{28}$$

$$m(f^{N_k}(I_k)) = 1/2. \tag{29}$$

Once this inductive construction is done, the result follows easily. Actually pick $\xi \in \bigcap_j I_j$. Note that (27) gives $d_k \to 0$ as $k \to \infty$. Then, identity (29) and Corollary 2.3 give that $F_k(\xi) \to w$ as $k \to \infty$. Applying (28) we deduce that $\sum_n a_n f^n(\xi)$ converges and its sum is w.

We now explain the construction of the blocks P_j, P_j^* and the arcs I_j. For $k = 1$ one can pick any partial sum P_1 and any arc $I_1 \subset \partial\mathbb{D}$ with $m(f^{N_1}(I_1)) = 1/2$. Assume by induction that the blocks $P_1, P_1^*, \ldots, P_{k-1}, P_{k-1}^*, P_k$ and the arcs $I_k \subset I_{k-1} \subset \cdots \subset I_1$ have been chosen. We will now define P_k^*, P_{k+1} and the arc $I_{k+1} \subset I_k$ and show that the estimates (27), (28) and (29) hold. Let $\eta = \eta(f) > 0$ be the constant appearing in Corollary 3.1. Fix a constant $0 < c_1 = c_1(\eta) < 1$ such that there exists $z_k^* \in \mathbb{D}$ with $\rho(z_k^*, z(I_k)) \le c_1$ such that $\eta^{-1} I(z_k^*) \subset I_k$. Observe that (29) and part (b) of Lemma 2.4 provide a constant $0 < \gamma_0 = \gamma_0(f) < 1$ such that $|f^{N_k}(z(I_k))| \le \gamma_0$. By Schwarz's Lemma, there exists a constant $0 < \gamma_1 = \gamma_1(\gamma_0, \eta) < 1$ such that $|f^{N_k}(z_k^*)| \le \gamma_1$. Since $f^n \to 0$ uniformly on compacts of $\mathbb{D}$, if the number of indices $T = T(f)$ of P_k^* is chosen sufficiently large (but independent of k), we deduce $|f^{M_{k+1}}(z_k^*)| \le \eta$, where $M_{k+1} = N_k + T$. This defines P_k^*. This choice of M_{k+1} allows to apply Corollary 3.1 to the point z_k^* and any partial sum starting at the index M_{k+1}. To define the next block P_{k+1} we distinguish two cases.

(I) Assume $\max_{j \ge N_k} |a_j| \le \eta d_k / 4$. In this case, let N_{k+1} be the minimal integer bigger than M_{k+1}

such that

$$\sum_{n=M_{k+1}}^{N_{k+1}} |a_n| \geq \eta d_k/2.$$

This defines the block P_{k+1}. Since $|a_{N_{k+1}}| \leq \eta d_k/4$, the minimality of N_{k+1} gives

$$3\eta d_k/4 \geq \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| \geq \eta d_k/2. \tag{30}$$

Let $\xi_k \in I_k$ such that $|w - F_k(\xi_k)| = d_k$. Apply Corollary 3.1 to the point $z_k^* \in \mathbb{D}$, the value $w - F_k(\xi_k)$ and the block P_{k+1} to obtain a point $\xi_{k+1} \in \eta^{-1}I(z_k^*) \subset I_k$ such that

$$\begin{aligned} |w - F_k(\xi_k) - P_{k+1}(\xi_{k+1})| &\leq d_k - \eta \sum_{n\in\mathcal{P}_{k+1}} |a_n| \\ &\leq d_k\left(1 - \frac{\eta^2}{2}\right). \end{aligned}$$

Hence

$$|w - F_k(\xi_{k+1}) - P_{k+1}(\xi_{k+1})| \leq d_k\left(1 - \frac{\eta^2}{2}\right) + \alpha_k,$$

where $\alpha_k = \max\{|F_k(\xi) - F_k(\xi')| : \xi, \xi' \in I_k\}$. We deduce

$$|w - F_{k+1}(\xi_{k+1})| \leq d_k\left(1 - \frac{\eta^2}{2}\right) + \alpha_k + \sum_{n\in\mathcal{P}_k^*} |a_n|. \tag{31}$$

Since f is expanding and $N_{k+1} > N_k$, the identity (29) gives $m(f^{N_{k+1}}(I_k)) > 1/2$. Let I_{k+1} be an arc with $\xi_{k+1} \in I_{k+1} \subset I_k$ and $m(f^{N_{k+1}}(I_{k+1})) = 1/2$. We now check (27), (28) and (29). Note that (29) holds by construction. Define $\delta_k = \alpha_k + \sum_{n\in\mathcal{P}_k^*} |a_n|$. The estimate (31) gives $d_{k+1} \leq d_k(1-\eta^2/2) + \delta_k$, which is (27) with $\gamma = 1 - \eta^2/2$. Corollary 2.3 gives that α_k tends to zero and since $\mathcal{P}_k^*$ contains a fix number of indices, it follows that δ_k tends to zero. Finally note that (30) and the fact that d_k tends to 0 as k tends to infinity, give (28).

(II) Assume $\max_{j\geq N_k} |a_j| > \eta d_k/4$. In this case, we choose P_{k+1} having a single term, that is, $N_{k+1} = M_{k+1}$. Let $\xi_k \in I_k$ such that $|w - F_k(\xi_k)| = d_k$. As in case (I), let I_{k+1} be an arc with $\xi_k \in I_{k+1} \subseteq I_k$ and $m(f^{N_{k+1}}(I_{k+1})) = 1/2$. Hence (29) follows by construction. Since in this case $\mathcal{P}_k^* \cup \mathcal{P}_{k+1}$ has $T+1$ elements, (28) follows from the assumption that $\{a_n\}$ tends to 0. Define $\delta_{k+1} = \left(4\eta^{-1} + T + 1\right)\max_{j\geq N_k} |a_j|$. Note that δ_k tends to 0 as k tends to infinity. Then

$$\begin{aligned} d_{k+1} \leq |F_{k+1}(\xi_k) - w| &\leq |F_k(\xi_k) - w| + |P_{k+1}(\xi_k)| + |P_k^*(\xi_k)| \\ &\leq d_k + (T+1)\max_{j\geq N_k} |a_j| \leq \delta_{k+1}. \end{aligned}$$

This gives (27). □

The proof of Theorem 1.2 has some similarities with the proof of Theorem 1.1 but requires some new ideas.

Proof of Theorem 1.2. Let $\eta = \eta(f) > 0$ be the constant appearing in Corollary 3.1. Fix any $\xi_1 \in \partial\mathbb{D}$. Let N_1 be a positive integer to be fixed later and consider $L_1 = \sum_{n=1}^{N_1} a_n f^n$. We will show that for any $w \in \mathbb{C}$ satisfying

$$|w - L_1(\xi_1)| \leq \frac{\eta}{10} \sum_{n=N_1+1}^{\infty} |a_n|, \tag{32}$$

there exists $\xi \in \partial\mathbb{D}$ such that $\sum_{n=1}^{\infty} a_n f^n(\xi)$ converges and its sum is w.

Take $M_1 = 1$. We first explain the choice of the positive integer N_1. Let $0 < \gamma_1 = \gamma_1(\eta) < 1$ be a number to be fixed later. Since f^n tends to 0 uniformly on compacts of $\mathbb{D}$, there exists an integer $T = T(\eta) > 0$ such that

$$|f^T(z)| \leq \eta \quad \text{if} \quad |z| < \gamma_1. \tag{33}$$

Also note that the assumption (3) gives

$$\lim_{j\to\infty} \frac{\sum_{n=j}^{j+T} |a_n|}{\sum_{n=j+T}^{\infty} |a_n|} = 0.$$

Hence there exists an integer $N_1 > 0$ such that

$$\sum_{n=j}^{j+T} |a_n| \leq \frac{\eta^2}{100} \sum_{n=j+T}^{\infty} |a_n|, \quad j \geq N_1 \tag{34}$$

and

$$|a_j| \leq \frac{\eta^2}{300} \sum_{k>j} |a_k|, \quad j \geq N_1. \tag{35}$$

Fix any $\xi_1 \in \partial\mathbb{D}$ and let I_1 be an arc containing ξ_1 such that $m(f^{N_1}(I_1)) = 1/2$. Fix $w \in \mathbb{C}$ satisfying (32). Note that

$$d_1 := \inf\{|w - L_1(\xi)| : \xi \in I_1\} < \frac{\eta}{10} \sum_{n=N_1+1}^{\infty} |a_n|.$$

By induction we will construct a sequence of nested arcs $I_k \subset \partial\mathbb{D}$ and positive integers $M_k < N_k$ with $M_1 = 1$ and $N_k < M_{k+1}$, $k \geq 1$, such that the partial sums $F_1 = L_1$, $F_k = \sum_{j=1}^{k} L_j + \sum_{j=1}^{k-1} S_j$, $k > 1$, where

$$L_j = \sum_{n=M_j}^{N_j} a_n f^n, \quad S_j = \sum_{n=N_j+1}^{M_{j+1}-1} a_n f^n, \quad j \geq 1,$$

satisfy

$$m(f^{N_k}(I_k)) = 1/2, \quad k \geq 1, \tag{36}$$

$$d_k := \inf\left\{\left|w - F_k(\xi)\right| : \xi \in I_k\right\} \leq \frac{\eta}{10} \sum_{n=N_k+1}^{\infty} |a_n|, \quad k \geq 1, \tag{37}$$

Note that this would end the proof. Since $m(f^{N_k}(I_k)) = 1/2$, Corollary 2.3 gives

$$\max\{|F_k(\xi) - F_k(\xi')| : \xi, \xi' \in I_k\} \underset{k\to\infty}{\longrightarrow} 0. \tag{38}$$

Pick $\xi \in \bigcap_k I_k$. Note that (37) gives that d_k tends to zero. Then (38) gives that $F_k(\xi)$ converges to w. Since $\sum_n |a_n| < \infty$, this would finish the proof.

The construction of the blocks L_j, S_j and the arcs I_j is done by induction and has some similarities to the construction used in the proof of Theorem 1.1. Again the number of terms of each S_j will be a large number independent of j, actually $M_{j+1} - N_j = T$ for any j, while the number of terms of the blocks L_j may be arbitrarily large, that is, $N_j - M_j$ may tend to ∞ as $j \to \infty$. However the proof is more involved and several technical adjustments are needed. We have already constructed $F_1 = L_1$ and the arc I_1 satisfying (36) and (37) for $k = 1$. Assume, by induction, that the blocks $L_1, \dots, L_k$, $S_1, \dots, S_{k-1}$ and the arcs $I_k \subset I_{k-1} \subset \cdots \subset I_1$ have been constructed verifying the induction assumptions. We will complete the inductive step constructing the blocks S_k, L_{k+1} and the arc I_{k+1}.

Fix a constant $0 < c_1 = c_1(\eta) < 1$ such that there exists $z_k^* \in \mathbb{D}$ with $\rho(z_k^*, z(I_k)) \leq c_1$ such that $\eta^{-1} I(z_k^*) \subset I_k$. Observe that (36) and part (b) of Lemma 2.4 provide a constant $0 < \gamma_0 = \gamma_0(f) < 1$ such that $|f^{N_k}(z(I_k))| \leq \gamma_0$. By Schwarz's Lemma there exists a constant $0 < \gamma_1 = \gamma_1(\gamma_0, \eta) < 1$ such that $|f^{N_k}(z_k^*)| \leq \gamma_1$. Since $M_{k+1} = N_k + T$, estimate (33) gives $|f^{M_{k+1}}(z_k^*)| \leq \eta$. This defines S_k and allows to apply Corollary 3.1 to any partial sum starting at the index M_{k+1}. The construction of the block L_{k+1} and the arc I_{k+1} requires to consider two cases:

(I) Assume

$$\sum_{n=N_k+1}^{\infty} |a_n| \leq \frac{100 d_k}{\eta}. \tag{39}$$

Note that the induction hypothesis (37) gives

$$\sum_{n=N_k+1}^{\infty} |a_n| \geq \frac{10 d_k}{\eta}.$$

Since $M_{k+1} = N_k + T$, estimate (34) gives

$$\sum_{n=M_{k+1}}^{\infty} |a_n| \geq \frac{9 d_k}{\eta}.$$

Let N_{k+1} be the smallest positive integer bigger than M_{k+1} such that

$$\sum_{n=M_{k+1}}^{N_{k+1}} |a_n| \geq \frac{d_k \eta}{2}.$$

This defines the block L_{k+1}. Note that

$$\sum_{n=M_{k+1}}^{N_{k+1}} |a_n| \leq \frac{d_k \eta}{2} + |a_{N_{k+1}}|.$$

By estimate (35),

$$|a_{N_{k+1}}| \leq \frac{\eta^2}{300} \sum_{n > N_{k+1}} |a_n|.$$

Hence estimate (39) gives

$$\frac{5 d_k \eta}{6} \geq \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| \geq \frac{d_k \eta}{2}. \tag{40}$$

Let $\xi_k \in I_k$ such that $d_k = |w - F_k(\xi_k)|$. Apply Corollary 3.1 to the point z_k^*, the value $w - F_k(\xi_k)$ and the block L_{k+1}, to find $\xi_{k+1} \in \eta^{-1} I(z_k^*) \subset I_k$ such that

$$|w - F_k(\xi_k) - L_{k+1}(\xi_{k+1})| \leq d_k - \eta \sum_{n=M_{k+1}}^{N_{k+1}} |a_n|.$$

Hence

$$|w - F_k(\xi_k) - S_k(\xi_{k+1}) - L_{k+1}(\xi_{k+1})| \le d_k - \eta \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| + \sum_{n=N_k+1}^{M_{k+1}-1} |a_n|. \tag{41}$$

Since $m(f^{N_k}(I_k)) = 1/2$ and f is expanding, one can find an arc I_{k+1} with $\xi_{k+1} \in I_{k+1} \subset I_k$ and $m(f^{N_{k+1}}(I_{k+1})) = 1/2$. We complete the inductive step showing

$$d_{k+1} \le \frac{\eta}{10} \sum_{n=N_{k+1}+1}^{\infty} |a_n|. \tag{42}$$

Let $K = \min\{|f'(\xi)| : \xi \in \partial\mathbb{D}\}$. Since $f(0) = 0$ and f is not a rotation, we have $K > 1$. Note that by Lemma 2.1 we have

$$|F_k(\xi_k) - F_k(\xi_{k+1})| \le \pi \sum_{n=1}^{N_k} |a_n| K^{n-N_k}.$$

Hence (41) gives

$$d_{k+1} \le d_k - \eta \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| + \sum_{n=N_k+1}^{M_{k+1}-1} |a_n| + \pi \sum_{n=1}^{N_k} |a_n| K^{n-N_k}.$$

Applying the inductive assumption, the estimate (42) would follow from

$$\frac{\eta}{10} \sum_{n=N_k+1}^{\infty} |a_n| - \eta \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| + \sum_{n=N_k+1}^{M_{k+1}-1} |a_n| + \pi \sum_{n=1}^{N_k} |a_n| K^{n-N_k} \le \frac{\eta}{10} \sum_{n=N_{k+1}+1}^{\infty} |a_n|.$$

Hence, we need to check

$$\frac{-9\eta}{10} \sum_{n=M_{k+1}}^{N_{k+1}} |a_n| + \left(1 + \frac{\eta}{10}\right) \sum_{n=N_k+1}^{M_{k+1}-1} |a_n| + \pi \sum_{n=1}^{N_k} |a_n| K^{n-N_k} \le 0. \tag{43}$$

Now, (40) says that the first sum is comparable to d_k. By Lemma 2.8 and the assumption (39), last sum of (43) can be absorbed by the fist one. On the other hand $M_{k+1} = N_k + T$ and hence by (34), the second sum of (43) is also absorbed by the first one. We see that (43) holds and the inductive step is completed in this case.

(II) Assume

$$\sum_{n=N_k+1}^{\infty} |a_n| \ge \frac{100 d_k}{\eta}. \tag{44}$$

Let $\xi_k \in I_k$ such that $d_k = |w - F_k(\xi_k)|$ and pick $N_{k+1} = M_{k+1}$, that is, in this case L_{k+1} has a single term. Since $m(f^{N_k}(I_k)) = 1/2$ and f is expanding, one can take an arc I_{k+1} with $\xi_k \in I_{k+1} \subset I_k$ and $m(f^{N_{k+1}}(I_{k+1})) = 1/2$. Note that

$$d_{k+1} \le d_k + \sum_{n=N_k+1}^{M_{k+1}-1} |a_n| + |a_{N_{k+1}}|.$$

Applying (44) we deduce

$$d_{k+1} \le \frac{\eta}{100} \sum_{n=N_k+1}^{\infty} |a_n| + \sum_{n=N_k+1}^{M_{k+1}} |a_n|$$

and to complete the inductive step, it is sufficient to check

$$\frac{\eta}{100}\sum_{n=N_k+1}^{\infty}|a_n|+\sum_{n=N_k+1}^{M_{k+1}}|a_n|\le\frac{\eta}{10}\sum_{n=N_{k+1}+1}^{\infty}|a_n|.$$

The choice $M_{k+1}-N_k=T+1$ and estimate (34) finish the proof. □

4 Concluding remarks

Given $\xi\in\partial\mathbb{D}$ and $M>1$, let $\Gamma(\xi,M)=\{z\in\mathbb{D}:|z-\xi|\le M(1-|z|)\}$ be the Stolz angle with vertex at ξ and aperture depending on M. A function $F:\mathbb{D}\to\mathbb{C}$ has a non tangential limit at the point $\xi\in\partial\mathbb{D}$, which will be denoted by $\lim_{\substack{z\to\xi\\ \not<}}F(z)$, if for any $M>1$ the limit

$$\lim_{z\in\Gamma(\xi,M),z\to\xi}F(z)$$

exists and is finite.

In the statements of Theorems 1.1 and 1.2 one can replace the sum $\sum_n a_nf^n(\xi)$ by the non-tangential limit

$$\lim_{\substack{z\to\xi\\ \not<}}\sum_n a_nf^n(z).$$

This follows from our next result which can be understood as a version of the classical Abel's Theorem in our context.

Theorem 4.1. *Let f be a finite Blaschke product with $f(0)=0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers and let $\xi\in\partial\mathbb{D}$ such that $\sum_n a_nf^n(\xi)$ converges. Then the non-tangential limit*

$$\lim_{\substack{z\to\xi\\ \not<}}\sum_{n=1}^{\infty}a_nf^n(z)$$

exists and it is equal to $\sum_{n=1}^{\infty}a_nf^n(\xi)$.

Proof. Let $K_1=\max\{|f'(\xi)|:\xi\in\partial\mathbb{D}\}$ and pick $M>K_1$. Fix $0<\delta<(10K_1)^{-1}$. For $z\in\Gamma(\xi,M)$ let $N(z)$ be the smallest positive integer such that $m(f^{N(z)}(I(z)))\ge\delta$. Apply Lemma 2.6 to obtain a constant $C=C(\delta,f)>0$ such that

$$\sum_{k=1}^{N(z)}|f^k(\xi)-f^k(z)|\le C,\quad \xi\in I(z). \tag{45}$$

Note that $N(z)\to\infty$ as $|z|\to1$ and $m(f^{N(z)}(I(z)))\le\delta K_1$. Part (b) of Lemma 2.4 gives a constant $0<\gamma=\gamma(\delta,f)<1$ such that $|f^{N(z)}(z)|\le\gamma$. Pommerenke estimates of the Denjoy–Wolff Theorem (Lemma 2 of [P]) provide constants $C_1=C_1(\gamma)>0$ and $0<a=a(f)<1$ such that

$$|f^n(z)|\le C_1a^{n-N(z)},\quad n\ge N(z). \tag{46}$$

We are now ready to prove the result. Fix $\varepsilon>0$. We want to show that there exists $N_0=N_0(\varepsilon)>0$ such that for any $z\in\Gamma(\xi,M)$ sufficiently close to ξ, we have

$$\left|\sum_{n=N}^{\infty}a_n(f^n(z)-f^n(\xi))\right|\le\varepsilon,\quad N\ge N_0. \tag{47}$$

Summing by parts

$$\sum_{n=N}^{\infty} a_n(f^n(z) - f^n(\xi)) = \sum_{n=N+1}^{\infty} S_n(\xi)\left(\frac{f^n(z)}{f^n(\xi)} - \frac{f^{n-1}(z)}{f^{n-1}(\xi)}\right) + S_N(\xi)\left(\frac{f^N(z)}{f^N(\xi)} - 1\right),$$

where

$$S_n(\xi) = \sum_{k=n}^{\infty} a_k f^k(\xi).$$

Next, we will show that there exists a constant $C_2 = C_2(\delta, f) > 0$ such that

$$\sum_{n=1}^{\infty}\left|\frac{f^n(z)}{f^n(\xi)} - \frac{f^{n-1}(z)}{f^{n-1}(\xi)}\right| \leq C_2. \tag{48}$$

Note that estimate (46) gives a constant $C_3 = C_3(\delta, f) > 0$ such that,

$$\sum_{n=1}^{\infty}\left|\frac{f^n(z)}{f^n(\xi)} - \frac{f^{n-1}(z)}{f^{n-1}(\xi)}\right| \leq \sum_{n=1}^{N(z)} \left|f^n(z)f^{n-1}(\xi) - f^{n-1}(z)f^{n-1}(\xi)\right| + C_3.$$

Adding and substracting $f^n(\xi)f^{n-1}(\xi)$ and applying estimate (45), we obtain (48). By assumption, we can choose N_0 such that $|S_n(\xi)| \leq \varepsilon$ if $n \geq N_0$. This gives (47) and finishes the proof. □

Another turn of the screw of the methods in the proof of Theorem 1.1 leads to our next result which is formally, a generalization of Theorem 1.1. An analogous result in the context of lacunary series can be found in [KWW]. Given a sequence of complex numbers $\{b_n\}$ let $\{\sum b_n\}'$ denote the set of accumulation points of its partial sums, that is,

$$\{\sum b_n\}' = \bigcap_{n=1}^{\infty} \overline{\left\{\sum_{k=1}^{m} b_k \,:\, m \geq n\right\}}.$$

Note that if $\{b_n\}$ tends to 0, then $\{\sum b_n\}'$ is a closed connected set of the extended complex plane.

Theorem 4.2. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers tending to 0 such that $\sum |a_n| = \infty$. Then, for any closed connected set K of the extended complex plane, there exists a point $\xi \in \partial\mathbb{D}$ such that*

$$K = \{\sum a_n f^n(\xi)\}'.$$

Proof of Theorem 4.2. Since the proof only requires an iteration of the ideas of the proof of Theorem 1.1, we will only sketch it. Given the set K, let us consider a sequence $\{w_n\}$ of complex numbers with $|w_{n+1} - w_n| \to 0$ as $n \to \infty$, such that

$$K = \bigcap_{n=1}^{\infty} \overline{\{w_m \,:\, m \geq n\}}.$$

We use the notation $S_N(\xi) = \sum_{n=1}^{N} a_n f^n(\xi)$. Using the methods in the proof of Theorem 1.1 one can construct a closed arc $I_1 \subset \partial\mathbb{D}$ and an integer N_1 so that for any $\xi \in I_1$, the partial sum $S_{N_1}(\xi)$ is close to w_1 and $S_m(\xi)$ is not far from the segment $[0, w_1]$ for any $0 < m \leq N_1$. Once one has constructed this arc, using again the same argument, one constructs a closed subarc $I_2 \subset I_1$ and an integer $N_2 > N_1$ such that for any $\xi \in I_2$, the partial sum $S_{N_2}(\xi)$ is close to w_2 and $S_m(\xi)$ is not far from the segment $[w_1, w_2]$ for any $N_1 \leq m \leq N_2$. Iterating this process and taking $\xi \in \bigcap I_j$ one obtains the desired result. □

Theorem 4.2 has the corresponding counterpart in the context of radial behaviour of the function $\sum a_n f^n$. Given an analytic function $g : \mathbb{D} \to \mathbb{C}$, the radial cluster set $\mathrm{Cl}_r(g, \xi)$ of the function g at the point $\xi \in \partial\mathbb{D}$ is defined to be

$$\mathrm{Cl}_r(g, \xi) = \bigcap_{r<1} \overline{\{g(s\xi) \,:\, s \geq r\}}.$$

We will need the following easy consequence of Lemma 2.6.

Lemma 4.3. *Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Fix $0 < \delta < (10 \max\{|f'(\xi)| : \xi \in \partial\mathbb{D}\})^{-1}$. Then there exits a constant $C = C(\delta, f) > 0$ such that the following statements hold.*

(a) Given $z \in \mathbb{D}$ let $N(z)$ be the smallest positive integer such that $m(f^{N(z)}(I(z))) \geq \delta$. Then

$$\left| \sum_{n=1}^{\infty} a_n f^n(z) - \sum_{n=1}^{N(z)} a_n f^n(\xi) \right| \leq C \sup_n |a_n|, \quad \xi \in I(z).$$

(b) Fix $\xi \in \partial\mathbb{D}$. Given a positive integer N, consider $r_N = \sup\{0 < r < 1 \ : m(f^N(I(r\xi))) \geq \delta\}$. Then

$$\left| \sum_{n=1}^{N} a_n f^n(\xi) - \sum_{n=1}^{\infty} a_n f^n(r_N \xi) \right| \leq C \sup_n |a_n|.$$

Proof. Apply Lemma 2.6 to obtain a constant $C = C(\delta, f) > 0$ such that

$$\sum_{n=1}^{N(z)} |a_n||f^n(\xi) - f^n(z)| \leq C \sup_n |a_n|, \quad \xi \in I(z).$$

The exponential decay (46) given by Pommerenke's result, provides a constant $C_1 = C_1(\delta) > 0$ such that

$$\sum_{n=N(z)}^{\infty} |f^n(z)| \leq C_1.$$

This finishes the proof of (a). Part (b) follows similarly. □

Let $\{a_n\}$ be a sequence of complex numbers tending to 0 and $F = \sum a_n f^n$. Note that in the previous Lemma we have that $N(z) \to \infty$ as $|z| \to 1$ and $r_N \to 1$ as $N \to \infty$. We deduce that $\mathrm{Cl}_r(F, \xi) = \{\sum a_n f^n(\xi)\}'$ for any $\xi \in \partial\mathbb{D}$. Hence next result follows from Theorem 4.2.

Theorem 4.4. *Let f and $\{a_n\}$ be as in Theorem 4.2 and let F be the analytic function defined by $F(z) = \sum_{n=1}^{\infty} a_n f^n(z)$, $z \in \mathbb{D}$. Then, for any closed connected set K of the extended complex plane there exists a point $\xi \in \partial\mathbb{D}$ such that $Cl_r(F, \xi) = K$.*

We finally mention some related open problems we have not explored.

1. Let the function f and the coefficients $\{a_n\}$ be as in the statement of Theorem 1.1. Fix $w \in \mathbb{C}$. It is natural to study the size of the set $E(w)$ of points $\xi \in \partial\mathbb{D}$ such that $\sum a_n f^n(\xi) = w$. Note that if we assume $\sum |a_n|^2 = \infty$, then $\sum a_n f^n$ is a function in the Bloch space that has radial limit at almost no point of the unit circle (see [N]). In this situation Rohde proved that the set

$$\left\{ \xi \in \partial\mathbb{D} : \lim_{r\to 1} \sum_{n=1}^{\infty} a_n f^n(r\xi) = w \right\}$$

has Hausdorff dimension 1. See [Ro]. It is also natural to ask for the size of the set of points $\xi \in \partial\mathbb{D}$ where the conclusion of Theorem 4.2 holds.

2. Let $F(z) = \sum a_n z^{k_n}$ be a lacunary power series with radius of convergence equals to 1, such that $\sum |a_n| = \infty$. Murai proved that for any $w \in \mathbb{C}$ there are infinitely many $z \in \mathbb{D}$ such that $F(z) = w$ ([Mu3]). It is natural to seek for analogus results in our context. More concretely, let f and $\{a_n\}$ be as in the statement of Theorem 1.1. Is it true that for any $w \in \mathbb{C}$ there exists $z \in \mathbb{D}$ such that $\sum a_n f^n(z) = w$?

3. The High Indices Theorem of Hardy and Littlewood provides a converse of Abel's Theorem in the setting of lacunary series. See [K]. Hence it is natural to ask for the converse of Theorem 4.1.

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