Bull. London Math. Soc. 43 (2011) 536–546
doi:10.1112/blms/bdq121

Smoothness of sets in Euclidean spaces

Artur Nicolau and Daniel Seco

Abstract

We study some properties of smooth sets in the sense defined by Hungerford. We prove a sharp form of Hungerford's theorem on the Hausdorff dimension of their boundaries on Euclidean spaces and show the invariance of the definition under a class of automorphisms of the ambient space.

1. *Introduction*

The Lebesgue density theorem tells us that the density of a measurable set approximates the characteristic function of the set at almost every point. We are going to study sets whose densities at small scales vary uniformly.

In this paper, a cube will mean a cube in the Euclidean space $\mathbb{R}^n$ with sides parallel to the axis. Two cubes $Q, Q' \subset \mathbb{R}^n$ with the same sidelength $l(Q) = l(Q')$ are called consecutive if the intersection of their closures is one of their faces. Given a measurable set $A \subset \mathbb{R}^n$, let $|A|$ denote its Lebesgue measure and $D(Q)$ its density in a cube $Q \subset \mathbb{R}^n$, that is, $D(Q) = |A \cap Q|/|Q|$. A measurable set $A \subset \mathbb{R}^n$ is called *smooth* (in $\mathbb{R}^n$) if

$$\lim_{\delta \to 0} \sup |D(Q) - D(Q')| = 0,$$

where the supremum is taken over all pairs of consecutive cubes Q, Q' with $l(Q) = l(Q') \leqslant \delta$. In dimension $n = 1$, this notion was introduced by Hungerford [3] in relation to the small Zygmund class. Actually, a set $A \subset \mathbb{R}$ is smooth if and only if its distribution function is in the small Zygmund class, or equivalently, the restriction of the Lebesgue measure to the set A is a smooth measure in the sense of Kahane [4].

Sets $A \subset \mathbb{R}^n$ with $|A| = 0$ or $|\mathbb{R}^n \setminus A| = 0$ are trivially smooth but Hungerford provided non-trivial examples, using a nice previous recursive construction by Kahane; see [3]. Other sharper examples are given in [1].

In dimension $n = 1$, Hungerford proved that the boundary of a non-trivial smooth set has full Hausdorff dimension [3, 5]. His argument shows that if A is a smooth set in $\mathbb{R}$ with $|A| > 0$ and $|\mathbb{R} \setminus A| > 0$, then the set of points $x \in \mathbb{R}$ for which there exists a sequence of intervals $\{I_j\}$ containing x such that

$$\lim_{j \to \infty} D(I_j) = 1/2$$

still has Hausdorff dimension 1. The main goal of this paper is to sharpen this result and to extend it to Euclidean spaces. It is worth mentioning that Hungerford arguments cannot be extended to several dimensions since it is used that the image under a non-trivial linear mapping of an interval is still an interval, or more generally, that an open connected set is an interval and this obviously does not hold for cubes in $\mathbb{R}^n$, for $n > 1$. Given a point $x \in \mathbb{R}^n$ and

Received 2 June 2010; revised 12 October 2010; published online 27 January 2011.

2000 *Mathematics Subject Classification* 28A05 (primary), 28A80 (secondary).

Both authors are supported partially by the grants MTM2008-00145 and 2009SGR420.

$h > 0$, let $Q(x,h)$ denote the cube centred at x of sidelength h. With this notation, our main result is the following theorem.

THEOREM 1. *Let A be a smooth set in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \setminus A| > 0$. Fix $0 < \alpha < 1$. Then the set*

$$E(A, \alpha) = \{x \in \mathbb{R}^n : \lim_{h\to 0} D(Q(x,h)) = \alpha\}$$

has Hausdorff dimension n.

Our result is local, meaning, given a cube $Q \subset \mathbb{R}^n$ with $0 < |A \cap Q| < |Q|$, we have that $E(A,\alpha) \cap Q$ has full Hausdorff dimension. As a consequence the Hausdorff dimension of $\partial A \cap Q$ is n.

Section 2 contains a proof of Theorem 1. A Cantor-type subset of $E(A,\alpha)$ will be constructed and its dimension will be computed using a standard result. The generations of the Cantor set will be defined recursively by means of a stopping time argument. The good averaging properties of the density are used to estimate the dimension of the Cantor set.

The definition of smooth set concerns the behaviour of the density of the set on the grid of cubes in $\mathbb{R}^n$ with sides parallel to the axis. We consider two natural questions arising from this definition. First, we study how the definition depends on the grid of cubes, that is, if other natural grids, such as dyadic cubes or general parallelepipeds would lead to the same notion. Second, we consider whether the class of smooth sets is preserved by regular mappings. It turns out that these questions are related and in Section 3 we provide a positive answer to both of them. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is bilipschitz if there exists a constant $C \geqslant 1$ such that $C^{-1}\|x-y\| \leqslant \|\phi(x) - \phi(y)\| \leqslant C\|x-y\|$ for any $x, y \in \mathbb{R}^n$.

THEOREM 2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz $\mathcal{C}^1$ mapping with uniformly continuous Jacobian. Let $A \subset \mathbb{R}^n$ be a measurable set. Then the following are equivalent*:

(a) *A is a smooth set*;
(b) *$\phi^{-1}(A)$ is a smooth set*;
(c) *A verifies the smoothness condition taking, instead of the grid of cubes, their images through ϕ, that is,*

$$\lim_{|Q|\to 0} \frac{|A \cap \phi(Q)|}{|\phi(Q)|} - \frac{|A \cap \phi(Q')|}{|\phi(Q')|} = 0.$$

As part (c) states, one could replace in the definition of smooth set, the grid of cubes by other grids such as the grid of dyadic cubes or the grid of general parallelepipeds with bounded eccentricity whose sides are not necessarily parallel to the axis or even the pullback by ϕ of the grid of cubes. One can combine Theorems 1 and 2 to conclude the following corollary.

COROLLARY 1. *Let A be a smooth set in $\mathbb{R}^n$ with $|A| > 0$ and $|\mathbb{R}^n \setminus A| > 0$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz $\mathcal{C}^1$ mapping with uniformly continuous jacobian. Fix $0 < \alpha < 1$. Then the set*

$$\left\{x \in \mathbb{R}^n : \lim_{h\to 0} \frac{|A \cap \phi(Q(x,h))|}{|\phi(Q(x,h))|} = \alpha\right\}$$

has Hausdorff dimension n.

2. *Proof of Theorem* 1

2.1. *Preliminary results*

We begin with a preliminary result on the Hausdorff dimension of certain Cantor-type sets which will be used in the proof of Theorem 1. In dimension $n = 1$, the result was given by Hungerford in [**3**]; see also [**6**, Theorem 10.5]. The proof in the higher dimensional case only requires minor adjustments and it will be omitted.

LEMMA 1. *For $s = 0, 1, 2, \ldots$ let $G(s)$ be a collection of closed dyadic cubes in $\mathbb{R}^n$ with pairwise disjoint interiors. Assume that the families are nested, that is,*

$$\bigcup_{Q \in G(s+1)} Q \subseteq \bigcup_{Q \in G(s)} Q.$$

Suppose that there exist two positive constants $0 < P < C < 1$ such that the following two conditions hold.

(a) *For any cube $Q \in G(s+1)$ with $Q \subset \tilde{Q} \in G(s)$ one has $|Q| \leqslant P|\tilde{Q}|$.*
(b) *For any $\tilde{Q} \in G(s)$ one has*

$$\sum |Q| \geqslant C|\tilde{Q}|,$$

where the sum is taken over all cubes $Q \in G(s+1)$ contained in $\tilde{Q}$.
Let $E(s) = \bigcup Q$, where the union is taken over all cubes in $G(s)$ and $E \equiv \bigcap_{s=0}^{\infty} E(s)$. Then $\dim E \geqslant n(1 - \log_P C)$.

The next auxiliary result is the building block of the Cantor set on which the set has a fixed density. Recall that, given a measurable set $A \subset \mathbb{R}^n$, its density on a cube Q is $D(Q) = |A \cap Q|/|Q|$. Given a continuous increasing function $\omega : [0, 1] \to [0, \infty)$ with $\omega(0) = 0$, a set $A \subset \mathbb{R}^n$ is called *ω-smooth* (in $\mathbb{R}^n$) if

$$|D(Q) - D(Q')| \leqslant \omega(l(Q)),$$

for any pair of consecutive cubes $Q, Q' \subset \mathbb{R}^n$ of sidelength $l(Q) = l(Q')$.

LEMMA 2. *Let A be an ω-smooth set of $\mathbb{R}^n$ with $0 < |A \cap [0, 1]^n| < 1$. Let Q be a dyadic cube. Fix a constant $\varepsilon > 0$ such that $n\omega(l(Q)) < \varepsilon < \min\{D(Q), 1 - D(Q)\}$. Let $\mathfrak{A}(Q)$ be the family of maximal dyadic cubes Q_k contained in Q such that*

$$|D(Q_k) - D(Q)| \geqslant \varepsilon. \tag{2.1}$$

Then:

(a) *for any $Q_k \in \mathfrak{A}(Q)$ one has*

$$|Q_k| \leqslant 2^{-\varepsilon/\omega(l(Q))}|Q|;$$

(b) *let $\mathfrak{A}^+(Q)$ (respectively $\mathfrak{A}^-(Q)$) be the subfamily of $\mathfrak{A}(Q)$ formed by those cubes $Q_k \in \mathfrak{A}(Q)$ for which $D(Q_k) - D(Q) \geqslant \varepsilon$ (respectively $D(Q) - D(Q_k) \geqslant \varepsilon$); then*

$$\sum |Q_k| \geqslant |Q|/4,$$

where the sum is taken over all the cubes $Q_k \in \mathfrak{A}^+(Q)$ (respectively $Q_k \in \mathfrak{A}^-(Q)$).

Proof. If $Q_1 \subset Q_2 \subset Q$ are two dyadic cubes with $l(Q_1) = l(Q_2)/2$, then $|D(Q_1) - D(Q_2)| \leqslant n\omega(l(Q))$. So if (2.1) holds, one deduces that $\log_2 l(Q_k)^{-1} \geqslant \log_2 l(Q)^{-1} + \varepsilon/n\omega(l(Q))$. Therefore, (a) is proved.

To prove (b), we observe first that, by the Lebesgue density theorem, one has $\sum_{\mathfrak{A}(Q)} |Q_k| = |Q|$. Also

$$\sum_{\mathfrak{A}(Q)} (D(Q_k) - D(Q))|Q_k| = 0. \tag{2.2}$$

We argue by contradiction. Assume that $\sum_{Q_k \in \mathfrak{A}^+(Q)} |Q_k| < |Q|/4$ and hence $\sum_{Q_k \in \mathfrak{A}^-(Q)} |Q_k| \geqslant 3|Q|/4$, which gives us

$$\sum_{Q_k \in \mathfrak{A}^-(Q)} (D(Q_k) - D(Q))|Q_k| \leqslant -3\varepsilon|Q|/4.$$

The maximality of Q_k tells us that $|D(Q_k) - D(Q)| \leqslant \varepsilon + n\omega(l(Q)) < 2\varepsilon$. Therefore,

$$\sum_{Q_k \in \mathfrak{A}^+(Q)} (D(Q_k) - D(Q))|Q_k| \leqslant \varepsilon|Q|/2,$$

which contradicts (2.2). The same argument works for $\mathfrak{A}^-(Q)$. □

2.2. *The dyadic case*

Our next goal is to prove a dyadic version of Theorem 1, which already contains its core. Let $Q_k(x)$ be the dyadic cube of generation k which contains the point $x \in \mathbb{R}^n$.

PROPOSITION 1. *Let A be a smooth set in $\mathbb{R}^n$ with $0 < |A \cap [0,1]^n| < 1$. For $0 < \alpha < 1$ consider the set $E_1(A, \alpha) = \{x \in [0,1]^n : \lim_{k\to\infty} D(Q_k(x)) = \alpha\}$. Then* $\dim E_1(A, \alpha) = n$.

Proof. Fix $0 < \alpha < 1$. A Cantor-type set contained in $E_1(A, \alpha)$ will be constructed and Lemma 1 will be used to calculate its Hausdorff dimension. The Cantor-type set will be constructed using generations $G(s)$ which will be defined using Lemma 2, yielding the estimates appearing in Lemma 1.

Given the smooth set $A \subset \mathbb{R}^n$, consider the function

$$\omega(t) = \sup |D(Q) - D(Q')|, \quad 0 < t \leqslant 1,$$

where the supremum is taken over all pairs of consecutive cubes Q and Q' of the same sidelength $l(Q) = l(Q') \leqslant t$. Observe that $\lim \omega(t) = 0$ as $t \to 0$. Pick a positive integer k_0 such that $\omega(2^{-k_0}) < \min\{\alpha, 1-\alpha\}/20$. Define an increasing sequence $\{c_k\}$ with $c_k \to \infty$ as $k \to \infty$ and $c_k \geqslant 2n$ for any k, satisfying $\varepsilon_k = c_k\omega(2^{-k-k_0}) \to 0$ as $k \to \infty$. We can also assume $\varepsilon_k < \min\{\alpha, 1-\alpha\}/10$ for any $k = 1, 2, \ldots$. Since $0 < |A \cap [0,1]^n| < 1$, there are some small dyadic cubes in $[0,1]^n$ with density close to 0 and others with density close to 1. Since A is smooth, we can choose a dyadic cube Q_1 with $l(Q_1) \leqslant 2^{-k_0-1}$ and $|D(Q_1) - \alpha| < \varepsilon_1/2$. Then define the first generation $G(1) = \{Q_1\}$. The next generations are constructed inductively as follows. Assume that the kth generation $G(k)$ has been defined so that the following two conditions are satisfied: $l(Q) \leqslant 2^{-k-k_0}$ and $|D(Q) - \alpha| < \varepsilon_k/2$ for any cube $Q \in G(k)$. The generation $G(k+1)$ is constructed in two steps. Roughly speaking, we first find cubes whose density is far away from α and later we find subcubes with density close to α. For $Q \in G(k)$ consider the family $\mathfrak{R}(Q)$ of maximal dyadic cubes $R \subset Q$ such that $|D(R) - D(Q)| \geqslant \varepsilon_k$. Observe that, by the Lebesgue density theorem, $\sum |R| = |Q|$, where the sum is taken over all cubes $R \in \mathfrak{R}(Q)$. Fix $R \in \mathfrak{R}(Q)$. Since the set A is ω-smooth, the difference of densities between two dyadic cubes $Q_1 \subset Q_2 \subset Q$, with $l(Q_2) = 2l(Q_1)$, is smaller than $n\omega(l(Q))$. Hence, to achieve such a cube R, we need to go through at least $\varepsilon_k/n\omega(2^{-k-k_0}) = c_k/n$ dyadic steps.

Hence,

$$|R| \leqslant 2^{-c_k}|Q|. \tag{2.3}$$

The maximality and the estimate $l(Q) \leqslant 2^{-k-k_0}$ give that $|D(R) - D(Q)| \leqslant \varepsilon_k + n\omega(2^{-k-k_0-1})$. Since

$$|D(R) - \alpha| > \varepsilon_k/2 > n\omega(2^{-k-k_0}) \geqslant n\omega(l(R)),$$

one can apply Lemma 2 with the parameter $\varepsilon = |D(R) - \alpha|$. In this way, one obtains two families $\mathfrak{A}^-(R)$ and $\mathfrak{A}^+(R)$ of dyadic cubes contained in R, according to whether their densities are smaller or bigger than $D(R)$, but we shall only be interested in one of them which will be called $G_{k+1}(R)$. If $D(R) > \alpha$, then we choose $G_{k+1}(R) = \mathfrak{A}^-(R)$. Otherwise, take $G_{k+1}(R) = \mathfrak{A}^+(R)$. Fix, now, $Q^* \in G_{k+1}(R)$. The maximality gives that $|D(Q^*) - \alpha| \leqslant n\omega(l(R))$. Since $l(R) \leqslant 2^{-k-k_0-1}$, we deduce that $|D(Q^*) - \alpha| < \varepsilon_{k+1}/2$. Also, $l(Q^*) < l(R)/2 \leqslant 2^{-k-k_0-1}$. Note that any dyadic cube $\tilde{Q}$ with $Q^* \subset \tilde{Q} \subset Q$ satisfies

$$|D(\tilde{Q}) - \alpha| \leqslant 6\varepsilon_k. \tag{2.4}$$

The generation $G(k+1)$ is defined as

$$G(k+1) = \bigcup_{Q \in G(k)} \bigcup_{R \in \mathfrak{R}(Q)} G_{k+1}(R).$$

Next we compute the constants appearing in Lemma 1. Let $Q \in G(k)$ and $R \in \mathfrak{R}(Q)$. Part (b) of Lemma 2 says that $\sum |Q_j| \geqslant |R|/4$ where the sum is taken over all cubes $Q_j \in G_{k+1}(R)$. Since $\sum_{\mathfrak{R}(Q)} |R| = |Q|$, one deduces that

$$\sum |Q_j| \geqslant |Q|/4, \tag{2.5}$$

where the sum is taken over all cubes $Q_j \in G(k+1), Q_j \subset Q$. Also, if $Q_j \in G(k+1)$ and $Q_j \subset Q \in G(k)$, then estimate (2.3) guarantees that

$$|Q_j| \leqslant 2^{-c_k}|Q|. \tag{2.6}$$

For $k = 1, 2, \ldots$ let $E(k)$ be the union of the cubes of the family $G(k)$ and let $E = \bigcap E(k)$ be the corresponding Cantor-type set.

Next we show that E is contained in $E_1(A, \alpha)$. To do this, fix $x \in E$ and, for any $k = 1, 2, \ldots$ pick the cube $Q_k \in G(k)$ containing x. Let Q be a dyadic cube that contains x, and k be the integer for which $Q_{k+1} \subset Q \subset Q_k$. Observe that $k \to \infty$ as $l(Q) \to 0$. By (2.4) one deduces that $|D(Q) - \alpha| \leqslant 6\varepsilon_k$ and therefore $x \in E_1(A, \alpha)$.

Finally, we apply Lemma 1 to show that the dimension of E is n. Actually (2.5) and (2.6) give that one can take $C = 1/4$ and $P = 2^{-c_k}$ in Lemma 1. Hence, the dimension of E is bigger than $n(1 - (2/c_k))$. Since $c_k \to \infty$ as $k \to \infty$, we deduce that $\dim E = n$. □

2.3. *Affine control and proof of Theorem* 1

We want to study the density of a smooth set in non-dyadic cubes. The proof of Theorem 1 will be based on Proposition 1 and on the following auxiliary result on the behaviour of the densities with respect to affine perturbations.

Lemma 3. *Let A be a smooth set in $\mathbb{R}^n$. Consider the function*

$$\omega(t) = \sup |D(Q) - D(Q')|,$$

where the supremum is taken over all pairs of consecutive cubes $Q, Q' \subset \mathbb{R}^n$, of sidelength $l(Q) = l(Q') \leqslant t$.

(a) *Let $Q, \tilde{Q}$ be two cubes in $\mathbb{R}^n$ with non-empty intersection such that $l(Q) = l(\tilde{Q})$. Then $|D(Q) - D(\tilde{Q})| \leqslant 3n^2\omega(l(Q))$.*

(b) *Let Q be a cube in $\mathbb{R}^n$ and let tQ denote the cube with the same centre as Q and sidelength $tl(Q)$. Then, for any $1 \leqslant t \leqslant 2$, one has*

$$|D(Q) - D(tQ)| \leqslant c(n)\omega(l(Q)).$$

Here $c(n)$ is a constant that only depends on the dimension.

Proof. To prove (a), suppose, without loss of generality, that $Q = [0,1]^n$. Let $Q' = [x, 1+x] \times [0,1]^{n-1}$, where $-1 < x < 1$. We will show that

$$|D(Q) - D(Q')| \leqslant 3n\omega(1). \tag{2.7}$$

Since any Q' intersecting Q is of the form $Q' = [x_1, 1+x_1] \times \ldots \times [x_n, 1+x_n]$, part (a) follows using (2.7) n times.

To show (2.7), decompose $[x, 1+x]$ into dyadic intervals, that is, $[x, 1+x] = \bigcup I_k$, where I_k is a dyadic interval of length 2^{-k} for $k = 1, 2, \ldots$. Consider the parallelepiped $R_k = I_k \times [0,1]^{n-1}$ and the density $D(R_k)$ of the set A on R_k, meaning that $D(R_k) = |R_k \cap A|/|R_k|$, $k = 1, 2, \ldots$. The set R_k can be split into a family $\mathfrak{F}_k$ of $2^{k(n-1)}$ pairwise disjoint cubes S of sidelength 2^{-k}. Since $|D(S) - D(Q)| \leqslant n\omega(1)(k+1)$ for any $S \in \mathfrak{F}_k$ and $D(R_k)$ is the mean of $D(S), S \in \mathfrak{F}_k$, we deduce that $|D(R_k) - D(Q)| \leqslant n\omega(1)(k+1)$. Since

$$D(Q') = \sum_{k=1}^{\infty} 2^{-k} D(R_k),$$

we deduce that $|D(Q') - D(Q)| \leqslant 3n\omega(1)$, which is (2.7).

We turn now to (b). We can assume $Q = [-1/2, 1/2]^n$. Consider the binary decomposition of t, that is, $t = 1 + \sum_{k=1}^{\infty} t_k 2^{-k}$, with $t_k \in \{0, 1\}$. For $m = 1, 2, \ldots$ let Q_m be the cube with the same centre as Q but with sidelength $l(Q_m) = 1 + \sum_{k=1}^{m} t_k 2^{-k}$. Then $tQ = \bigcup_{m=0}^{\infty} R_m$ where $R_0 \equiv Q_0 \equiv Q$ and $R_m = Q_m \setminus Q_{m-1}$ for $m \geqslant 1$. So, for $m \geqslant 1, R_m$ is empty whenever $t_m = 0$ and otherwise we estimate $|D(R_m) - D(Q)|$. With this aim, assume that $t_m = 1$ and split R_m into a family $\mathfrak{F}_m$ of pairwise disjoint cubes S of sidelength 2^{-m-1}. Thus, $|D(S) - D(Q)| \leqslant (n(m+1)+1)\omega(l(Q))$ for any $S \in \mathfrak{F}_m$. Since $D(R_m)$ is the mean of $D(S), S \in \mathfrak{F}_m$, this implies that

$$|D(R_m) - D(Q)| \leqslant (n(m+1)+1)\omega(l(Q)) \quad m = 1, 2, \ldots. \tag{2.8}$$

As we also have

$$D(tQ) = \sum_{m=0}^{\infty} \frac{|R_m|}{|tQ|} D(R_m),$$

using (2.8) we obtain

$$|D(tQ) - D(Q)| \leqslant \omega(l(Q)) \sum_{m=0}^{\infty} \frac{|R_m|}{|tQ|}(n(m+1)+1).$$

Since $|R_m| \leqslant C(n)2^{-m}$, the sum is convergent and the proof is complete. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Applying Proposition 1, we only need to check that

$$\lim_{h \to 0} D(Q(x,h)) = \alpha,$$

for any $x \in E_1(A,\alpha)$. Given $h > 0$, let k be the unique integer such that $2^{-k} \leqslant h < 2^{-k+1}$. Consider the cube $h2^k Q_k(x)$ and apply Lemma 3 to deduce that

$$\lim_{h\to 0} |D(Q(x,h)) - D(Q_k(x))| = 0. \qquad \square$$

3. *Equivalent definitions and invariance*

The definition of a smooth set involves the density of the set on the grid of all cubes in $\mathbb{R}^n$ with sides parallel to the axis. The main purpose of this section is to study the situation for perturbations of this grid of cubes. The first step consists in considering linear deformations of the family of cubes, obtaining a certain grid of parallelepipeds in $\mathbb{R}^n$. Afterwards, we consider the more general case of the grid arising from a bilipschitz image of the family of cubes.

PROPOSITION 2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping, and let $A \subset \mathbb{R}^n$ be a smooth set. Then $\phi(A)$ is smooth.*

Proof. We can assume that ϕ is a linear isomorphism. Given the smooth set A, consider

$$\omega(t) = \sup |D(Q) - D(Q')|, \quad 0 < t \leqslant 1,$$

where the supremum is taken over all pairs of consecutive cubes Q and Q' of the same sidelength $l(Q) = l(Q') \leqslant t$. Since $|\phi(A) \cap Q| = c(\phi)|A \cap \phi^{-1}(Q)|$, where $c(\phi)$ is a constant that only depends on ϕ, it is sufficient to show the smoothness condition, taking, instead of cubes, their preimages through ϕ, that is,

$$\lim_{|Q|\to 0} \frac{|A \cap \phi^{-1}(Q)| - |A \cap \phi^{-1}(Q')|}{|Q|} = 0. \tag{3.1}$$

Apply the Singular Value Decomposition (see, for instance, [**2**, Theorem 7.3.5]) to ϕ, which allows us to write $\phi = V\Sigma W$ where V, W are orthogonal mappings and Σ is a diagonal mapping, that is, the matrix of Σ is diagonal. Moreover, the elements of the diagonal of Σ are the positive square roots of the eigenvalues of $\phi\phi^*$. It will prove useful later that the elements $\lambda \in \mathbb{R}$ of the diagonal of Σ verify $\|\phi^{-1}\| \leqslant |\lambda| \leqslant \|\phi\|$ where $\|\phi\|$ denotes the norm of ϕ as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. So, we proceed to prove the proposition for these two cases: (a) ϕ is an orthogonal mapping and (b) ϕ is a dilation.

We study first case (a). As any orthogonal application is either a rotation or the composition of a rotation and a reflection by a subspace parallel to the axis (leaving invariant the grid of cubes), we can reduce the orthogonal case to rotations. Furthermore, we can assume that ϕ is the identity on a subspace of dimension $n-2$ generated by elements of the canonical basis of $\mathbb{R}^n$ and a rotation of angle $\alpha \in [\pi/6, \pi/4]$ on its orthogonal complement. Actually, any rotation can be written as the composition of at most $3n(n-1)$ rotations of this form (see, for instance, [**2**]). Let $\tilde{Q}$ and $\widetilde{Q'}$ be the cubes centred at the centres of $\phi^{-1}(Q)$ and $\phi^{-1}(Q')$, respectively, of sidelength $l(Q)$ with sides parallel to the axis. Observe that $\widetilde{Q'}$ is a translation of $\tilde{Q}$ by a vector of norm less than $nl(Q)$, so Lemma 3 implies that $|D(\tilde{Q}) - D(\widetilde{Q'})| \leqslant 3n^3\omega(l(Q))$. Hence, to show (3.1), it is enough to prove that

$$\lim_{|Q|\to 0} \frac{|A \cap \phi^{-1}(Q)| - |A \cap \tilde{Q}|}{|Q|} = 0. \tag{3.2}$$

We study first the case $n = 2$. We are going to decompose $\phi^{-1}(Q)$ into squares as follows. Let Q_0 be the maximal square with sides parallel to the axis contained in $\phi^{-1}(Q)$, and write $\mathfrak{F}_0 = \{Q_0\}$. Observe that the ratio of the area of Q_0 to that of $\phi^{-1}(Q)$ is $C = 1/(1 + \sin(2\alpha))$ and that $0.5 \leqslant C \leqslant 4 - 2\sqrt{3}$. Then $\phi^{-1}(Q) \setminus Q_0$ is the union of eight right-angled triangles

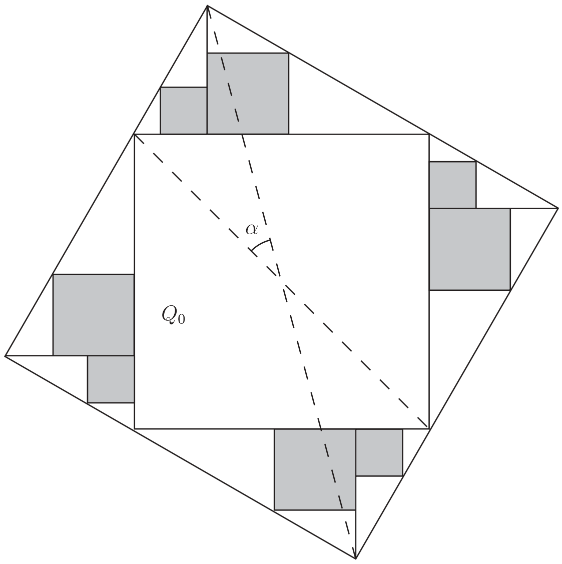

FIGURE 1. *The shaded squares are the eight elements of* $\mathfrak{F}_1$ *for* $\alpha = \pi/6$.

whose hypotenuse is contained in $\partial\phi^{-1}(Q)$; see Figure 1. Take again the maximal square with sides parallel to the axis contained in each triangle, obtaining a family $\mathfrak{F}_1$ of eight squares of total area $(1-C)^2|Q|$. Thus, $\phi^{-1}(Q) \setminus (\mathfrak{F}_0 \cup \mathfrak{F}_1)$ is the union of sixteen right-angled triangles and we continue inductively, constructing, for $k = 1, 2, \dots$, a family $\mathfrak{F}_k$ of 2^{k+2} squares of total area $C^{k-1}(1-C)^2|Q|$. Observe that

$$D(\phi^{-1}(Q)) - D(\tilde{Q}) = \sum_k \sum_{R\in\mathfrak{F}_k} \frac{|R|}{|Q|}(D(R) - D(\tilde{Q})).$$

Let R be a square in $\mathfrak{F}_k$. Since A is smooth, there exists a constant $C_1 > 0$ such that $|D(R) - D(\tilde{Q})| \leqslant C_1 k\omega(l(Q))$. We deduce that

$$\begin{aligned}|D(\phi^{-1}(Q)) - D(\tilde{Q})| &\leqslant C_1\omega(l(Q)) \sum_k k \frac{|\bigcup_{\mathfrak{F}_k} R|}{|\phi^{-1}(Q)|} \\ &\leqslant C_1(1-C)^2 C^{-1}\omega(l(Q)) \sum_k kC^k = C_1\omega(l(Q)).\end{aligned}$$

This implies (3.2) and completes the proof in dimension 2 when ϕ is a rotation.

We now study the higher dimensional case $n > 2$. Recall that ϕ^{-1} is a rotation on a two-dimensional subspace E and ϕ^{-1} is the identity on its orthogonal complement. Without loss of generality, we may assume that E is generated by the two first vectors of the canonical basis of $\mathbb{R}^n$. Consider the orthogonal projection Π of $\mathbb{R}^n$ onto E and decompose $\Pi(\phi^{-1}(Q))$ as in the two-dimensional case, that is, $\Pi(\phi^{-1}(Q)) = \bigcup_{k=0}^{\infty} \mathfrak{F}_k$, where $\mathfrak{F}_k$ is, as before, the union of 2^{k+2} (two-dimensional) squares with sides parallel to the axis of total area $C^{k-1}(1-C)^2 l(Q)^2$. Since $\phi^{-1}(Q) = \Pi(\phi^{-1}(Q)) \times B$, where B is a cube in $\mathbb{R}^{n-2}$ with sides parallel to the axis, we have $\phi^{-1}(Q) = \bigcup_{k=0}^{\infty} G_k$, where $G_k = \bigcup_{R\in\mathfrak{F}_k} R \times B$. Using the smoothness condition, one can show that there exists a constant $C_2 > 0$ such that $|D(R\times B) - D(\tilde{Q})| \leqslant C_2 k\omega(l(Q))$ for any square $R \in \mathfrak{F}_k$. Then

$$|D(\phi^{-1}(Q)) - D(\tilde{Q})| \leqslant C_2\omega(l(Q)) \sum_{k=0}^{\infty} k \frac{|G_k|}{|Q|}.$$

Since $|G_k| = (1-C)^2 C^{k-1}|Q|$, with $0.5 < C < 4 - 2\sqrt{3}$, we deduce that $|D(\phi^{-1}(Q)) - D(\tilde{Q})| \leqslant C_2\omega(l(Q))$ and thus (3.1) is satisfied, completing the proof in the case that ϕ is a rotation.

Let us now assume that ϕ is a dilation, that is, ϕ has a diagonal matrix. Without loss of generality, we can assume $\phi^{-1}(x_1, \ldots, x_n) = (\lambda x_1, x_2, \ldots, x_n)$ for some $\lambda \in \mathbb{R}$. Assume $\lambda > 1$. To prove (3.1), it is sufficient to find a constant $C(\lambda) > 0$ such that, for any cube $Q \subset \mathbb{R}^n$ and any cube $\tilde{Q} \subset \phi^{-1}(Q)$ with $l(\tilde{Q}) = l(Q)$, we have

$$|D(\phi^{-1}(Q)) - D(\tilde{Q})| \leqslant C(\lambda)\omega(l(Q)). \tag{3.3}$$

The proof of (3.3) resembles that of part (a) of Lemma 3. We can assume that $\tilde{Q}$ is the unit cube. Let $[\lambda]$ be the integer part of λ and write the interval $[[\lambda], \lambda)$ as a union of maximal dyadic intervals $\{I_k\}$ with $|I_k| = 2^{-k}$, that is, $[[\lambda], \lambda) = \bigcup I_k$. Consider $R_k = I_k \times [0,1]^{n-1}$, $k = 1, 2, \ldots$. Observe that

$$|D(\phi^{-1}(Q)) - D(\tilde{Q})| = \sum_{j=0}^{[\lambda]-1} \frac{1}{\lambda}(D([j, j+1) \times [0,1]^{n-1}) - D(\tilde{Q})) + \sum_{k=1}^{\infty} \frac{2^{-k}}{\lambda}(D(R_k) - D(\tilde{Q})).$$

For $j = 0, 1, \ldots, [\lambda] - 1$, we have $|D([j, j+1) \times [0,1]^{n-1}) - D(\tilde{Q})| \leqslant \lambda\omega(1)$. Since R_k can be split into a family of dyadic cubes of generation k, we deduce that $|D(R_k) - D(\tilde{Q})| \leqslant (\lambda + 1 + k)\omega(1)$. Therefore,

$$|D(\phi^{-1}(Q)) - D(\tilde{Q})| \leqslant (\lambda + 1 + 3/\lambda)\omega(1),$$

which proves (3.3). An analogous argument can be used in the case $\lambda < 1$. □

Remark 1. The first part of the proof shows that there exists a constant $C = C(n) > 0$ such that, for any rotation ϕ in $\mathbb{R}^n$ and any ω-smooth set $A \subset \mathbb{R}^n$, its image $\phi(A)$ is $C\omega$-smooth. When ϕ is a dilation in a single direction with parameter $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}^n$ is an ω-smooth set, the proof shows that $\phi(A)$ is $C(\lambda)\omega$-smooth, with $C(\lambda) \leqslant 4(\lambda + 1/\lambda)$.

Remark 2. Let $\{T_i\}$ be a countable family of linear isomorphisms in $\mathbb{R}^n$ for which there exists a constant $M > 0$ such that $M^{-1}\|x\| \leqslant \|T_i(x)\| \leqslant M\|x\|$ for any $x \in \mathbb{R}^n$ and $i = 1, 2, \ldots$. Then there exists a constant $C = C(M, n) > 0$ such that, for any ω-smooth set A and any i, one has

$$\frac{\||T_i(Q) \cap A| - |T_i(Q') \cap A|\|}{|Q|} \leqslant C\omega(l(Q)).$$

Remark 3. Proposition 2 and part (a) of Lemma 3 give that affine mappings preserve smooth sets.

We could have defined smooth sets using the grid of dyadic cubes or, in the opposite direction, using the grid of all cubes, even without taking them parallel to the axis. The previous results imply that both grids would lead to equivalent definitions.

Corollary 2. *Let A be a measurable set in $\mathbb{R}^n$. The following are equivalent.*

(a) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|D(Q) - D(Q')| \leqslant \varepsilon$ for any pair of consecutive dyadic cubes Q, Q', of the same sidelength $l(Q) = l(Q') < \delta$.*

(b) *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|D(Q) - D(Q')| \leqslant \varepsilon$ *for any pair of consecutive cubes* Q, Q' *with sides non-necessarily parallel to the axis, of the same sidelength* $l(Q) = l(Q') < \delta$.

(c) *The measurable set* A *is a smooth set.*

Proof. Since any cube in $\mathbb{R}^n$ is the affine image of a dyadic cube, Lemma 3 shows that (a) implies (b). The other implications are obvious. □

Observe that bilipschitz mappings do not preserve smoothness in general. Applying locally Proposition 2, we can extend it to certain diffeomorphisms, but we need extra assumptions to guarantee that the local bounds that we obtain are satisfied uniformly. We are now ready to proceed with the proof of Theorem 2.

Proof of Theorem 2. Since ϕ is bilipschitz, $|\phi(Q)|$ is comparable to $|Q|$. Also, $|J\phi|$ is uniformly bounded from above and below. Therefore, $J\phi^{-1}$ is uniformly continuous as well. We also need that

$$\lim_{|Q|\to 0} \frac{|\phi(Q)| - |\phi(Q')|}{|Q|} = 0. \tag{3.4}$$

To show this, observe that this quantity is

$$\frac{1}{|Q|}\left(\int_Q J\phi - \int_{Q'} J\phi\right),$$

which tends to 0 uniformly when $l(Q) \to 0$ because of the uniform continuity of $J\phi$.

We first show that (b) is equivalent to (c). A change of variables gives that

$$|\phi^{-1}(A) \cap Q| - |\phi^{-1}(A) \cap Q'| = \int J\phi^{-1}(x)(\mathbb{1}_{A\cap\phi(Q)}(x) - \mathbb{1}_{A\cap\phi(Q')}(x))\, dx.$$

Let $p(Q)$ be a point in $\overline{\phi(Q)} \cap \overline{\phi(Q')}$. Given $\varepsilon > 0$, if $l(Q)$ is sufficiently small, then one has $\|J\phi^{-1}(x) - J\phi^{-1}(p(Q))\| < \varepsilon$ for any $x \in \phi(Q)$. Hence, the uniform continuity of $J\phi^{-1}$ gives us that

$$\lim_{|Q|\to 0} \frac{|\phi^{-1}(A) \cap Q| - |\phi^{-1}(A) \cap Q'|}{|Q|} = \lim_{|Q|\to 0} \frac{(|A \cap \phi(Q)| - |A \cap \phi(Q')|)J\phi^{-1}(p(Q))}{|Q|}.$$

Let $D(\phi(Q))$ be the density of A in $\phi(Q)$, that is, $D(\phi(Q)) = |A \cap \phi(Q)|/|\phi(Q)|$. Applying (3.4), we have

$$\lim_{|Q|\to 0} \frac{|\phi^{-1}(A) \cap Q| - |\phi^{-1}(A) \cap Q'|}{|Q|} = \lim_{|Q|\to 0} (D(\phi(Q)) - D(\phi(Q')))J\phi^{-1}(p(Q)).$$

Since $J\phi^{-1}$ is uniformly bounded both from above and below, we deduce that (b) and (c) are equivalent.

We now show that (a) implies (c). Observe that, applying (3.4), it is sufficient to show

$$\lim_{|Q|\to 0} \frac{|A \cap \phi(Q)| - |A \cap \phi(Q')|}{|Q|} = 0. \tag{3.5}$$

Let $z(Q)$ be a point in Q with dyadic coordinates. Let $T = T(Q)$ be the affine mapping defined by $T(x) = \phi(z(Q)) + D\phi(z(Q))(x - z(Q))$ for any $x \in \mathbb{R}^n$, where $D\phi$ denotes the differential of ϕ. Given $\varepsilon > 0$, the uniform continuity of $J\phi$ tells us that $|\phi(x) - T(x)| \leqslant \varepsilon l(Q)$ for any $x \in Q \cup Q'$ if $l(Q)$ is sufficiently small. Thus, there exists a constant $C_1(n) > 0$ such that if $l(Q)$ is sufficiently small, then

$$|(\phi(Q) \setminus T(Q)) \cup (T(Q) \setminus \phi(Q))| \leqslant C_1(n)\varepsilon|Q|,$$

and similarly for Q'. So we deduce that (3.5) is equivalent to

$$\lim_{|Q|\to 0} \frac{|A\cap T(Q)| - |A\cap T(Q')|}{|Q|} = 0. \tag{3.6}$$

Now (3.6) follows from Remark 2 because, since ϕ is bilipschitz, there exists a constant $M > 0$ such that $M^{-1}\|x\| \leqslant \|D\phi(z(Q))(x)\| \leqslant M\|x\|$ for any $x \in \mathbb{R}^n$ and any cube Q in $\mathbb{R}^n$. This completes the proof that (a) implies (c). The proof that (b) implies (a) follows applying the previous part to ϕ^{-1}. □

Acknowledgements. We would like to thank Professors F. Cedó and W. Dicks for their help with linear algebra techniques.

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Artur Nicolau and Daniel Seco
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Spain

artur@mat.uab.cat
dseco@mat.uab.cat