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THE CLOSURE OF THE HARDY SPACE IN THE BLOCH NORM

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Dedicated to V. P. Havin on the occasion of his 75th birthday

Abstract. A description of the closure in the Bloch norm of the Bloch functions that are in a Hardy space is given. The result uses the classical estimates for the Lusin area function.

§1. Introduction

This paper is devoted to the description of the closure in the Bloch norm of the space $\mathsf{H}^p \cap \mathsf{Bloch}$. First, we recall some definitions.

For $0 < p < \infty$, the Hardy space H^p is the space of analytic functions f in the unit disk such that

$$\|f\|_{\mathsf{H}^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty.$$

As usual, H^∞ is the space of bounded analytic functions in the unit disk.

We also recall the definition of the space BMO of functions with bounded mean oscillation. Let u be an integrable function on the unit circle $\mathbb{T}$, and let u_I denote the mean of u over the arc $I \subset \mathbb{T}$, that is,

$$u_I = \frac{1}{|I|} \int_I u(\xi) \, |d\xi|.$$

Here $|d\xi|$ is the normalized Lebesgue measure on $\mathbb{T}$. The function u is in BMO if

$$\|u\|_{\mathsf{BMO}} = \sup \frac{1}{|I|} \int_I |u(\xi) - u_I| \, |d\xi| < +\infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. An analytic function f in the unit disk $\mathbb{D}$ is in the space BMOA if it is the Poisson extension to the disk of a function in BMO.

Finally, recall that a function f is in the Bloch space, denoted by Bloch, if f is analytic in $\mathbb{D}$ and

$$\|f\|_{\mathsf{Bloch}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

It is well known that $\mathsf{H}^\infty \subsetneq \mathsf{BMOA} \subsetneq \mathsf{Bloch} \cap \mathsf{H}^p$ for any $p < \infty$. Therefore, it is natural to ask for a description of the closure of the spaces H^∞, BMOA, and $\mathsf{H}^p \cap \mathsf{Bloch}$ in the Bloch space.

Garnett and Jones gave a description of the closure of L^∞ in the BMO norm, and also of H^∞ in BMOA, based on the John–Nirenberg inequality [6], stating that a function

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$u \in \mathsf{L}^1(\mathbb{T})$ is in BMO if and only if there exists $\varepsilon > 0$ and a constant $C > 0$ such that for any arc $I \subset \mathbb{T}$ and any $\lambda > 0$ we have

$$|\{\xi \in I : |u(\xi) - u_I| > \lambda\}| \le Ce^{-\lambda/\varepsilon}|I|.$$

In [3], Garnett and Jones showed that a function u is in the closure of L^∞ in the BMO norm if and only if for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that the inequality above is satisfied.

A characterization of the closure of BMOA in the Bloch norm is also due to P. Jones. Given a function $f \in \mathsf{Bloch}$ and $\varepsilon > 0$, define

$$\Omega_\varepsilon(f) = \{z \in \mathbb{D} : (1 - |z|^2)|f'(z)| \ge \varepsilon\}.$$

Then f is in the closure of BMOA in the Bloch norm if and only if for every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that

$$\int_{Q\cap\Omega_\varepsilon(f)} \frac{dA(z)}{1 - |z|^2} \le C\ell(Q)$$

for any Carleson square Q of the form

$$Q = \left\{re^{i\theta} : 0 < 1 - r < \ell(Q),\ |\theta - \theta_0| < \ell(Q)\right\}, \quad 0 \le \theta_0 < 2\pi.$$

In [5], Ghatage and Zheng gave a proof of this result and they attributed it to P. Jones. Our main purpose in this paper is to adapt this proof to give a description of the Bloch functions that can be approximated in the Bloch norm by functions in $\mathsf{Bloch} \cap \mathsf{H}^p$. This result is stated in Theorem 1 below.

We start with some notation. Given a set $\Omega \subseteq \mathbb{D}$, let $A_h(\Omega)$ be the hyperbolic area of Ω, that is,

$$A_h(\Omega) = \int_\Omega \frac{dA(z)}{(1 - |z|^2)^2}.$$

Also, for fixed $M > 1$ and for $\xi \in \mathbb{T}$, let $\Gamma(\xi) = \{z \in \mathbb{D} : |z - \xi| < M(1 - |z|)\}$ be the Stolz angle with vertex at ξ. Our main result is as follows.

Theorem 1. *Let $1 < p < \infty$, and let f be a function in the Bloch space. Then f is in the closure in the Bloch norm of $\mathsf{Bloch} \cap \mathsf{H}^p$ if and only if for any $\varepsilon > 0$ the function $A_h^{1/2}(\Omega_\varepsilon(f) \cap \Gamma(\xi))$ is in $\mathsf{L}^p(\mathbb{T})$.*

In the case where $p = 2$, this condition can be written in a more pleasant way. The Fubini theorem allows us to restate our result as follows: A function $f \in \mathsf{Bloch}$ is in the closure of $\mathsf{H}^2 \cap \mathsf{Bloch}$ if and only if for any $\varepsilon > 0$ we have

$$\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{1 - |z|^2} < \infty.$$

Notice that the condition in Peter Jones' result is the conformally invariant version of the previous one.

The necessity in our result follows easily from well-known estimates on the Lusin area function. For $f \in \mathsf{H}^p$, the area function of f at the point $\xi \in \mathbb{T}$ is defined as

$$A(f)(\xi) = \left(\int_{\Gamma(\xi)} |f'(z)|^2\, dA(z)\right)^{1/2}.$$

The following characterization of H^p spaces in terms of the area function will be used (see [7] and [8, p. 224]).

Theorem A. *Let $0 < p < +\infty$, and let f be an analytic function in the unit disk. Then $f \in \mathsf{H}^p$ if and only if $A(f) \in \mathsf{L}^p(\mathbb{T})$. Moreover, the norms $\|f\|_{\mathsf{H}^p}$ and $\|A(f)\|_{\mathsf{L}^p}$ are comparable.*

The proof of the sufficiency in Theorem 1 is more difficult. Here we proceed as in the proof of Jones' theorem given in [5]. If $f \in \mathsf{Bloch}$ and $f(0) = f'(0) = 0$, then for any $z \in \mathbb{D}$ we have the following reproducing formula:

$$f(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)f'(w)}{(1-\bar{w}z)^2\bar{w}}\, dA(w);$$

see [2]. We shall show that the integral over $\mathbb{D} \setminus \Omega_\varepsilon(f)$ has small Bloch norm, so that we shall need to check that the integral over $\Omega_\varepsilon(f)$ is in H^p. This will be accomplished by a duality argument, again with the use of estimates of the Lusin area function.

It is important to remark that our arguments do not apply to the case of $p = \infty$, because Theorem A fails for $p = \infty$. The problem of describing the closure of H^∞ in the Bloch norm, first stated in [1], remains an open problem.

§2. Proof of Theorem 1

Fix $1 < p < \infty$. First, we show necessity. So, let f be a function in the closure of the space $\mathsf{H}^p \cap \mathsf{Bloch}$. Then, given $\varepsilon > 0$, we can find $g \in \mathsf{H}^p \cap \mathsf{Bloch}$ such that $\|f - g\|_{\mathsf{Bloch}} \leq \varepsilon/2$. Since $\Omega_\varepsilon(f) \subseteq \Omega_{\varepsilon/2}(g)$, for any $\xi \in \mathbb{T}$ we have

$$A_h(\Omega_\varepsilon(f) \cap \Gamma(\xi)) \leq \int_{\Omega_{\varepsilon/2}(g)\cap\Gamma(\xi)} \frac{dA(z)}{(1-|z|^2)^2} \leq \int_{\Gamma(\xi)} \frac{4}{\varepsilon^2}|g'(z)|^2\, dA(z).$$

Since $g \in \mathsf{H}^p$, Theorem A shows that its area function is in $\mathsf{L}^p(\mathbb{T})$, and we deduce that $A_h^{1/2}(\Omega_\varepsilon(f) \cap \Gamma(\xi)) \in \mathsf{L}^p(\mathbb{T})$.

Conversely, fix p with $1 < p < \infty$, and let f be a function in the Bloch space such that for any $\varepsilon > 0$ the function $A_h^{1/2}(\Omega_\varepsilon(f) \cap \Gamma(\xi))$, as a function of $\xi \in \mathbb{T}$, is in $\mathsf{L}^p(\mathbb{T})$. Fix $\varepsilon > 0$. We are going to construct a function $f_1 \in \mathsf{H}^p \cap \mathsf{Bloch}$ such that $\|f - f_1\|_{\mathsf{Bloch}} < \varepsilon$. We proceed as in [5]. We may assume that $f(0) = f'(0) = 0$ and $\|f\|_{\mathsf{Bloch}} = 1$. In [2] it was proved that

$$f(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)f'(w)}{(1-\bar{w}z)^2\bar{w}}\, dA(w) \tag{1}$$

for all $z \in \mathbb{D}$. We denote $\Omega_\varepsilon = \Omega_\varepsilon(f)$, split the integral into two parts, and define

$$f_1(z) = \int_{\Omega_\varepsilon} \frac{(1-|w|^2)f'(w)}{(1-\bar{w}z)^2\bar{w}}\, dA(w)$$

and

$$f_2(z) = \int_{\mathbb{D}\setminus\Omega_\varepsilon} \frac{(1-|w|^2)f'(w)}{(1-\bar{w}z)^2\bar{w}}\, dA(w),$$

so that $f = f_1 + f_2$. Since

$$|f_2'(z)| \leq 2\varepsilon \int_{\mathbb{D}} \frac{dA(z)}{|1-\bar{w}z|^3}, \quad z \in \mathbb{D},$$

we deduce that $\|f_2\|_{\mathsf{Bloch}} \leq C\varepsilon$. Hence, we only need to show that f_1 is in H^p. This will be accomplished as in [5] by a duality argument.

Without loss of generality we may assume that $f_1(0) = 0$. Let $\overline{\mathsf{H}^q}$ be the space of antianalytic functions in the unit disk such that $\bar{g} \in \mathsf{H}^q$. Consider the operator

$$T(g) = \int_{\mathbb{D}} \overline{\bar{g}'(z)} f_1'(z) \log\frac{1}{|z|}\, dA(z), \quad g \in \overline{\mathsf{H}^q}.$$

The argument below will show that there exists a fixed constant $C = C(\varepsilon) > 0$ such that $|T(g)| \leq C\|\bar{g}\|_{\mathsf{H}^q}$ for any $g \in \overline{\mathsf{H}^q}$. Once this inequality is established, we shall see that T

gives rise to a bounded linear functional on $\overline{\mathsf{H}^q}$. Then, by duality, there exists $F \in \mathsf{H}^p$ such that

$$T(g) = \int_{\mathbb{T}} F(e^{i\theta}) g(e^{i\theta})\, d\theta$$

for any $g \in \overline{\mathsf{H}^q}$. Here $d\theta$ is the normalized angular measure. Since $T(1) = 0$, we have $F(0) = 0$. By the Littlewood–Paley integral formula (see [4, p. 228]), we have

$$T(g) = \int_{\mathbb{D}} F'(z) \overline{\overline{g}'(z)} \log \frac{1}{|z|}\, dA(z)$$

for all $g \in \overline{\mathsf{H}^q}$. Then $F = f_1$, which shows that $f_1 \in \mathsf{H}^p$. So, we need to check that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\left| \int_{\mathbb{D}} \overline{\overline{g}'(z)} f_1'(z) \log \frac{1}{|z|}\, dA(z) \right| \leq C \|\overline{g}\|_{\mathsf{H}^q}, \quad g \in \overline{\mathsf{H}^q}. \tag{2}$$

First, observe that the Fubini theorem gives

$$T(g) = 2 \int_{\Omega_\varepsilon} (1 - |w|^2) f'(w) \int_{\mathbb{D}} \frac{\overline{\overline{g}'(z)}}{(1 - \overline{w} z)^3} \log \frac{1}{|z|}\, dA(z)\, dA(w).$$

Next, we show that the integral

$$\int_{\mathbb{D}} \frac{\overline{g}'(z)}{(1 - w\overline{z})^3} \log \frac{1}{|z|}\, dA(z) \tag{3}$$

is essentially the derivative of a certain function in H^q. To check this, we fix $w \in \mathbb{D} \setminus \{0\}$ and apply the Littlewood–Paley integral formula to the functions $\overline{g}(z)$ and $\left(2w(1 - w\overline{z})^2\right)^{-1}$ to obtain

$$\int_{\mathbb{D}} \frac{\overline{g}'(z)}{(1 - w\overline{z})^3} \log \frac{1}{|z|}\, dA(z) = \int_0^{2\pi} \frac{\overline{g}(e^{i\theta})}{2w(1 - we^{-i\theta})^2}\, d\theta - \frac{\overline{g}(0)}{2w}.$$

Now we can use the variable $\xi = e^{i\theta}$, and by the Cauchy integral formula we can express the right-hand side of the last identity as

$$\frac{1}{2wi} \int_{\mathbb{T}} \frac{(\overline{g}(\xi) - \overline{g}(0))\xi}{(\xi - w)^2}\, d\xi = \frac{1}{2w} h'(w),$$

where $h(w) = (\overline{g}(w) - \overline{g}(0))w$. So, finally we obtain

$$T(g) = \int_{\mathbb{D}} (1 - |w|^2) f'(w) \chi_{\Omega_\varepsilon}(w) \frac{1}{\overline{w}} \overline{h'(w)}\, dA(w).$$

Since $f'(0) = 0$, there exists $C_1 = C_1(\varepsilon)$ such that $\Omega_\varepsilon \subset \{w \in \mathbb{D} : |w| \geq C_1\}$. Therefore, taking modules, we deduce that

$$\begin{aligned} |T(g)| &\leq \frac{1}{C_1} \int_{\mathbb{D}} (1 - |w|^2) |f'(w)|\, |h'(w)| \chi_{\Omega_\varepsilon}(w)\, dA(w) \\ &\leq \frac{1}{C_1} \int_{\mathbb{D}} |h'(w)| \chi_{\Omega_\varepsilon}(w)\, dA(w). \end{aligned}$$

By the Fubini theorem, we deduce that there exists a constant $C_2 > 0$ such that

$$|T(g)| \leq C_2 \int_{\mathbb{T}} \int_{\Gamma(\xi) \cap \Omega_\varepsilon} \frac{|h'(w)|}{1 - |w|^2}\, dA(w)\, |d\xi|.$$

Applying the Cauchy–Schwarz inequality to the inner integral, we get

$$
\begin{aligned}
|T(g)| &\le C_2 \int_{\mathbb{T}} \left(\int_{\Omega_\varepsilon \cap \Gamma(\xi)} \frac{dA(w)}{(1-|w|^2)^2} \right)^{\frac{1}{2}} \left(\int_{\Gamma(\xi)} |h'(w)|^2 \, dA(w) \right)^{\frac{1}{2}} |d\xi| \\
&= C_2 \int_{\mathbb{T}} A_h^{1/2}(\Omega_\varepsilon \cap \Gamma(\xi)) A(h)(\xi) \, |d\xi|.
\end{aligned}
$$

By the hypothesis that $A_h^{1/2}(\Omega_\varepsilon \cap \Gamma(\xi))$ is in $\mathsf{L}^p(\mathbb{T})$ and by Theorem A, the area function $A(h)$ is in $\mathsf{L}^q(\mathbb{T})$; so using finally the Hölder inequality and the hypothesis, we deduce that

$$
\begin{aligned}
|T(g)| &\le C_2 \left\| A_h^{1/2}(\Omega_\varepsilon \cap \Gamma(\xi)) \right\|_{\mathsf{L}^p(\mathbb{T})} \left\| A(h) \right\|_{\mathsf{L}^q(\mathbb{T})} \\
&\le C_3 \|h\|_{\mathsf{L}^q(\mathbb{T})} \le 2C_3 \|g\|_{\mathsf{L}^q(\mathbb{T})}
\end{aligned}
$$

for any $g \in \overline{\mathsf{H}^q}$. This inequality gives (2), so that $T(g)$ determines a bounded linear functional on $\overline{\mathsf{H}^q}$. Hence, Theorem 1 is proved.

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